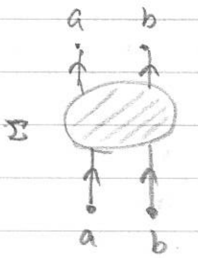
§ 6.3 BS wavefunction in a QFT process



$$=: (2\pi)^4 \delta^4(P'_{CM} - P_{CM})\, G(P_{CM}; P', P)$$

$$G(P_{CM}; P', P) = \sum_n \chi_n(p', \vec{P}'_{CM}) \frac{i}{(P_{CM}^2 - M_n^2 + i\varepsilon)} \chi_n^*(p, \vec{P}_{CM})$$

$$+ (\text{non-bound-state contribution})$$

$$\langle \Omega | T\{\psi_a(-p_3)\psi_b(-p_4)\} | n; \vec{P}_{CM}\rangle = (2\pi)^4 \delta^4(p_3 + p_4 - P_{CM})\, \chi_n(p', \vec{P}_{CM})$$

$\Uparrow$ BS wavefcn.

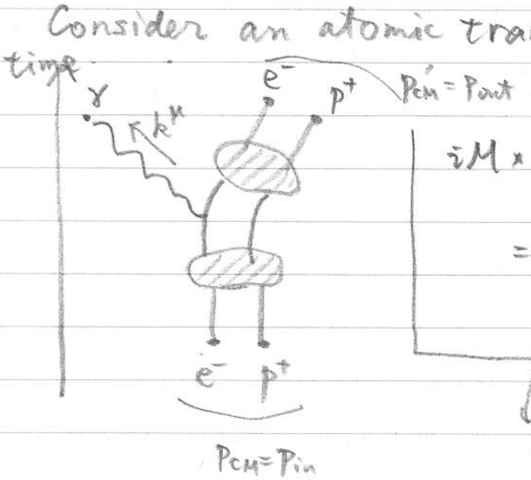
$$\psi(p) := \int d^4x\, \psi(x)\, e^{-ip\cdot x}$$

Under the non-relativistic & weak coupling approximat'n

$$\chi_n(p', \vec{P}_{CM}) \cong \frac{i}{\left[\eta_a(\Delta E) + \omega' - \frac{(\eta_a \vec{P}_{CM} + \vec{p}')^2}{2m_a} + i\varepsilon\right]} \frac{i}{\left[\eta_b(\Delta E) - \omega' - \frac{(\eta_b \vec{P}_{CM} - \vec{p}')^2}{2m_b} + i\varepsilon\right]} \underline{\underline{\chi_n}}(\vec{p}', \vec{P}_{CM})$$

$$\psi_n(\vec{p}', \vec{P}_{CM}) := \int \frac{d\omega'}{2\pi} \chi_n(p', \vec{P}_{CM}) \cong \frac{i}{\left[\Delta E - \frac{(\vec{P}_{CM})^2}{2(m_a + m_b)} - \frac{(\vec{p}')^2}{2\mu_{ab}}\right]} \underline{\underline{\chi_n}}(\vec{p}', \vec{P}_{CM})$$

Consider an atomic transition process $\boxed{\text{bnd state (i)} \longrightarrow \text{bnd state (f)} + \gamma}$



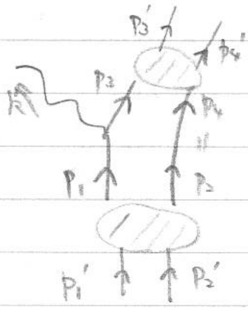
$$i\mathcal{M} \times (2\pi)^4 \delta^4(p_{in} - p_{out} - k)$$

$$= \text{Residue}\left[\langle\Omega| T\{A_\mu(-k)\, \psi_{e^-}(-p_3)\, \psi_{p^+}(-p_4)\, \psi^\dagger_{p^+}(p_1)\, \psi^\dagger_{e^-}(p_2)\} |\Omega\rangle\right]$$

(LSZ formula)

residue ~~of~~ the poles $\left(\frac{1}{k^2}\right), \left(\frac{1}{P_{CM}^2 - M_i^2}\right), \left(\frac{1}{P_{CM}^2 - M_f^2}\right)$.

