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§9.3 Background field method

☆ To compute $\Gamma_{th.eq}[\langle O(\vec{x})\rangle]$ or $\Gamma_{vac}[\langle O_I(\vec{x})\rangle]$

with $\langle O(\vec{x})\rangle \neq 0$ for only elementary fields (O's in $\mathcal{D}O$ in path integral)

$$\phi(x) =: \langle\phi(x)\rangle + \text{fluctuat'n.} =: \phi_{cl}(x) + \phi_{fl}(x)$$

$$\mathcal{D}\phi(x) = \mathcal{D}\phi_{fl}(x)$$

$$\left(\int \mathcal{D}\phi\, e^{i\int d^{d+1}x(\mathcal{L}+J\cdot\phi)}\right) \times e^{-i\int d^{d+1}x\, J\cdot\phi_{cl}} = e^{i\Gamma_{vac}[\phi_{cl}]}$$

$$\left(\int \mathcal{D}\phi\, e^{i\int d^{d+1}x(\mathcal{L}+J\cdot\phi)}\right) \times e^{-i\int d^{d+1}x\, J\cdot\phi_{cl}} = e^{-i\int d^{d+1}x\,(F[J]+J\phi_{cl})}$$

(solve $\frac{\partial F}{\partial J} = -\phi_{cl}$ w.r.p.t. J to rewrite J in favor of ϕ_{cl} on the L.H.S.)

The path integral $(\cdots)$ on the L.H.S. is

$$\int \mathcal{D}\phi_{fl}\; e^{i\int d^{d+1}x\left\{(\mathcal{L}+J\phi)_{cl} + \left(\frac{\partial\mathcal{L}}{\partial\phi}+J\right)_{cl}\cdot\phi_{fl} + \frac{1}{2}\left(\frac{\delta^2\mathcal{L}}{\partial\phi\partial\phi}\right)_{cl}\phi_{fl}^2 + \cdots\right\}}$$

- $(\mathcal{L}+J\phi)_{cl}$: ϕ's in $(\mathcal{L}+J\phi)$ replaced by ϕ_{cl}.
  $\longrightarrow$ indep. of ϕ_{fl}.
- $\left(\frac{\partial\mathcal{L}}{\partial\phi}+J\right)_{cl}$, $\left(\frac{\delta^2\mathcal{L}}{\partial\phi\partial\phi}\right)_{cl}$ etc. : dep. only on ϕ_{cl} (and J) not on ϕ_{fl}.

So.

$$e^{i\Gamma_{vac}} = e^{i\int d^{d+1}x\,((\mathcal{L}+J\phi)_{cl} - J\phi_{cl})}\left(\int \mathcal{D}\phi_{fl}\, e^{i\int d^{d+1}x\,(\text{Lagrangian of } \phi_{fl})}\right)$$

(ϕ_{cl}-dep Lagrangian of ϕ_{fl})

$$i\Gamma_{vac} = i\int d^{d+1}x\,\mathcal{L}_{cl} + (\text{conn. vac. bubbles of } \phi_{fl}\text{ QFT})$$

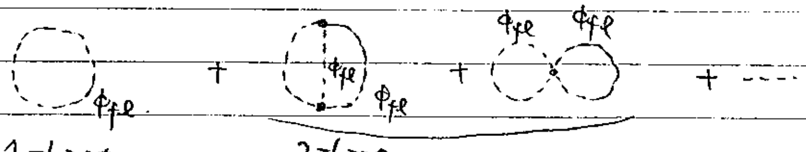
(OK to ignore $\phi_{fl}\cdot\left(\frac{\partial\mathcal{L}}{\partial\phi}+J\right)_{cl}$ because J is chosen (by def)

$$\frac{\partial\Gamma}{\partial\phi_{cl}} = -J. \quad \Downarrow \quad \overbrace{i\left(\frac{\partial\mathcal{L}}{\partial\phi_{cl}}+J\right) + \frac{\partial}{\partial\phi_{cl}}(\text{conn. vac. bubble graphs})}^{\int\mathcal{D}\phi\, e^{i(S+\int J\phi)}} \overbrace{-iJ}^{e^{-i\int J\phi_{cl}}} = -iJ$$

$$\Longrightarrow \quad i\left(\frac{\partial\mathcal{L}}{\partial\phi_{cl}}+J\right)_{cl} + (\phi_{fl}\text{ tadpole graphs}) = 0.$$
)

PLUS

connected vacuum bubble graphs :



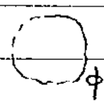
$-(\text{mass})^2 = \left(\frac{\delta^2 \mathcal{L}}{\partial\phi\,\partial\phi}\right)_{cl}$ $\quad$ $(3\text{pt coupling}) = \left(\frac{\delta^3 \mathcal{L}}{\partial\phi\,\partial\phi\,\partial\phi}\right)_{cl}$ $\quad$ ....

dep. on ϕ_{cl}.

