§9. Path integral formulation of QFT's

§9.1 Repeating the derivation.

Example 1 a free scalar boson on $(d+1)$-dimensional space-time

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \iff \mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$[\phi(\vec{x},t), \pi(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y})$$

Fourier transformation $\Rightarrow$ collection of infinitely many harmonic oscillators w/ $\omega = \sqrt{k^2+m^2}$ labeled by $\vec{k} \in \mathbb{R}^d$.

$$\boxed{\langle\Omega| T\{\phi(x_1^\mu)\phi(x_2^\mu)\cdots \pi(y_1^\mu)\pi(y_2^\mu)\cdots\}|\Omega\rangle = \frac{\int\mathcal{D}\phi\mathcal{D}\pi_{+i\varepsilon}\, e^{i\int d^dx\,dt\{-\mathcal{H}+\pi(\partial_t\phi)\}}\left(\phi(x_1)\phi(x_2)\cdots\pi(y_1)\pi(y_2)\cdots\right)}{\int\mathcal{D}\phi\mathcal{D}\pi_{+i\varepsilon}\, e^{i\int d^dx\,dt\{-\pi+\pi(\partial_t\phi)\}}}}$$

The same story when $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^4 + \frac{\lambda}{4!}\phi^4$.

Example 2 Dirac theory $\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$.

$$\boxed{\langle\Omega| T\{\Psi(x_1)\Psi(x_2)\cdots \bar{\Psi}(y_1)\bar{\Psi}(y_2)\cdots\}|\Omega\rangle = \frac{\int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}_{+i\varepsilon}\, e^{i\int d^4x\,\bar{\Psi}(i\gamma^\mu\partial_\mu-m)\Psi}\left(\Psi(x_1)\Psi(x_2)\cdots\bar{\Psi}(y_1)\bar{\Psi}(y_2)\cdots\right)}{\int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}_{+i\varepsilon}\, e^{i\int d^4x\,\bar{\Psi}(i\gamma^\mu\partial_\mu-m)\Psi}}}$$

Ψ & $\bar{\Psi}$: Grassmann coordinates (fields).

Supplementary Notes.

scalar:

$$\phi(\vec{x}) = \Sigma_{\vec{k}}\ e^{i\vec{k}\cdot\vec{x}}(\phi^R_{\vec{k}} + i\phi^I_{\vec{k}}) \qquad \left(\phi^I_{-\vec{k}} = -\phi^I_{\vec{k}},\ \phi^R_{-\vec{k}} = \phi^R_{\vec{k}}\right)$$

$$H = \int d^3x \left(\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right)$$

$$= vol(space) \cdot \sum_{\vec{k}} \left(\frac{1}{2}\left\{(\dot{\phi}^R_{\vec{k}})^2 + E_k^2(\phi^R_{\vec{k}})^2\right\} + \frac{1}{2}\left\{(\dot{\phi}^I_{\vec{k}})^2 + E_k(\phi^I_{\vec{k}})^2\right\}\right)$$

In the operator formalism.

$$\phi^R_{\vec{k}} = \frac{1}{\sqrt{2E_{\vec{k}}}}\frac{1}{2}(a_{\vec{k}} + a^\dagger_{\vec{k}} + a^\dagger_{-\vec{k}} + a_{-\vec{k}})$$

$$\phi^I_{\vec{k}} = \frac{1}{\sqrt{2E_{\vec{k}}}}\frac{1}{2i}(a_{\vec{k}} - a^\dagger_{\vec{k}} + a^\dagger_{-\vec{k}} - a_{-\vec{k}}) \qquad [a_{\vec{k}}, a^\dagger_{\vec{q}}] = \delta_{\vec{k},\vec{q}}$$

The path integral measure should be

$$\mathcal{D}\phi\,\mathcal{D}\Pi := \prod_{\vec{k}/\{\pm1\}}\left[\mathcal{D}(\phi^R_{\vec{k}}(t))\,\mathcal{D}(\Pi^R_{\vec{k}}(t))\,\mathcal{D}(\phi^I_{\vec{k}}(t))\,\mathcal{D}(\Pi^I_{\vec{k}}(t))\right] \approx \prod_{k^\mu}\left[d\phi(k)\,d\Pi(k)\right]$$

$$\left(\Pi^{R/I}_{\vec{k}} = vol(space)\ \dot{\phi}^{R/I}_{\vec{k}}\right)$$

Dirac theory:

In the operator formalism $\Psi(\vec{x}) = \sum_{\vec{k},s}\frac{1}{\sqrt{2E_{\vec{k}}}}\left(u_{\vec{k},s}a_{\vec{k},s}e^{i\vec{k}\cdot\vec{x}} + v_{\vec{k},s}b^\dagger_{\vec{k},s}e^{-i\vec{k}\cdot\vec{x}}\right)$

$$\mathcal{H}_{\text{Hilb.sp.}} \cong \bigotimes_{\substack{\vec{k},s=\uparrow,\downarrow \\ f=e^-,e^+}}\left(\mathbb{C}^2_{(\vec{k},s,f)}\right) \otimes \mathcal{H}_{\text{others}}$$

$\Uparrow$ 2-state system in Week 2. $\Leftarrow$ γ, etc. if there is any.