set notation of the kinematics



$$\left[k^\mu = (k, \vec{k})\right] + \left[p_3^\mu = \left(m_e + \eta_e(\Delta E_f(-\vec{k})) + \omega', -\eta_e\vec{k} + \vec{p}'\right)\right] \quad \left[p_4^\mu = \left(m_p + \eta_p(\Delta E_f(-\vec{k})) - \omega', -\eta_p\vec{k} - \vec{p}'\right)\right]$$

$$\left[p_1^\mu = \left(m_e + \eta_e(\Delta E_i) + \omega, \vec{0} + \vec{p}\right)\right] \qquad \left[p_2^\mu = \left(m_p + \eta_p(\Delta E_i) - \omega, \vec{0} - \vec{p}\right)\right]$$

from $\vec{p}_2 = \vec{p}_4$: $\quad \vec{p} = \eta_p\vec{k} + \vec{p}'$

from $(p_1+p_2)^0 = (k+p_3+p_4)^0$: $\quad \underset{\text{neg.}}{(\Delta E_i)} = \underset{\text{pos}}{k} + (\Delta E_f(-\vec{k}))$ $\quad \longrightarrow (\Delta E_i) - (\Delta E_f(-\vec{k})) > 0$

even more neg.

from $p_2^0 = p_4^0$: $\quad \omega = \omega' + \eta_p\{(\Delta E_i) - (\Delta E_f(-\vec{k}))\} = \omega' + \eta_p k$

Now write down the amplitude by picking up the residues.

set. $\vec{p}_{CM.in} = \vec{0}$ $\quad$ (so $\vec{p}_{CM.out} = -\vec{k}$)

$$\Delta(iM) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d\omega}{2\pi}\, \chi_f^*(\omega', \vec{p}'; -\vec{k}) \left(\frac{i}{\eta_p(\Delta E_i) - \omega - \frac{(\vec{p})^2}{2m_p} + i\epsilon}\right)^{-1} \chi_i(\omega, \vec{p}; \vec{0})$$

γ on e line.

$$\times \left(\frac{eQ_e}{2m_e}\, i(\vec{p}_{e.in} + \vec{p}_{e.out}) \cdot \vec{\varepsilon}^*(-\vec{k})\right)$$

$\mathcal{L}_{int} > \frac{eQ_e}{2m_e}\vec{A}\cdot(\vec{p}_{e.in} + \vec{p}_{out})$

$\vec{p}_{e.in} = \eta_e \cdot \vec{0} + \vec{p} \qquad \vec{p}_{e.out} = -\eta_e\vec{k} + \vec{p}' = \vec{p} - \vec{k}$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\frac{eQ_e}{2m_e}\vec{\varepsilon}^*(-\vec{k}) \cdot (\vec{p}2 - \vec{k})\right) \underline{\underline{\chi_f^*}}(\vec{p} - \eta_p\vec{k}; -\vec{k})\ \underline{\underline{\chi_i}}(\vec{p}; \vec{0})$$

$$\times \int \frac{d\omega}{2\pi} \frac{1}{\left(\eta_e(\Delta E_i) + \omega - \frac{(\vec{p})^2}{2m_e} + i\epsilon\right)\left(\eta_e(\Delta E_f(-\vec{k})) + \underset{\omega'}{(\omega - \eta_p k)} - \frac{(\vec{p}-\vec{k})^2}{2m_e} + i\epsilon\right)\left(\eta_p(\Delta E_i) - \omega - \frac{(\vec{p})^2}{2m_p} + i\epsilon\right)}$$

$(-\eta_e\vec{k}) + \vec{p}' = \vec{p} - \vec{k}$

The ω-integration is straight forward.

by using the following straightforward calculation

$$\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{1}{(\omega+A+i\epsilon)(\omega+B+i\epsilon)(-\omega+C+i\epsilon)}$$

$$=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{-1}{(A-B)(A+C)(B+C)}\left\{\frac{(B+C)}{(\omega+A+i\epsilon)}-\frac{(A+C)}{(\omega+B+i\epsilon)}+\frac{(A-B)}{(\omega-C-i\epsilon)}\right\}$$

$$=(\text{log divergence})\times\frac{\{-(B+C)+(A+C)-(A-B)=0\}}{(A-B)(A+C)(B+C)}+\frac{(-\pi i)}{2\pi}\frac{-(B+C)+(A+C)+(A-B)}{(A-B)(A+C)(B+C)}$$