The 1-loop contribution $\left(= \text{an approx. to ignore } (\phi_{fl})^3\right)$

$$= -\frac{1}{2}\, \text{vol}_{d+1} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \ln\left(p^2 - \text{mass}^2\right)$$ for a real scalar ϕ_{fl}.

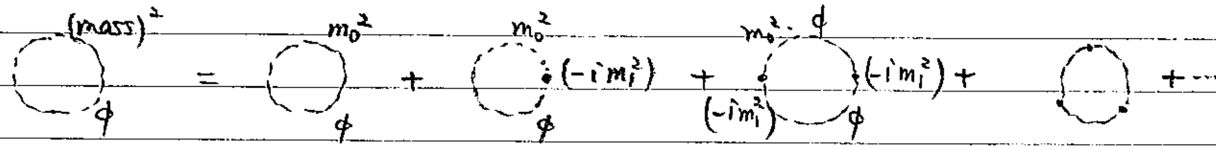
justification (A)

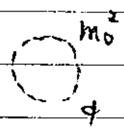


$$= \ln\left(Z \text{ of a free real scalar}\right) = \ln\left(\int \prod_{p^\mu} [d\phi(p)]\, e^{i\int\left(\frac{d^{d+1}p}{2\pi}\right)\frac{1}{2}\phi^*(p)(p^2-m^2)\phi(p)} \times(\text{const})\right)$$

$$= \text{vol}_{d+1} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \ln\left(\sqrt{\frac{2\pi i}{p^2-m^2}}\right) + (\text{const}) = -\frac{1}{2}\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \ln(p^2 - m^2) + \text{const}'$$
$\times(\text{vol}_{d+1})$

justificat'n (B) $\quad$ split $(\text{mass})^2 = m_0^2 + m_1^2$

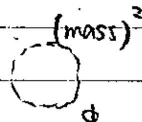




$$= \text{(bubble)}^{m_0^2}_{\phi} + \frac{1}{2}(-im_1^2)\int\frac{d^{d+1}p}{(2\pi)^{d+1}}\left(\frac{i}{p^2-m_0^2}\right) + \frac{1}{4}(-im_1^2)^2\int\frac{d^{d+1}p}{(2\pi)^{d+1}}\left(\frac{i}{p^2-m_0^2}\right)^2$$

$$+ \cdots + \frac{1}{2n}(-im_1^2)^n\int\frac{d^{d+1}p}{(2\pi)^{d+1}}\left(\frac{i}{p^2-m_0^2}\right)^n + \cdots$$

then



~~must be~~ $-\frac{1}{2}\int\frac{d^{d+1}p}{(2\pi)^{d+1}} \ln(p^2 - m_0^2 - m_1^2) + (\text{const.})$

$\times$(trivially satisfied $(2\pi)^{d+1}\delta^{d+1}(p\text{'s}) \Rightarrow \text{vol}_{d+1}$)

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☆ When one wishes to consider $\langle \mathcal{O} \rangle \neq 0$ of a composite operator $\mathcal{O}$.

