If $\left| e^{2i\delta_\ell(E)} \right| < 1$ for some ℓ (not a pure phase shift)

then $\sum_\gamma \left| S^{2\to n\geq 3}_{\gamma\alpha} \right|^2 > 0$

When the particles in the Fock space have non-0 spins.

$\underline{S^{2\to 2}_{\beta\alpha}}_{\text{red}}$ should be parametrized by $e^{2i\delta_j(E_{cm})}$

summed over the total angular momentum $\vec{J}$

(see Weinberg I. §3.7)

<Let us keep the story simple in this lecture note>

A relation between the partial wave unitarity constraint and perturbative calculation

We may compute $i\mathcal{M}^{2\to 2}_{\beta\alpha}$ using Feynman rule.

$$S^{2\to 2}_{\beta.\alpha} = \mathbb{1}_{\beta.\alpha} + (2\pi)^4 \delta^4(p_\beta - p_\alpha)\, i\mathcal{M}_{\beta.\alpha}$$

$$\parallel$$

$$(2\pi)^4\delta^4(p_\beta - p_\alpha)\cdot(4\pi)^2 \sqrt{\frac{v_{12}}{v_1 v_2}} \sqrt{\frac{v_{34}}{v_3 v_4}}\, \underline{S}^{2\to 2}_{\text{red}}$$

Similarly we introduce $i\mathcal{M}^{2\to 2}_{\beta.\alpha} = (4\pi)^2 \sqrt{\frac{v_{12}}{v_1 v_2}} \sqrt{\frac{v_{34}}{v_3 v_4}} \left(\underline{i\mathcal{M}}^{2\to 2}_{\beta.\alpha\,\text{red}} \right).$

Then $\boxed{\underline{i\mathcal{M}}^{2\to 2}_{34.12\,\text{red}} = \sum_{\ell.m} Y_{\ell.m}(\theta.\phi_3)\left(e^{2i\delta_\ell} - 1\right)\left(Y_{\ell.m}(\theta.\phi_1)\right)^*}$

It follows that.

$$d\sigma^{2\to 2} = \frac{1}{(2E_1)(2E_2)\, v_{12}} \int \frac{d^3p_3}{(2\pi)^3}\frac{1}{(2E_3)} \frac{d^3p_4}{(2\pi)^3}\frac{1}{(2E_4)} (2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\, |\mathcal{M}|^2$$

$$= \cdots = \frac{\pi}{p_1^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| e^{2i\delta_\ell(E_{cm})} - 1 \right|^2$$

☆ In a QFT model with small coupling constants

Feynman rule calculations often yield

$$i\mathcal{M}^{2\to2}_{\beta\alpha} \sim \text{dimensionless } \overbrace{\text{imaginary}}^{\text{almost pure}} \text{ and small.}$$

$$e^{2i\delta_\ell} = 1 + i(\mathcal{M})_{\ell\text{ component}} \simeq 1.$$

$\delta_\ell \sim$ small phase shift.

☆ In a QFT model where known contributions to $i\mathcal{M}^{2\to2}$

do not satisfy the partial wave unitarity,

(for some ℓ and at certain energy scale)

$$\left| 1 + \frac{R}{2(4\pi)^2} \int d^2\Omega(\hat{p}_3) \, (\Delta\, i\mathcal{M}^{2\to2})(\theta,\phi_3) \left(Y_{\ell,m}(\theta,\phi_3)\right)^* \cdot \left(Y_{\ell,0}(\theta=0) = \sqrt{\frac{2\ell+1}{4\pi}}\right)^{-1} \right| > 1,$$

$$\left(R := \sqrt{\frac{v_1 v_2}{(v_{12}/2)}} \sqrt{\frac{v_3 v_4}{(v_{34}/2)}} \right)$$

then that is an indication that there must be

additional contributions (operators) (interactions terms)

in the model.

example

$$\mathcal{L} \sim G_F \left(\bar{\Psi} \gamma^\mu \left(\frac{1-\gamma_5}{2}\right) \Psi\right) \left(\bar{\Psi} \gamma_\mu \left(\frac{1-\gamma_5}{2}\right) \Psi\right) \Rightarrow \mathcal{M} \sim G_F E_{CM}^2.$$

$$\mathcal{L} \sim \frac{1}{M} (\ell h)(\ell h) \Rightarrow \mathcal{M} \sim \left(\frac{E_{CM}}{M}\right)$$

something has to happen @ $\begin{cases} E \sim 1/\sqrt{G_F} \\ E \sim M \end{cases}$