$$=\frac{(-i)}{(A+C)(B+C)}$$

$$\Delta(i\mathcal{M})=\int\frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{-ieQ_e}{2m_e}\vec{\varepsilon}^*(-\vec{k})\cdot(2\vec{p}-\vec{k})\right)\frac{\chi_i(\vec{p};\vec{0})}{\left[(\Delta E_i)-\frac{(\vec{p})^2}{2\mu_{ep}}\right]}\frac{\chi_f^*(\vec{p}',-\vec{k})}{\left[(\Delta E_f(-\vec{k}))-\frac{(\vec{k})^2}{2(m_e+m_p)}-\frac{(\vec{p}')^2}{2\mu_{ep}}\right]}$$

$$=\int\frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{ieQ_e}{2m_e}\vec{\varepsilon}^*(-\vec{k})\cdot(2\vec{p}-\vec{k})\right)\psi_f^*(\underset{\substack{\| \\ (\vec{p}-\eta_p\vec{k})}}{\vec{p}'};-\vec{k})\,\psi_i(\vec{p};\vec{0})$$

$$=ieQ_e\;2(m_e+m_p)\int\frac{d^3\vec{p}}{(2\pi)^3}\psi_{NR.f}^*(\vec{p}-\eta_p\vec{k};-\vec{k})\frac{\vec{\varepsilon}^*(-\vec{k})\cdot(2\vec{p}-\vec{k})}{2m_e}\psi_{NR.i}(\vec{p};\vec{0})$$

$$\left([i\mathcal{M}]=\quad +1\quad +3\quad -3/2\quad\quad 0\quad\quad -3/2\;=+1\right.$$
as expected in $1\to 2$ body decay)

Roughly speaking, $i\mathcal{M}\sim m_H\times 2ieQ_e\,\vec{\varepsilon}^*\cdot\langle\vec{v}\rangle$. $\langle\vec{v}\rangle\sim\langle f|\frac{\vec{p}}{m_e}|i\rangle$

Further evaluation

☆ Because $k\sim\Delta E_i-\Delta E_f\sim\mathcal{O}(m_e\alpha^2)$

$\vec{p}\sim 1/r_B\sim m_e\alpha$ so $k\ll|\vec{p}|$.

the $\vec{p}$-integral may be approximated

$$\left(\begin{array}{l}\cdot\;(2\vec{p}-\vec{k})\Rightarrow 2\vec{p}\\ \cdot\;\chi_f^*(\vec{p}-\eta_p\vec{k};-\vec{k})\Rightarrow\chi_f^*(\vec{p};\vec{0})\end{array}\right)$$

to be. $$\int\frac{d^3\vec{p}}{(2\pi)^3}\psi_{NR.f}^*(\vec{p})\frac{\vec{\varepsilon}^*\cdot\vec{p}}{m_e}\psi_{NR.i}(\vec{p})=\int d^3\vec{x}\,\psi_{NR.f}^*(\vec{x})\frac{\vec{\varepsilon}^*\cdot(-i\vec{\partial})}{m_e}\psi_{NR}(\vec{x}).$$

☆ $\frac{\vec{p}}{m_e}=-i[\vec{x}_e,H]$ so $\langle f|\frac{\vec{p}}{m_e}|i\rangle=-i\langle f|[\vec{x},H]|i\rangle=-i(\Delta E_i-\Delta E_f)\langle f|\vec{x}|i\rangle$

$\left(\vec{x}=i\frac{\partial}{\partial\vec{p}}\right)$

so,

$$\Delta(i\mathcal{M}) \cong 2m_H\big(\overset{k}{\overset{\parallel}{(\Delta E_i) - (\Delta E_f)}}\big)\int d^3\vec{x}\ \psi^*_{NR,f}(\vec{x})\,(eQ_e\vec{x})\,\psi_{NR,i}(\vec{x})\ \cdot\big(\vec{\varepsilon}^*(-\vec{k})\big)$$

$$= 2m_H\ k\ \langle f|\vec{d}\,|i\rangle\cdot\vec{\varepsilon}^*(-\vec{k})$$

↑ electric dipole $eQ_e\langle\vec{x}\rangle$ matrix element (off-diagonal)

called dipole approximation.