Hubbard – Stratonovich transformation.

e.g.

$$\mathcal{L} = \psi^\dagger (i\partial_t - \varepsilon(\nabla))\psi + \frac{1}{\Lambda^2}(\psi_\downarrow^\dagger \psi_\uparrow^\dagger \psi_\uparrow \psi_\downarrow) \qquad \mathcal{D}\psi\,\mathcal{D}\bar{\psi}$$

$$\Updownarrow$$

$$\mathcal{L}' = \psi^\dagger (i\partial_t - \varepsilon(\nabla))\psi - \sigma^2 - \sigma\, \psi_\uparrow \psi_\downarrow \frac{1}{\Lambda} - \sigma\, \psi_\downarrow^\dagger \psi_\uparrow^\dagger \frac{1}{\Lambda}$$

$$\mathcal{D}\sigma\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}$$

complete a square w.r.p.t. σ

and integrate out σ. (Gaussian integral peaked at $\sigma = \frac{1}{\Lambda}\mathrm{Re}(\psi_\uparrow \psi_\downarrow)$)

Instead of thinking of $P_{vac}[\langle \psi_\uparrow \psi_\downarrow \rangle]$ of $(\mathcal{L}, \mathcal{D}\psi\,\mathcal{D}\bar{\psi})$,

one may think of $P_{vac}[\langle \sigma \rangle]$ of $(\mathcal{L}', \mathcal{D}\sigma\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi})$

PLUS

§9.4 Thermal field theory (imaginary time formalism)

★ Study $\Gamma_{\text{th.eq}}[\beta, \langle \mathcal{O} \rangle]$ at 1-loop approximation ($\bigcirc_{\phi_{fe}}$ only)

$\Leftrightarrow$ ignore $(\phi_{fe})^3$ and higher

(look at excitat'n spectrum)

$\Leftrightarrow$ mean field ($\langle \mathcal{O} \rangle = \mathcal{O}_{cl}$) approximation

When $(\phi_{fe})^3$ ignored, we have

$$\boxed{(\mathcal{L}_{\text{Mink}})_{\partial_t \to i\partial_\tau} =: -\mathcal{L}_{\text{Eucl}}}$$

$$\beta \Gamma^{\text{1-loop}}_{\text{th.eq}}[\beta, \langle \mathcal{O} \rangle] = +\beta \int d^d x \left(-\mathcal{L}^{cl.}_{\text{Eucl}}\right) + \ln\left(Z^{\text{1-loop}}_{\phi_{fe}}\right)$$

$$\ln\left(Z^{\text{1-loop}}_{\phi_{fe}}\right) = (\text{Vol}_{d\text{-dim}}) \int \frac{d^d k}{(2\pi)^d} \ln\left(\operatorname{tr}_n \left[e^{-\beta E_{\vec{k}}(n+\frac{1}{2})} \right] \right)$$

1-loop approx = free theory approx.

$\Rightarrow \bigotimes_{\vec{k} \in \mathbb{R}^d}$ (harmonic oscill., $\omega = E_{\vec{k}}$)

$$= (\text{Vol}_d) \int \frac{d^d k}{(2\pi)^d} \ln\left(e^{-\beta E/2} \frac{1}{(1 - e^{-\beta E_{\vec{k}}})} \right)$$

$$= (\text{Vol}_d) \int \frac{d^d k}{(2\pi)^d} \left(-\ln\left(e^{\beta E/2} - e^{-\beta E/2} \right) \right)$$

$$= (\text{Vol}_d) \int \frac{d^d k}{(2\pi)^d} \left\{ -\frac{\beta E_{\vec{k}}}{2} - \ln\left(1 - e^{-\beta E/2}\right) \right\}$$

$\Uparrow$ the 0-pt oscillation (Casimir energy) $\qquad$ $\Uparrow$ thermal contribut'n

◎ The thermal contribution

$$\text{vol}_d \times \begin{cases} \text{if} \quad m_{\text{eff}}^2 = \left(\frac{\partial^2 \mathcal{L}}{\partial\phi\partial\phi}\right)_{cl} \ll T^2 & + \mathcal{O}(1) \cdot T^d \\ \text{if} \quad m_{\text{eff}}^2(\phi_{cl}) \gg T^2 & + \mathcal{O}(1) \times (m_{\text{eff}} T)^{\frac{d}{2}} e^{-m_{\text{eff}}/T} \end{cases}$$

(as well known in statistical mechanics)

Now this contribution is regarded as a function of ϕ_{cl}.

◎ The Casimir energy contribut'n to $\left(\Gamma^{\text{1-loop}}_{\text{th.eq}}\right)$ is β-indep. (manifest).

It is divergent: subtlety in interpretat'n. [renormalizat'n. $\Rightarrow$ TEP course]

but known to have weak ϕ_{cl}-dependence.