$\Rightarrow$ Introduce Grassmann-odd variables and make replacements

$$a_{\vec{k},s} \to \theta_{\vec{k},s,e^-} \qquad a^\dagger_{\vec{k},s} \to \bar{\theta}_{\vec{k},s,e^-}$$

$$b_{\vec{k},s} \to \theta_{\vec{k},s,e^+} \qquad b^\dagger_{\vec{k},s} \to \bar{\theta}_{\vec{k},s,e^+}$$

In the path integral formulation,

$$\Psi(\vec{x},t) = \Sigma_{\vec{k},s}\frac{1}{\sqrt{2E_{\vec{k}}}}\left(u_{\vec{k},s}\theta_{\vec{k},s,e^-}(t)e^{i\vec{k}\cdot\vec{x}} + v_{\vec{k},s}\bar{\theta}_{\vec{k},s,e^+}(t)e^{-i\vec{k}\cdot\vec{x}}\right)$$

and the measure is

$$\mathcal{D}\Psi\,\mathcal{D}\bar{\Psi} := \prod_{\substack{\vec{k},s=\uparrow\downarrow \\ f=e^-,e^+}}\left[\mathcal{D}\theta_{\vec{k},s,f}(t)\,\mathcal{D}\bar{\theta}_{\vec{k},s,f}(t)\right] \approx \prod_{\substack{k^\mu,s=\uparrow\downarrow \\ f=e^-,e^+}}\left[d\theta_{k^\mu,s,f}\,d\bar{\theta}_{k^\mu,s,f}\right]$$

§9.2 Partition function, free energy, effective action (formal)

We call the followings as the partion function:

$$\boxed{Z(\beta,\text{theory}) := \mathrm{tr}_{\text{Hilb.sp}}\left(e^{-\beta H}\right) = \sum_{\text{all states}} e^{-\beta E_i}}$$

$$\boxed{= \int \mathcal{D}\Pi\mathcal{D}\phi \; e^{\left[-\int d^dx \int_0^\beta d\tau\, \mathcal{H} + i\int_0^\beta d\tau \int d^dx\, \Pi(\partial_\tau \phi)\right]}}$$

$\beta\to\infty$ ($T\to 0$) limit.

$$\boxed{Z(\text{theory}) := \int_{+i\varepsilon\text{ prescr.}} \mathcal{D}\Pi\mathcal{D}\phi \; e^{\int d^dx\, dt\left\{-i\mathcal{H} + i\Pi(\partial_t\phi)\right\}}}$$

The following generalizations are useful and are also called partition functions.

$$\boxed{Z(\beta,\text{theory}, J_I(\vec{x})) := \int \mathcal{D}\Pi\mathcal{D}\phi \; e^{\int d^dx \int_0^\beta d\tau\left\{-\mathcal{H} + i\Pi(\partial_\tau\phi) + \Sigma_I\, \mathcal{O}_I(\vec{x}) J_I(\vec{x})\right\}}}$$

$$\boxed{Z(\text{theory}, J_I(\vec{x},t)) := \int_{+i\varepsilon} \mathcal{D}\Pi\mathcal{D}\phi \; e^{i\int d^dx\int dt\left\{-\mathcal{H} + \Pi(\partial_t\phi) + \mathcal{O}_I(\vec{x},t) J_I(\vec{x},t)\right\}}}$$

They are the generating functions of various τ-/t- ordered correlation functions.

$$\boxed{\left\langle \mathcal{O}_I(\vec{x})\right\rangle_{\text{th.eq}}^{J_I=0} = \frac{1}{\beta}\frac{\partial}{\partial J_I(\vec{x})} \ln\left(Z(\beta,\text{theory}, J_I(\vec{x}))\right)\Big|_{J=0}}$$