To bring all together

$$\Delta(d\Gamma) = \underset{\substack{\uparrow\\ \text{initial}\\ \text{state}}}{\frac{1}{2m_H}}\ \underbrace{\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2k}\frac{1}{(2E_f)}(2\pi)\delta\big(k-((\Delta E_i)-(\Delta E_f))\big)}_{\text{final state phase space \& 4-momentum conservation}}\ |\mathcal{M}|^2$$

bnd state (i) at rest

$$\cong \frac{(2m_H)^2}{(2m_H)^2}\frac{d^2\Omega(\hat{k})}{(2\pi)^3}\,dk\,\delta(k-\cdots)\,\frac{k^2}{k}\,\pi\times k^2\left|\vec{\varepsilon}^*\langle\vec{d}\rangle\right|^2$$

$$= \frac{d^2\Omega}{8\pi^2}\,k^3\left|\vec{\varepsilon}^*\cdot\langle\vec{d}\rangle\right|^2 \sim \alpha k^3 r_B^2 \sim \alpha\times(m_e\alpha^2)^3\times\frac{1}{(m_e\alpha)^2}\sim m_e\alpha^5.$$

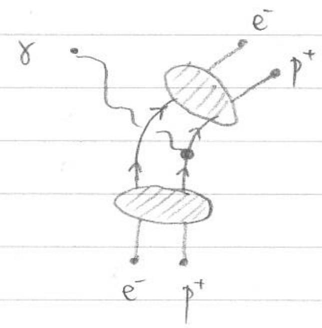
↑ dipole formula.

The decay width $\Gamma\sim m_e\alpha^5$ is smaller than

the fine structure splitting $O(m_e\alpha^4)$

but is larger than the hyperfine splitting.

— Another contribution



is evaluated similarly.

It turs out that this contribution is for

$$\frac{ieQ_e}{2m_e}\vec{\varepsilon}^{\,*}(-\vec{k})\cdot(2\vec{p}-\vec{k}) \Longrightarrow \frac{ieQ_p}{2m_p}\vec{\varepsilon}^{\,*}(-\vec{k})\cdot(2\vec{p}+\vec{k}).$$

So we can ignore this contribution due to $\left(m_e/m_p\right) \ll 1$.

— The selection rule in the electric dipole transition processes:

$\langle f|\vec{d}|i\rangle \neq 0$ only when $L_f \otimes (L=\text{spin-}1) \otimes L_i$ irreducible decomposition of $SU(2) \simeq SO(3)$ space rotation contains the $(L=0)$ component.

$$\Rightarrow L_i \otimes (L=1) = \begin{cases} (L_i+1)\oplus(L_i)\oplus(L_i-1) & \text{if } L_i \geq 1 \; \left(\text{p-wave or higher}\right) \\ (L=1) & \text{if } L_i = 0 \; (\text{s-wave}) \end{cases}$$

on spherical harmonics (not about earthquakes!)

⇒ If the initial bound state is in an S-wave state (e.g. 2S, 3S, etc.)

then the dipole transition is possible only to a p-wave state. (2S : nowhere to go)

If the initial bound state is not in an s-wave state (e.g. 2p, 3p, 3d, ...)

then the dipole transition is possible when $|L_f - L_i| \leq 1$.

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$$\Delta(i\mathcal{M}) = \int \frac{d\vec{p}}{(2\pi)^3} \frac{d\omega}{2\pi}\, \chi_f^*(\omega', \vec{p}\,'; -\vec{k}) \left(\frac{i}{\eta_e(\Delta E_i) + \omega - \frac{(\vec{p})^2}{2m_e} + i\varepsilon} \right)^{-1} \chi_i(\omega, \vec{p}; \vec{0})$$

$\uparrow$
γ to p^+ line

$$\times \left(\frac{eQ_p}{2m_p}\, i\left(\vec{p}_{p.in} + \vec{p}_{p.out}\right) \cdot \vec{\varepsilon}^{\,*}(-\vec{k}) \right)$$

$$\left[\begin{array}{l} \vec{p} = -\eta_e \vec{k} + \vec{p}\,' \qquad \omega = \omega' - \eta_e \left\{ (\Delta E_i) - \Delta E_f(\vec{k}) \right\} = \omega' - \eta_e k \\ \vec{p}_{p.in} = \vec{0} - \vec{p} \qquad \vec{p}_{p.out} = -\eta_p \vec{k} - \vec{p}\,' = -\eta_p \vec{k} - \eta_e \vec{k} - \vec{p} = -\vec{k} - \vec{p} \\ (\vec{p}_{p.in} + \vec{p}_{p.out}) = (-2\vec{p} - \vec{k}) \end{array} \right.$$