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memo

for a free real scalar field ϕ

$$\left(Z_{free\,\phi}(\beta)\right) = \int \mathcal{D}\pi\mathcal{D}\phi\; e^{-\int_0^\beta d\tau \int d^dx \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right)}$$

$$\propto \int \mathcal{D}\phi\; e^{-\int_0^\beta d\tau \int d^dx\; \frac{1}{2}\left((\partial_\tau\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right)}$$

$$\propto \prod_{n\in\mathbb{Z}} \prod_{k} \left((2\pi T n)^2 + k^2 + m^2\right)^{-1/2}.$$

$$\ln(Z_{free\,\phi}(\beta)) = vol_d \int \frac{d^dk}{(2\pi)^d} \ln\left(\prod_{n\in\mathbb{Z}}\left((2\pi T n)^2 + E_k^2\right)^{-1/2} \cdot (const)\right)$$

should be equal to

$$vol_d \int \frac{d^dk}{(2\pi)^d} \ln\left(e^{-\beta E_k/2} \frac{1}{(1-e^{-\beta E_k})}\right). \quad \text{How?}$$

The square of the argument of log:

$$(const)^2 \prod_{n\in\mathbb{Z}} \frac{1}{(2\pi T n)^2 + (E_k)^2} = (const')^2 \prod_{n\in\mathbb{Z}} \left(\frac{T^2}{(2\pi T n)^2 + E^2}\right)$$

$$= (const')^2 \left(\frac{T^2}{E^2}\right) \times \prod_{n=1}^{\infty} \left[\left(\frac{(2\pi T n)^2}{(2\pi T n)^2 + E^2}\right) \times \frac{1}{(2\pi n)^2}\right]^2$$

$$= (const')^2 \left(\frac{T^2}{E^2}\right) \times \left(\prod_{n=1}^{\infty} \frac{1}{1 + \left(\frac{E}{2\pi T n}\right)^2}\right)^2 \left(\prod_{n=1}^{\infty} \frac{1}{(2\pi n)^2}\right)$$

$$= (const')^2 \left(\frac{T^2}{E^2}\right) \left(\frac{-iE/2T}{\sin(-iE/2T)}\right)^2 \prod_{n=1}^{\infty} \frac{1}{(2\pi n)^2}$$

$$= \left(\frac{1}{2\,\mathrm{sh}(E/2T)}\right)^2 \left(const' \cdot \prod_{n=1}^{\infty} \frac{1}{(2\pi n)^2}\right)^2.$$

So, if $(const) = \prod_{n=1}^{\infty} (2\pi T n)^2$, then both derivations yield the same result.

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(memo) Here is the rough story of how to evaluate the Casimir energy contribution.

$$\ln\left(Z^{\text{1-loop}}\right) = \text{vol}_d \int \frac{d^d k}{(2\pi)^d} \ln\left(\prod_{n\in\mathbb{Z}} \left((2\pi T n)^2 + E_k^2\right)^{-1/2} \cdot (\text{const}) \right)$$

$$\frac{1}{\beta}\ln\left(Z^{\text{1-loop}}\right) = \text{vol}_d \int \frac{d^d k}{(2\pi)^d} \left[\sum_n \frac{1}{\beta} \ln\left(\left\{ ((2\pi/\beta)n)^2 + E_k^2 \right\}^{-1/2} \right) + \frac{1}{\beta}\ln(\text{const}) \right]$$

$$= \text{vol}_d \cdot \Big(\underbrace{(\beta \to +\infty \text{ limit})}_{\text{Casimir}} + \text{thermal contribut'n} \Big).$$

$$\boxed{\frac{1}{\beta}\ln\left(Z^{\text{1-loop}}\right)^{\text{Cas.}} \frac{1}{\text{vol}_{d=4}} = -\frac{1}{2}\int \frac{d^4 k_E}{(2\pi)^4} \left\{ \ln\left(k_E^2 + m^2(\phi)\right) + (\text{const}') \right\}}$$

Think of the case $m^2(\phi) = m_0^2 + \lambda \phi_{cl}^2$.

$$= -\frac{1}{2}\int \frac{d^4 k_E}{(2\pi)^4} \left\{ \ln\left(\frac{k_E^2 + m^2(\phi)}{k_E^2 + \mu^2}\right) + \text{const}''(\mu) \right\}$$

for an arbitrary chosen energy scale μ.