☆ The black disc limit : observed in hadron–hadron scattering

$\overset{\text{def}}{\Leftrightarrow}$ $|e^{2i\delta_\ell(E)}| \ll 1$ for some (range of) ℓ's at $\overbrace{E = E_{CM}}^{\text{energy}}$

That is when $e^{2i\delta_\ell} \simeq (1 + i \times \underset{\approx}{i})$

$$\mathcal{M} \sim \Sigma_{\ell,m} Y_{\ell,m}(\theta,\phi_3) \, \underset{\approx}{i} \, (Y_{\ell,m}(\theta,\phi_3))^{cc}$$

$$\Rightarrow \left\{ \begin{array}{l} \sum_\beta^{2\text{-body}} |\mathcal{M}_{\beta\alpha}|^2 = 1 \\ \sum_\beta^{n\geq3\text{ body}} |S_{\beta\alpha}|^2 = 1 \end{array} \right\} \Rightarrow \left[\sigma_\ell^{2\to2} \simeq \sigma_\ell^{2\to(n\geq3)} \right]$$

§7.2 Optical theorem

The unitarity of the S-matrix also implies

$$\mathbb{1} = S^\dagger S = \left(\mathbb{1} + (2\pi)^4\delta^4(p_\gamma - p_\beta)(-i\mathcal{M}^*_{\beta\gamma})\right)\left(\mathbb{1} + (2\pi)^4\delta^4(p_\beta - p_\alpha)\, i\mathcal{M}_{\beta\alpha}\right)$$

$$= \mathbb{1} + (2\pi)^4\delta^4(p_\gamma - p_\alpha)\left\{ i\left[\mathcal{M}_{\gamma\alpha} - (\mathcal{M}_{\alpha\gamma})^*\right] + \sum_\beta (\mathcal{M}_{\beta\gamma})^*(\mathcal{M}_{\beta\alpha})(2\pi)^4\delta^4(p_\beta - p_\alpha)\right\}$$

so

$$\boxed{\frac{\mathcal{M}_{\gamma\alpha} - (\mathcal{M}_{\alpha\gamma})^*}{(i)} = \int \prod_{j=1}^{N_\beta}\left[\frac{d\vec{p}^{\,3}_j}{(2\pi)^3}\frac{1}{(2E_j)}\right](2\pi)^4\delta^4(p_\beta - p_\alpha)(\mathcal{M}_{\beta\gamma})^*(\mathcal{M}_{\beta\alpha})}$$

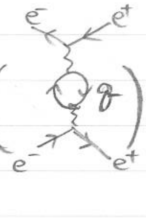
As a particular case $\gamma = \alpha$,

$$\boxed{2\,\mathrm{Im}(\mathcal{M}_{\alpha\alpha}) = \int \prod_{j=1}^{N_\beta}\left[\frac{d\vec{p}^{\,3}_j}{(2\pi)^3}\frac{1}{(2E_j)}\right](2\pi)^4\delta^4(p_\beta - p_\alpha)\left|\mathcal{M}_{\beta\alpha}\right|^2 = \begin{cases}\sigma_{tot}\cdot(4E_1E_2v_{12}) \\ \text{or} \\ \Gamma_{tot}\cdot(2E)\end{cases}}$$

optical theorem

Useful because....

☆ $\boxed{\mathrm{Im}(\mathcal{M}_{\alpha\alpha}) \Longrightarrow \Gamma_{tot},\ \sigma_{tot}}$

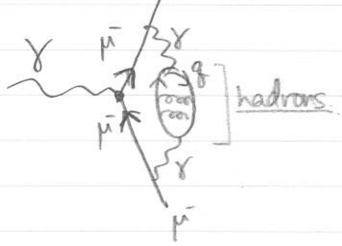
$e^+ + e^- \to \gamma$ or $Z \to$ hadrons $\quad (\sigma_{tot}\cdot s) \simeq 2\,\mathrm{Im}\left[\mathcal{M}\left(\right)\right]$ 

at $E_{CM} \gg$ GeV

[Perturbative calcutations are available for "inclusive enough" observables.]
such as σ_{tot}.