$$= \int \frac{d^3 p}{(2\pi)^3} \left(\frac{eQ_p}{2m_p} \vec{\varepsilon}^{\,*}(-\vec{k}) \cdot (-2\vec{p} - \vec{k}) \right) \underline{\underline{\chi_f^*}}(\vec{p} + \eta_e \vec{k}; -\vec{k})\ \underline{\underline{\chi_i}}(\vec{p}; \vec{0})$$

$$\int \frac{d\omega}{2\pi} \frac{1}{\left(\eta_p(\Delta E_i) - \omega - \frac{(\vec{p})^2}{2m_p} + i\varepsilon \right) \left(\eta_p(\Delta E_f(\vec{k})) - \underbrace{\omega'}_{\hookrightarrow (\omega + \eta_e k)} - \frac{(-\vec{k} - \vec{p})^2}{2m_p} + i\varepsilon \right) \left(\eta_e(\Delta E_i) + \omega - \frac{(\vec{p})^2}{2m_e} + i\varepsilon \right)}$$

$$\left[\begin{array}{l} \displaystyle\int \frac{d\omega}{2\pi} \frac{1}{(-\omega + A)(-\omega + B)(\omega + C)} \\ \displaystyle = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left\{ \frac{(B+C)}{(-\omega + A)} - \frac{(A+C)}{(-\omega + B)} + \frac{(B-A)}{(\omega + C)} \right\} \frac{1}{(B+C)(B-A)(C+A)} \\ \displaystyle = \frac{(\underline{\log\ div})}{2\pi} \times \frac{\left\{ -(B+C) + (A+C) + (B-A) = 0 \right\}}{(B+C)(B-A)(C+A)} + \frac{(-\pi i)}{2\pi} \frac{\left\{ -(B+C) + (A+C) - (B-A) = 2(A-B) \right\}}{(B+C)(B-A)(C+A)} \\ \displaystyle = \frac{i}{(B+C)(C+A)} \end{array} \right.$$

$$= \int \frac{d\vec{p}}{(2\pi)^3} \left(\frac{ieQ_p}{2m_p} \vec{\varepsilon}^{\,*}(-\vec{k}) \cdot (-2\vec{p} - \vec{k}) \right) \frac{\underline{\underline{\chi_f^*}}(\vec{p} + \eta_e \vec{k}; -\vec{k})}{\left[(\Delta E_f(\vec{k})) - \frac{(\vec{k})^2}{2(m_e + m_p)} - \frac{(\vec{p}\,')^2}{2\mu_{ep}} \right]} \frac{\underline{\underline{\chi_i}}(\vec{p}; \vec{0})}{\left[(\Delta E_i) - \frac{(\vec{p})^2}{2\mu_{ep}} \right]}$$

$$= \int \frac{d\vec{p}}{(2\pi)^3} \left(\frac{+ieQ_p}{2m_p} \vec{\varepsilon}^{\,*}(-\vec{k}) \cdot (2\vec{p} + \vec{k}) \right) \psi_f^*(\vec{p} + \eta_e \vec{k}; -\vec{k})\ \psi_i(\vec{p}; \vec{0})\ .$$

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§7. Unitarity

$\boxed{\mathcal{H}_{phys.}}$ (a vector space over $\mathbb{C}$) has a positive definite norm (no negative norm states)

§7.0 What if $\mathcal{H}_{phys}$ does not have a Fock-space structure?

(a Fock-space structure

$$\Longleftrightarrow \mathcal{H}_{phys} \cong \mathbb{C}|\Omega\rangle \oplus \left\{\begin{matrix}\text{1-particle} \\ \text{states}\end{matrix}\right\} \oplus \left\{\begin{matrix}\text{2-particle} \\ \text{states}\end{matrix}\right\} \oplus \cdots$$)

e.g. physics at a strongly coupled quantum critical point.

Still. symmetry of the system has its $\underbrace{\text{unitary}}_{\text{def. norm preserving}}$ representat'n on $\mathcal{H}_{phys}$.

translation. rotation. boost....