$$= -\frac{1}{2}\int \frac{d^4 k_E}{(2\pi)^4} \left\{ \ln\left(\frac{k_E^2 + m_0^2 + \lambda\phi_{cl}^2}{k_E^2 + \mu^2}\right) - \frac{(m_0^2 - \mu^2 + \lambda\phi_{cl}^2)}{k_E^2 + \mu^2} + \frac{1}{2}\left(\frac{m_0^2 - \mu^2 + \lambda\phi_{cl}^2}{k_E^2 + \mu^2}\right)^2 \right.$$
$$\left. + \left[\text{const}''(\mu) + \frac{m_0^2 - \mu^2}{k_E^2 + \mu^2} - \frac{1}{2}\left(\frac{m_0^2 - \mu^2}{k_E^2 + \mu^2}\right)^2\right] + \left[\frac{\lambda}{k_E^2 + \mu^2} - \frac{(m_0^2 - \mu^2)\lambda}{(k_E^2 + \mu^2)^2}\right]\phi_{cl}^2 - \frac{\lambda^2}{2}\frac{1}{(k_E^2 + \mu^2)^2}\phi_{cl}^4 \right\}$$

So

$$\left(-\mathcal{L}^{cl}_{\text{End}}\right) + \frac{1}{\text{vol}_{d=4}}\frac{1}{\beta}\ln\left(Z^{\text{1-loop}}_{\phi_{cl}}\right)^{\text{Cas}}$$
$$= \left(-\frac{1}{2}\int\frac{d^4 k_E}{(2\pi)^4}\left[\text{const}'' + \cdots\right]\right) - \frac{1}{2}\phi_{cl}^2\left(m_0^2 + \int\frac{d^4 k_E}{(2\pi)^4}\left[\cdots\cdots\right]\right) - \frac{\phi_{cl}^4}{4}\left(\lambda - \lambda^2\int\frac{d^4 k_E}{(2\pi)^4}\frac{1}{(k_E^2 + m_0^2)^2}\right)$$
$$-\frac{1}{2}\int\frac{d^4 k_E}{(2\pi)^4}\left\{\ln\left(\frac{k_E^2 + m_0^2 + \lambda\phi_{cl}^2}{k_E^2 + \mu^2}\right) - \frac{m_0^2 - \mu^2 + \lambda\phi_{cl}^2}{k_E^2 + \mu^2} + \frac{1}{2}\left(\frac{m_0^2 - \mu^2 + \lambda\phi_{cl}^2}{k_E^2 + \mu^2}\right)^2\right\}$$

- The integral in the 2nd line is convergent, and has weak ϕ_{cl}-dependence.
- The divergence of the integral has been (くりこまれた) [shoved] into the coefficients of the ϕ_{cl}-quartic polynomial in the 1st line.

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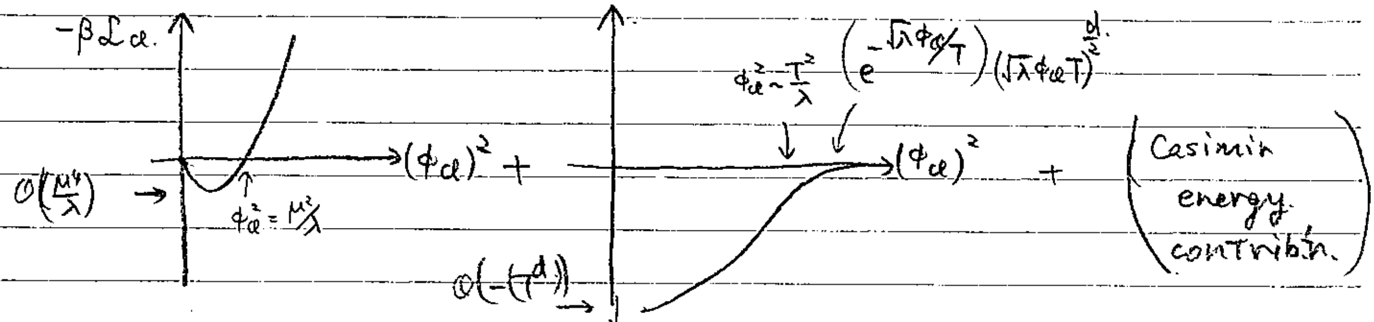
So. eg. for $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4$