If $\langle \mathcal{O}_I(\vec{x})\rangle_{\text{th.eq}}^{J\neq 0} \sim$ linear in $J_I(\vec{x}) \approx: \chi\cdot J_I(\vec{x})$

then

$$\boxed{\begin{aligned}\chi &= \left[\frac{\partial}{\partial J_J(\vec{y})}\left(\langle \mathcal{O}_I(\vec{x})\rangle_{\text{th.eq}}^{J\neq 0}\right)\right]\Big|_{J=0} = \left[\frac{1}{\beta}\frac{\partial^2}{\partial J_J(\vec{y})\,\partial J_I(\vec{x})}\ln\left(Z(\beta, J_K)\right)\right]\Big|_{J=0} \\ &= \beta\left(\langle \mathcal{O}_J(\vec{y})\,\mathcal{O}_I(\vec{x})\rangle_{\text{th.eq}}^{J=0} - \langle \mathcal{O}_J(\vec{y})\rangle_{\text{th.eq}}^{J=0}\langle \mathcal{O}_I(\vec{x})\rangle_{\text{th}}^{J=0}\right)\end{aligned}}$$

The linear respose relation.

(The susceptibility is given by a 2-pt function (fluctuation))

Ex: In a free theory $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$

$$\boxed{Z(m, J) :\cong \int_{+i\varepsilon} \mathcal{D}\phi \; e^{i\int d^{d+1}x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + J(x)\phi(x)\right)}}$$

$$= \int \prod_{p \in \mathbb{R}^{1,d}} \left[d\tilde{\phi}(p)\right] \; e^{i\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{2}\tilde{\phi}(p)^* (p^2 - m^2)\tilde{\phi}(p) + (\tilde{J}(p)^* \tilde{\phi}(p) + \tilde{J}(p)\tilde{\phi}(p)^*)\frac{1}{2}}$$

$$= \int \prod_{p \in \mathbb{R}^{1,d}} \left[d\tilde{\phi}(p)\right] \; e^{i\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{2}\left(\tilde{\phi}(p) + \frac{1}{p^2-m^2}\tilde{J}(p)\right)^* (p^2-m^2)\left(\tilde{\phi}(p) + \frac{1}{p^2-m^2}\tilde{J}(p)\right)}$$

$$e^{-i\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{2} \tilde{J}^*(p) \frac{1}{p^2-m^2} \tilde{J}(p)}$$

$$\propto e^{-i\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{2} \tilde{J}^*(p) \frac{1}{p^2-m^2} \tilde{J}(p)} \times \text{const}(m, \cancel{J})$$

So

$$\frac{\langle \Omega | T\{\tilde{\phi}(p)^* \tilde{\phi}(p')\} | \Omega \rangle}{\langle \Omega | \Omega \rangle_{(\text{often } \langle\Omega|\Omega\rangle = 1)}} = -\frac{\partial^2}{\partial \tilde{J}(p)\, \partial \tilde{J}(p')^*} \ln\left(Z(m, J)\right)\Big|_{J=0}$$

$$= -\frac{\partial^2}{\partial \tilde{J}(p)\, \partial \tilde{J}^*(p')} \left((-i)\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{2}\tilde{J}^*(p) \frac{1}{p^2-m^2}\tilde{J}(p)\right)\Big|_{J=0}$$

$$= (2\pi)^{d+1} \delta^{d+1}(p-p') \frac{i}{(p^2-m^2)}$$

as expected.

$$Z(\beta, \text{theory}, J_I(\vec{x})) =: e^{-\beta F(\beta, \text{theory}, J_I(\vec{x}))}$$

$$Z(\text{theory}, J_I(x^\mu)) =: e^{-i\int d^d\vec{x}\, dt\, \mathcal{F}(\text{theory}, J_I(\vec{y}, t'))}$$

F: called the free energy.

Two questions will come to our minds then.

- How is the Landau-Ginzburg theory related to $F(\beta, \text{thry}, J_I(\vec{x}))$?
- In thermodynamics / statistical mechanics, we say that the "phase 1" is realized when "$F(\beta, \text{thry}, \text{phase1}) < F(\beta, \text{thry}, \text{phase2})$".

  How do we compute "$F(\beta, \text{thry}, \text{phase})$"?

Here is a QFT version of such stories.

$$\boxed{\text{a phase} \Leftrightarrow \text{an ansatz on which set of operators have non-zero expectation values.}}$$

$$\left(\text{order parameters} \Leftrightarrow \text{hypothetically chosen operator expectation values.}\right)$$

Suppose that there is a choice $J_I(\vec{x})$ that reproduces an order parameter $\langle \mathcal{O}_J(\vec{y})\rangle^J_{th.eq} = -\frac{\partial}{\partial J_J(\vec{y})}\left(F(J_K(\vec{z}))\right)$.