(also scale invariance at a critical point)

Just like the theory of finite-dim. unitary representation of $SU(2)$ yields a constraint $|\langle L_z\rangle| \leq l$

some constraints can be derived from the theory of infinite-dim unitary representation of certain symmetry algebra.

A combination of the unitary representation theory of the conformal algebra and the crossing symmetry. can be a very powerful tool in determing the operator dimensions in a conformal field theory.

§7.1 Partial wave unitarity

Weakly coupled QFT's have two Fock-space structures:

the in-state basis & the out-state basis.

The Hermitian inner product on $\mathcal{H}_{phys}$ looks precisely the same in those two bases.

$$C_{\beta\alpha} = \delta_{\beta\alpha} \cdot \Pi_i \left((2\pi)^3 \delta^3(\vec{p}_i - \vec{q}_i)(2E_{\vec{q}_i}) \right) \quad \text{in both}$$

$$\downarrow$$

$$C^{-1}_{\delta\gamma} = \delta_{\delta\gamma} \; \Pi_{i=1}^{n_i} \left((2\pi)^3 \delta^3(\vec{p}_i - \vec{q}_i)(2E_{\vec{q}_i}) \right)$$

$\Rightarrow$ The basis change matrix $(P^{o/i})_{\gamma\alpha}$

$$|\alpha\rangle^{in} = |\gamma\rangle^{out} (P^{o/i})_{\gamma\alpha}$$

is unitary with respect to the norm above (norm-preserving)

$$(C)_{\delta\gamma} (P^{o/i})_{\gamma\alpha} \left((P^{o/i})_{\delta\beta} \right)^{c.c} = C_{\beta\alpha}$$

$$\Updownarrow$$

$$S_{\beta\alpha} := C_{\beta\gamma} (P^{o/i})_{\gamma\alpha} = {}^{out}\langle \beta | \alpha \rangle^{in}$$

$$(C^{-1})_{\delta\gamma} S_{\gamma\alpha} (S_{\delta\beta})^{c.c.} = C_{\beta\alpha}$$: called the S-matrix unitarity

C

S-matrix unitarity $\Rightarrow$ partial wave unitarity.

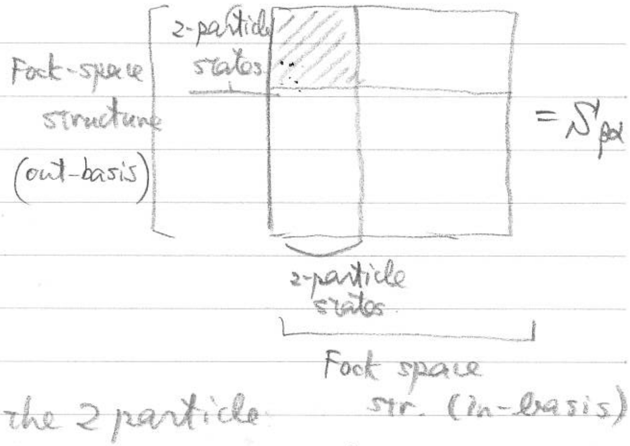
eg. a finite-dim unitary matrix U $\quad (U^\dagger U = \mathbb{1})$

$$\left.\begin{array}{l} \cdot\ \sum_{\text{all } i} |U_{ij}|^2 = 1 \\ \cdot\ \sum_{\text{some } i} |U_{ij}|^2 \le 1 \quad (*) \end{array}\right\} \text{ for any } j. \quad \left(\begin{array}{l}\text{for any column vector} \\ \text{of the matrix } U\end{array}\right)$$

Apply $(*)$ to the S-matrix

$$\cdot\ \sum_{\substack{\delta,\gamma: \\ \text{2-particle} \\ \text{states}}} C^{-1}_{\delta\gamma}\, S_{\gamma\alpha}\, (S_{\delta\alpha})^{c.c.} \le C_{\alpha\alpha}. \quad \left(\begin{array}{c}\text{for any 2 particle} \\ \text{state } \alpha\end{array}\right)$$

(partial wave unitarity)



Let us rewrite the sum over the 2 particle states to make the most of the partial wave unitarity.

◎ set the $\underbrace{\text{state } |\alpha\rangle^{in}}_{\text{2 particle}}$ to be in its center of mass frame.