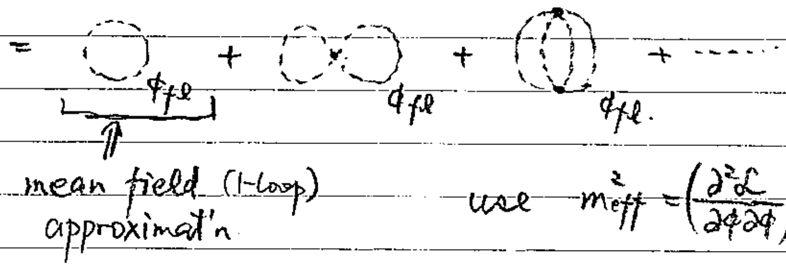
$(-\beta\mathcal{L}_{cl})$ $\quad + \int\frac{d^dk}{(2\pi)^d}\ln(1-e^{-\beta E_k})$

thermal contrib.

☆ To include perturbative higher order correct'ns

$$\beta\Gamma_{th.eq}[\beta.\phi_{cl}] = \beta\ \text{vol}._d \times (-\mathcal{L}^{cl}_{Eucl.}) + \ln(Z_{\phi_{fl}})$$

$$\ln(Z_{\phi_{fl}}) = \ln\left(\exp\left(\begin{matrix}\text{connected Feynman}\\ \text{graphs of } \phi_{cl}.\end{matrix}\right)\right)$$



use $m_{eff}^2 = \left(\frac{\partial^2\mathcal{L}}{\partial\phi\partial\phi}\right)_{cl}$ ← ϕ_{cl}-dep.

4pt coupling $= \left(\frac{\partial^4\mathcal{L}}{\partial\phi^4}\right)_{cl}$

The propagator is

$$\frac{Tr\left[T_{au}\{\phi_I(\tau,\vec{x})\,\phi_I(0,\vec{0})\}\,e^{-\beta H_0}\right]}{Tr\left[e^{-\beta H_0}\right]}$$

for $\phi_I(\tau,\vec{x}) = \int\frac{d^dk}{(2\pi)^d}\frac{1}{\sqrt{2E_k}}\left(e^{i\vec{k}\cdot\vec{x}-E_k\tau}a_{\vec{k}} + e^{-i\vec{k}\cdot\vec{x}+E_k\tau}a^\dagger_{\vec{k}}\right)$

$$= \int\frac{d^dk}{(2\pi)^d}\frac{1}{(2E_{\vec{k}})}\left(e^{i\vec{k}\cdot\vec{x}-E_k\tau}\cdot\frac{e^{-\beta E/2}\frac{1}{(1-e^{-\beta E/2})^2}}{e^{-\beta E/2}/(1-e^{-\beta E/2})} + e^{-i\vec{k}\cdot\vec{x}+E_k\tau}\cdot\frac{e^{-\beta E/2}\frac{e^{-\beta E}}{(1-e^{-\beta E})^2}}{e^{-\beta E/2}/(1-e^{-\beta E})}\right)$$

$$= \int\frac{d^dk}{(2\pi)^d}\frac{1}{(2E_{\vec{k}})}\left(e^{i\vec{k}\cdot\vec{x}-E_k\tau}\left(\frac{1}{e^{\beta E}-1}+1\right) + e^{-i\vec{k}\cdot\vec{x}+E\tau}\frac{1}{(e^{\beta E}-1)}\right)$$

⇑ Bose-Einstein distrib'n; ↑ vacuum contribution

cf

$$\begin{aligned}
&\sum_{n=0}^{\infty}e^{-\beta En}\langle n|a^\dagger a|n\rangle\\
&= \sum_{n=0}^{\infty}e^{-\beta En}\,n\\
&= \left[x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}x^n\right)\right]\Big|_{x=e^{-\beta E}}\\
&= \left[x\frac{\partial}{\partial x}\left(\frac{1}{1-x}\right)\right]\Big|_{x=e^{-\beta E}}\\
&= \frac{x}{(1-x)^2}
\end{aligned}$$

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[Le Bellac. p.65]

The thermal contribution in  is known to be.

$$\Delta\left(\frac{\Gamma^{2\text{-loop}}_{th.ef}[\beta, \phi_{cl}\sim 0]}{Vol_{d=3}}\right) = +\frac{T^4}{48}\frac{\lambda}{4}$$

$$\left[\begin{array}{l} d=3. \\ V(\phi)=\frac{m_0^2}{2}\phi^2+\frac{\lambda}{4!}\phi^4. \end{array}\right]$$

relatively to

$$\Delta\left(\frac{\Gamma^{1\text{-loop}}_{th.ef}[\beta, \phi_{cl}\sim 0]}{vol_{d=3}}\right) = -\frac{\pi^2}{90}T^4 \quad \text{from}$$ .