(we have a $J_I(\vec{x}) \Leftrightarrow \langle \mathcal{O}_J(\vec{y})\rangle_{th.eq}$ dictionary then.)

Define

$$\boxed{-\Gamma_{th.eq}[\langle \mathcal{O}_J(\vec{y})\rangle_{th.eq}] := \left(F[J_I(\vec{x})] + \int d^d\vec{x}\, J_I(\vec{x})\langle \mathcal{O}_I(\vec{x})\rangle^J_{th.eq}\right)}\ \text{all } J\text{'s translated to } \langle \mathcal{O}\rangle_{th.eq}\text{'s}$$

Legendre transformation w.r.t. $J_I(\vec{x})$

Then

$$\boxed{-\frac{\partial}{\partial\langle \mathcal{O}_J(\vec{y})\rangle_{th.eq}}\left(\Gamma_{th.eq}[\langle \mathcal{O}_K(\vec{y})\rangle_{th.eq}]\right) = J_J(\vec{y})}$$

so the order parameters that may be realized in the original theory ($J_I(\vec{x}) = 0$) should satisfy

$$\boxed{\frac{\partial}{\partial\langle \mathcal{O}_J(\vec{y})\rangle_{th.eq}}\Gamma_{th.eq}[\langle \mathcal{O}\rangle_{th.eq}] = 0}$$

No.

Date

The order parameter $\langle O_I(\vec{x})\rangle_{th.eq.}$ may have $\vec{x}$-dependence
$\left(\begin{array}{l}\text{e.g. Pierls transition} \\ \langle \text{phonon}(k = \pi/a)\rangle \end{array}\right)$

In a phase with $\langle O_I(\vec{x})\rangle_{th.eq.}$

$$\Gamma_{th.eq} =: -\int d^d\vec{x}\, \underbrace{V_{th.eq}(\langle O_I\rangle_{th.eq})}$$

↳ the Landau-Ginzbung potential
$(\beta, \text{theory}, \underset{\substack{\uparrow \\ \text{order parameter}}}{\langle O_I\rangle_{th.eq}})$

In each phase

$$\boxed{F(\beta, \text{theory}, \text{phase}) := -\Gamma_{th.eq}(\beta, \text{theory}, \langle O_I(\vec{x})\rangle)\Big|_{\frac{\partial\Gamma_{th.eq}}{\partial\langle O_I(\vec{x})\rangle}=0}}$$

and we can apply the thermodynamical principle

"phase 1" realized when $F(\text{phase 1}) < F(\text{phase 2})$.

Example: in a condensed matter system with

$$\mathcal{H} = \sum_{s=\uparrow,\downarrow}\left(\psi_s^\dagger(\vec{k})\,\varepsilon(\vec{k})\,\psi_s(\vec{k})\right) + \underset{\Uparrow}{(\psi_\uparrow^\dagger\psi_\uparrow)(\psi_\downarrow^\dagger\psi_\downarrow)\,c}$$

induced from $\begin{pmatrix}\text{photon}\\ \text{phonon}\end{pmatrix}$ exchange.

magnetic phase $\langle(\psi_\uparrow^\dagger\psi_\uparrow)(\vec{k})\rangle \neq 0$. $\langle(\psi_\downarrow^\dagger\psi_\downarrow)(\vec{k})\rangle = 0$ for some $\vec{k}$.

super conducting phase $\langle(\psi_\uparrow\psi_\downarrow)\rangle \neq 0$

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For $T \to 0$ $(\beta \to \infty)$ the vacuum case

$$\boxed{\langle \mathcal{O}_I(x^\mu) \rangle_{vac.} = - \frac{\partial}{\partial J_I(x^\mu)} \mathcal{F} = - \frac{\delta}{\delta J_I(x^\mu)} \left(\int d^{d+1}x \, \mathcal{F} \right)}$$

$$- \Gamma_{vac} [\langle \mathcal{O}_I(x^\mu) \rangle_{vac}] := \int d^{d+1}x \left\{ \mathcal{F} + J_I(x^\mu) \langle \mathcal{O}_I(x^\mu) \rangle_{vac} \right\}$$

(Legendre transformation)

The operator vacuum expectation value should satisfy $\boxed{\frac{\delta}{\delta \langle \mathcal{O}_I(x) \rangle_{vac}} \Gamma [\langle \mathcal{O}_I(x^\mu) \rangle_{vac}] = 0}$

$\Gamma_{vac} [\langle \mathcal{O}_I(x) \rangle_{vac}]$: called the effective action.

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