◎ get rid of common factors

$$\Rightarrow \quad \boxed{\frac{1}{(4\pi)^8}\, \frac{p_1}{M}\, \frac{p_3}{M} \int d^2\Omega(\vec{p}_3)\; \underline{S_{34,78}}^{cc}\; \underline{S_{34,12}} \le \delta^2(\Omega_{\Phi_2} - \Omega_{\Phi_7})}$$

$$S^{2\to2}_{\beta\alpha} =: (2\pi)^4 \delta^4(p_\beta - p_\alpha)\, \underline{S_{\beta\alpha}}$$

$\alpha \Rightarrow (E_1, \vec{p}_1) + (E_2, -\vec{p}_1)$

$\beta \Rightarrow (E_3, \vec{p}_3) + (E_4, -\vec{p}_3)$

$\alpha' \Rightarrow (E_7, \vec{p}_7) + (E_8, -\vec{p}_7)$

$E_1 = E_7,\ E_2 = E_8$

D

$$(S'_{\delta\alpha'})^{cc}\, C^{-1}_{\delta\gamma}\, S_{\gamma\alpha} \qquad \alpha \Rightarrow p_1^\mu + p_2^\mu$$

$$\gamma \Rightarrow p_3^\mu + p_4^\mu \qquad \delta \Rightarrow p_5^\mu + p_6^\mu \qquad \alpha' \Rightarrow p_7^\mu + p_8^\mu$$

The right hand side

$$C_{\alpha'\alpha} \Rightarrow (2\pi)^3 \delta^3(\vec{p}_1 - \vec{p}_7)(2\pi)^3 \delta^3(\vec{p}_{CM} - \vec{p}'''_{CM})(2E_1)(2E_2)$$

rewrite $\delta^3(\vec{p}_1 - \vec{p}_7) = \delta(p_1 - p_7) \frac{1}{p_1^2} \delta^2(\theta\phi_1 - \theta\phi_7)$

(here, $\theta\phi_1$ and $\theta\phi_7$ specify the direction of $\vec{p}_1$ & $\vec{p}_7$

$\int d^2\Omega_{(\vec{p}_7)}\, \delta^2(\theta\phi_1 - \theta\phi_7) f(\theta_7, \phi_7) = f(\theta_1, \phi_1)$

is the definition of $\delta^2(\theta\phi_1 - \theta\phi_7)$)

So, the RHS is $\delta^2(\theta\phi_1 - \theta\phi_7) \times (2\pi)^6 \frac{(2E_1)(2E_2)}{p_1^2} \delta^3(\vec{p}_{CM} - \vec{p}'''_{CM})\delta(p_1 - p_7)$

$\vec{p}_{CM} = \vec{p}_1 + \vec{p}_2$, $\vec{p}'''_{CM} = \vec{p}_7 + \vec{p}_8$

The left hand side

$$(S'_{\delta\alpha'})^{cc}\, C^{-1}_{\delta\gamma}\, (S_{\gamma\alpha})$$

C^{-1}

$$= \int \frac{d^3p_5}{(2\pi)^3} \frac{1}{2E_5} \frac{d^3p_6}{(2\pi)^3} \frac{1}{2E_6} \int \frac{d^3p_3}{(2\pi)^3} \frac{1}{2E_3} \frac{d^3p_4}{(2\pi)^3} \frac{1}{2E_4} \left((2\pi)^3 \delta^3(\vec{p}_3 - \vec{p}_5)(2\pi)^3 \delta^3(\vec{p}'_{CM} - \vec{p}''_{CM}) \times (2E_3)(2E_4) \right)$$

$$\times (2\pi)^4 \delta(E_7 + E_8 - E_1 - E_2)\delta^3(\vec{p}'''_{CM} - \vec{p}_{CM}) \quad \left(\underline{S_{\cancel{34}\,78}}\right)^{cc}$$

$$\times (2\pi)^4 \delta(E_3 + E_4 - E_1 - E_2)\delta^3(\vec{p}'_{CM} - \vec{p}_{CM}) \quad \left(\underline{S_{34,12}}\right)$$