☆ $\boxed{\Gamma_{tot},\ \sigma_{tot} \Longrightarrow \mathrm{Im}(\mathcal{M}_{\alpha\alpha})}$

to estimate contributions to anomalous magnetic moment of μ^-,



we can use $[\sigma_{tot}(e^+e^- \to \text{hadron})\cdot s]$ to determine $\mathrm{Im}(\mathcal{M}_{\gamma^*\gamma^*})$.
unitarity

and $\mathrm{Re}(\mathcal{M}_{\gamma^*\gamma^*})(s+i\varepsilon) = \int \frac{ds'}{\pi}\frac{\mathrm{Im}(\mathcal{M}_{\gamma^*\gamma^*}(s'))}{s'-s}$.

dispersion integral $\quad$ ($\mathcal{M}(s+i\varepsilon)$: holomorphic in s)

[Kramers-Kronig relation]

☆ $\boxed{\sigma_{tot} \longrightarrow \mathrm{Im}(\mathcal{M}_{\alpha\alpha})}$ for perturbative calculation

Think of a theory with $\mathcal{L} = \mathcal{L}_{kin.} + g\,\phi\,\bar{\Psi}\Psi$. ϕ: scalar

$$\Rightarrow \int \frac{d^3\vec{p}_\Psi}{(2\pi)^3}\frac{1}{2E_{p_\Psi}} \int \frac{d^3\vec{p}_{\bar{\Psi}}}{(2\pi)^3}\frac{1}{2E_{p_{\bar{\Psi}}}} (2\pi)^4\delta^4(p_\Psi + p_{\bar{\Psi}} - p_\phi)\, \underbrace{|\mathcal{M}|^2}_{\text{spin-sum}} = \frac{g^2}{4\pi}\left\{E^2 - (2m_\Psi)^2\right\}.$$

$$\left[\, i\mathcal{M}(\underset{\substack{\Downarrow \\ (E,\vec{0})\ \text{CM frame}}}{\phi} \longrightarrow \Psi + \bar{\Psi}) \; = \right.$$

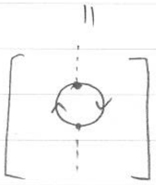
$]$ straightforward calculation. (tree level)

So...

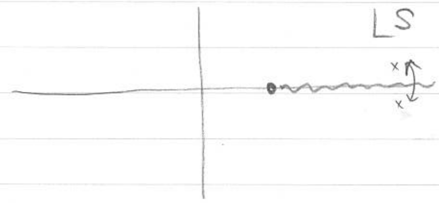
$$\boxed{2\,\mathrm{Im}\left[\mathcal{M}(\phi\to\phi)\right] \Leftarrow \frac{g^2}{4\pi}\left\{s - (2m_\Psi)^2\right\} \quad \text{unitarity}}$$

$$\to \mathcal{M}(\phi\to\phi) \text{ at } s = \frac{-g^2}{8\pi^2}\left\{s-(2m_\Psi)^2\right\}\ln\left(\frac{(2m_\Psi)^2 - s - i\varepsilon}{(2m_\Psi)^2}\right) + \left(\underset{(\text{for real } s)}{\text{rational. real}}\right)$$

$=$



$/(i)$

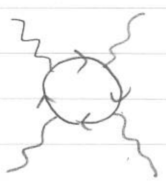


$\left[\begin{array}{l}\text{branch cut} \\ \text{at } (2m_\Psi)^2 \le s\end{array}\right]$

We have managed to obtain an expression for a 1-loop graph

without doing 1-loop computation.

More generally...



$= i\mathcal{M}(s,t)$ should be a holomorphic function of (s,t) except poles and branch cuts that satisfy all of $\left\{\begin{array}{l} s\text{-channel} \\ t\text{-channel} \\ u\text{-channel}\end{array}\right\}$ unitarity relation.

§ 8 Low-energy effective theory

Here, "theory" is in the sense of <u>model</u>.

$\boxed{\text{Example 1}}$ QED with $\gamma, e^{\pm}, \mu^{\pm}$ $\longrightarrow$ QED with $\gamma, e^{\pm}$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}_{(e)}(i\gamma^\mu D_\mu - m_{(e)})\Psi_{(e)} + \bar{\Psi}_{(\mu)}(i\gamma^\mu D_\mu - m_{(\mu)})\Psi_{(\mu)} \quad \text{———} (*)$$

Start from a theory (=model) above.

If we are interested in physics with energy below $m_{(\mu)} \sim 106$ MeV,

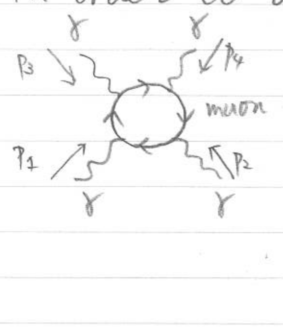
we do not have to maintain $\Psi_{(\mu)}$ in the Lagrangian.