$$= \int \frac{d^3p_3}{(2\pi)^3} \frac{1}{2E_3\, 2E_4} (2\pi)^2 \delta(E_3 + E_4 - E_1 - E_2)\delta(E_7 + E_8 - E_1 - E_2)\, \underline{S^{cc}_{34,78}}\, \underline{S_{34,12}} \times (2\pi)^3 \delta^3(\vec{p}'''_{CM} - \vec{p}_{CM})$$

$$= \int dp_3 \int d^2\Omega(\vec{p}_3) \frac{(p_3)^2}{2E_3\, 2E_4} \frac{E_3 E_4}{p_3 E_{CM}} \frac{1}{(2\pi)} \delta(p_3 - p_{3,\text{right value}}) \frac{E_1 E_2}{p_1 E_{CM}} \delta(p_7 - p_1) \times \underline{S^{cc}_{34,78}}\, \underline{S_{34,12}} \times (2\pi)^3 \delta^3(\vec{p}_{CM} - \vec{p}'''_{CM})$$

$$= \int d^2\Omega(\vec{p}_3) \frac{1}{2^4 (2\pi)^4} \frac{p_3}{E_{CM}} \frac{p_1}{E_{CM}} \cdot \underline{S^{cc}_{34,78}}\, \underline{S_{34,12}} \times \left[(2\pi)^6 \delta^3(\vec{p}_{CM} - \vec{p}'''_{CM})\, \delta(p_1 - p_7) \frac{(2E_1)(2E_2)}{p_1^2} \right]$$

$$\Rightarrow \boxed{\delta^2(\theta\phi_1 - \theta\phi_7) = \int d^2\Omega(\vec{p}_3) \frac{1}{(4\pi)^4} \frac{p_3}{E_{CM}} \frac{p_1}{E_{CM}} \underline{S^{cc}_{34,78}}\, \underline{S_{34,12}}}$$

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So. it is convenient to introduce

$$\underline{S^{2\to2}_{34.12}} =: (4\pi)^2 \sqrt{\frac{v_{12}}{v_1 v_2}} \sqrt{\frac{v_{34}}{v_3 v_4}} \; \underline{S^{2\to2}_{34.12}}\,\mathrm{red}.$$

$v_1 = P_1/E_1$, etc. $\quad v_{12} E_1 E_2 = |E_1 P_{z.2} - E_2 P_{z.1}|$

(when $\vec{p}_1 \parallel e_z$ and $\vec{p}_2 \parallel e_z$.)

$\rightarrow$ in the center of mass frame

$$\frac{v_{12}}{v_1 v_2} = \frac{|E_1 P_{z.1} - E_2 P_{z.2}|}{|P_{z.1}|\,|P_{z.2}|} = \frac{(E_1+E_2)}{P_1} = \frac{E_{CM}}{P_1}.$$

$(P_{z.1} = P_1. \quad P_{z.2} = -P_1)$

because the partial wave unitarity condition is rewritten as

$$\boxed{\int d^2\Omega(\vec{p}_3)\; \underline{S_{34.78}}_{\mathrm{red}}^{cc}\; \underline{S_{34.12}}_{\mathrm{red}} \le \delta^2(\theta.\phi_1 - \theta.\phi_7)}$$

- $\delta^2(\theta.\phi_1 - \theta.\phi_7) = \sum_{l.m} Y_{l.m}(\theta.\phi_7)\,(Y_{l.m}(\theta.\phi_1))^{cc.}$

- here. $\int d^2\Omega\; Y_{l.m}(\theta.\phi)\,(Y_{l'.m'}(\theta.\phi))^{cc} = \delta_{l.l'}\,\delta_{m.m'}$

$$Y_{l.0}(\theta.\phi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$$

- parametrize $\underline{S^{2\to2}_{34.12}}_{\mathrm{red}} =: \sum_{l.m} Y_{l.m}(\theta.\phi_3)\, e^{2i\delta_l(E_{CM})}\,(Y_{l.m}(\theta.\phi_1))^{c.c.}$

Then. $\sum_{l.m} Y_{l.m}(\theta.\phi_7)\; |e^{2i\delta_l(E_{CM})}|^2\,(Y_{l.m}(\theta.\phi_1))^{cc} \le \sum_{l.m} Y_{l.m}(\theta.\phi_7)\,(Y_{l.m}(\theta.\phi_1))^{c.c.}$

$$\boxed{|e^{2i\delta_l(E_{CM})}| \le 1}$$

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