But we have to use

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}_{(e)}(i\gamma^\mu D_\mu - m_{(e)})\Psi_{(e)} + \frac{e^4 C^{(1)}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}}{16\pi^2 m_{(\mu)}^4} F^{\mu_1\nu_1}F^{\mu_2\nu_2}F^{\mu_3\nu_3}F^{\mu_4\nu_4}$$

$$+ \cdots \quad \text{———} (**)$$

in order to account for the $\gamma+\gamma \to \gamma+\gamma$ scattering amplitude



$= i\mathcal{M}(m_{(\mu)}, p_i^\mu, \varepsilon_i^\nu)$ expand in power series of $p/m_{(\mu)}$

$$= i \frac{e^4 C^{(1)}_{\mu_1\nu_1\cdots\mu_4\nu_4}}{16\pi^2 m_{(\mu)}^4} (p_1^{\mu_1}\varepsilon_1^{\nu_1} - p_1^{\nu_1}\varepsilon_1^{\mu_1})(p_2^{\mu_2}\varepsilon_2^{\nu_2} - p_2^{\nu_2}\varepsilon_2^{\mu_2})(p_3^{\mu_3}\varepsilon_3^{\nu_3} - p_3^{\nu_3}\varepsilon_3^{\mu_3})(p_4^{\mu_4}\varepsilon_4^{\nu_4} - p_4^{\nu_4}\varepsilon_4^{\mu_4})$$

$$+ \mathcal{O}(p^6/m_{(\mu)}^6)$$

<u>The latter (**) is the low-energy effective theory $\underset{(=\text{model})}{}$ of the former (*).</u>

The latter theory with just the $\mathcal{O}(1/m_{(\mu)}^4)$ term

will violate partial wave unitarity at $E \sim m_{(\mu)}$.

But all the terms in the $p/m_{(\mu)}$ expansion are equally important

in the partial wave unitarity at $E \sim m_{(\mu)}$

We should use the high-energy theory (*) at $E \gtrsim m_{(\mu)}$

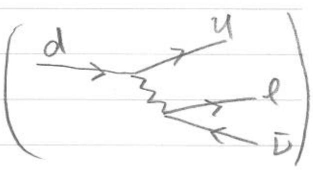
$\boxed{\text{Exampl 2}}$ The Standard Model $\longrightarrow$ QCD + QED + 4-fermi term

$$\mathcal{L}_i \supset \left[\bar{u}_i \, i\gamma^\mu \left(\frac{1-\gamma_5}{2}\right) ig_w \left(\frac{A^1_\mu + iA^2_\mu}{2}\right) d_j\right] V_{ij} \quad - \frac{1}{4g_w^2} \underbrace{\left(F^1_{\mu\nu}F^{1\mu\nu} + F^2_{\mu\nu}F^{2\mu\nu} + F^3_{\mu\nu}F^{3\mu\nu}\right)}_{\text{W-boson kinetic term}}$$

$$+ \left[\bar{\ell} \, i\gamma^\nu \left(\frac{1-\gamma_5}{2}\right) ig_w \left(\frac{A^1_\nu - iA^2_\nu}{2}\right) \nu\right] \quad + \cdots \qquad \text{———} (*)$$

- i: subscripts in u_i, d_i $\Rightarrow$ generation ($i = 1, 2, 3$)
- V_{ij}: a 3×3 unitary matrix (called Cabbibo Kobayashi Maskawa matrix (CKM))
- In the Peskin-Schroeder convention,

$$\gamma^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix} \qquad \gamma_5 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



This high-energy theory yields.

$$i\mathcal{M}(d_j \to u_i + \ell + \bar{\nu}) = \left[\bar{u}(\vec{p}_{u_i})\gamma^\mu(1-\gamma_5)u(\vec{p}_{d_j})\right] V_{ij} \times \frac{\left(\frac{ig_w}{4}\right)\left(\frac{ig_w}{4}\right)(-i \times \eta_{\mu\nu} \times 2)}{(p_d - p_u)^2 - m_W^2 + i\varepsilon}$$
$$\left[\bar{u}(\vec{p}_\ell)\gamma^\nu(1-\gamma_5)v(\vec{p}_\nu)\right].$$

Since $m_W \simeq 80\,\mathrm{GeV}$, it makes sense (when applied to low-energy physics) to expand in P/m_W

$$\frac{1}{(p_d - p_u)^2 - m_W^2} \Rightarrow \frac{1}{-m_W^2} - \frac{(p_d - p_u)^2}{(m_W^2)^2} - \frac{(p_d - p_u)^4}{(m_W)^6} - \cdots$$

and retain just a few terms.

The LO term in the amplitude is reproduced by

$$\mathcal{L}_{int} \mathrel{+}\cong -\left(\frac{g_w^2}{8m_W^2}\right)\left[\bar{u}_i\gamma^\mu(1-\gamma_5)d_j\right]\left[\bar{\ell}\gamma_\mu(1-\gamma_5)\nu\right] V_{ij} \qquad \text{———} (**)$$

to $\mathcal{L}_{QED \times QCD}$. (without $W^\pm$ bosons)

called 4-fermi interaction

$\boxed{(**) \text{ should be modified @ 1-Loop.}}$

$\boxed{\text{Example 3 = Question (homework)}}$ The seesaw mechanism simplified.

In a theory with one scalar ϕ and two Dirac fermions Ψ and Ψ',

suppose that $\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) + \bar{\Psi} i\gamma^\mu (\partial_\mu \Psi) + \bar{\Psi}'(i\gamma^\mu \partial_\mu - M)\Psi'$

$$+ \lambda \phi \bar{\Psi}\Psi' + \lambda^* \phi \bar{\Psi}'\Psi . \quad \text{———} (*)$$

So, ϕ and Ψ are massless, but Ψ' is massive.

Now, verify that the low-energy effective theory of $(*)$ at $E \ll M$ is given by

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) + \bar{\Psi} i\gamma^\mu (\partial_\mu \Psi) + \frac{|\lambda|^2}{M} \phi \bar{\Psi}\Psi \phi . \quad \text{———} (**)$$

analogy: ϕ: Higgs doublet
Ψ: left-handed neutrino
Ψ': right-handed neutrino

$\Big(\underbrace{\text{The SM + RH}\nu + \nu\text{ Yukawa int.}}_{(*)}\Big) \longrightarrow \Big(\underbrace{\text{The SM} + \frac{|\lambda|^2}{M}(\ell h)(\ell h)}_{(**)}\Big)$

In this case (the seesaw mechanism), we should deal with
ϕ as a complex boson
Ψ, Ψ' as Weyl spinors (2-component spinors) in fact.

- low-energy approximation
- derivative/mass expansion (and truncation)
- Born–Oppenheimer approximation.

Example 4 QCD $\longrightarrow$ hadrons...

quark + gluon $\rightarrow$ $\mathcal{L} = (\partial_\mu \pi^a)(\partial^\mu \pi^a) + \cdots$ (**)

$$- (\partial_\mu \rho_\nu^a - \partial_\nu \rho_\mu^a)(\partial^\mu \rho^{a\nu} - \partial^\nu \rho^{a\mu}) - \frac{1}{2} m^2 \rho_\mu^a \rho^{a\mu}$$

$$+ \cdots\cdots$$

(perturbative calculation cannot determine all the information of (**).)

Example 5 quantum hall system (e^-'s in $\langle \vec{B} \rangle \neq 0$ in 2+1 dim) $\longrightarrow$ Chern-Simons theory + scalar field

(hw E-4)

Example 6 QED $(\gamma, e^\pm)$ $\longrightarrow$ $E \ll m_e$

pair creation cannot take place anymore.

- Initial states with just photons $\Rightarrow$ an effective theory of γ.
- Initial states with just one e^- (+ γ's ?)

$$\Rightarrow \mathcal{L} \cong \psi^\dagger \left(i\partial_t - m - eQ_e\varphi - \frac{(i\vec{\partial} + eQ_e\vec{A})^2}{2m_e} - \cdots \right) \psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

(**)

This two component "ψ" is a part of the 4-component as

$$\underset{\text{4 component}}{\Psi(\vec{p}, t)} \cong \begin{pmatrix} \psi(\vec{p}, t) \\ \frac{\vec{p}\cdot\vec{\sigma}}{2m_e} \psi(\vec{p}, t) \end{pmatrix} + \underset{\text{(positron)}}{\cdots} .$$

- Even furtherer in low-energy (so there is no ionization)

$\Rightarrow$ an effective theory of bound states —— (***)

$$\mathcal{L} = \phi_{1s}^\dagger \left(i\partial_t - \frac{(i\vec{\partial})^2}{2m_{1s}} + \cdots \right) \phi_{1s} + \vec{\phi}_{2p}^\dagger \left(i\partial_t - \frac{(i\vec{\partial})^2}{2m_{2p}} + \cdots \right) \vec{\phi}_{2p} + \cdots$$

$$+ \mathcal{L}_{int.} - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

Sometimes the low-energy effective theory can be just an empty system.