So, for $\lambda \neq 0$ the pressure $p = \frac{\partial(F(V,T))}{\partial Vol} = -\frac{\partial \Gamma_{th.ef}[Vol, T]}{\partial Vol}$

the entropy density $s = \frac{1}{Vol_d}\frac{\partial F(V,T)}{\partial T}$

are T^4 and T^3 respectively. But the coefficients depend on λ.

Correlation functions in a canonical ensemble

$$\langle \mathcal{O}_I(\vec{x})\mathcal{O}_J(\vec{y})\rangle_\beta - \langle \mathcal{O}_I(\vec{x})\rangle_\beta \langle \mathcal{O}_J(\vec{y})\rangle_\beta = T^2 \frac{\partial^2}{\partial h_I(\vec{x})\,\partial h_J(\vec{y})}(\ln Z) = -T\frac{\partial^2}{\partial h^2}(F_{th})$$

$$\parallel$$

$$T\frac{\partial}{\partial h_I(\vec{x})}\langle \mathcal{O}_J(\vec{y})\rangle_{th}$$

How is this related to the "effective action" $\Gamma_{th.eq}$?

Remember that

$$-\frac{\partial \Gamma_{th.eq}}{\partial \langle \mathcal{O}_I(\vec{x})\rangle_{th}} = \frac{\partial}{\partial \langle \mathcal{O}_I(\vec{x})\rangle_{th}}\left(F + \int d^d\vec{x}\, h_I(\vec{x})\langle \mathcal{O}_I(\vec{x})\rangle\right)$$

$$= \frac{\partial h_K(\vec{z})}{\partial \langle \mathcal{O}_I(\vec{x})\rangle_{th}}\underbrace{\left(\frac{\partial F}{\partial h_K(\vec{z})} + \langle \mathcal{O}_K(\vec{z})\rangle_{th}\right)}_{=0} + h_I(\vec{x}) = h_I(\vec{x}),$$

we find that

$$-\beta\frac{\partial^2 \Gamma_{th.eq}}{\partial \langle \mathcal{O}_K(\vec{w})\rangle_{th}\,\partial \langle \mathcal{O}_I(\vec{x})\rangle} = \beta\frac{\partial}{\partial \langle \mathcal{O}_K(\vec{w})\rangle_{th}}(h_I(\vec{x}))$$

is the inverse matrix of $T\frac{\partial}{\partial h_I(\vec{x})}\langle \mathcal{O}_J(\vec{y})\rangle_{th}$.

So, if $\Gamma_{th.eq} \sim \int d^d x \left\{\frac{1}{2}(\vec{\nabla}\mathcal{O})(\vec{\nabla}\mathcal{O}) + \frac{1}{2}m_{eff}^2(\mathcal{O})^2\right\} + (\text{higher order in } \mathcal{O})$

then $$\frac{\partial^2 \Gamma_{th.eq}}{\partial \langle \mathcal{O}(-\vec{q})\rangle\,\partial \langle \mathcal{O}(\vec{p})\rangle} = (2\pi)^d \delta^d(\vec{p}-\vec{q})\,(p^2+m_{eff}^2).$$

Taking its inverse,

$$\langle \mathcal{O}(\vec{x})\,\mathcal{O}(\vec{y})\rangle = \int \frac{d^d p}{(2\pi)^d}\frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{(p^2+m_{eff}^2)} \sim \frac{e^{-m_{eff}|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|^{d-2}}.$$

$1/m_{eff}(\beta; g, m)$ is the correlation length.

Critical temperature $\Leftrightarrow$ $\beta_c = 1/T_c$ s.t. $m_{eff}^2(\beta_c; g, m) = 0$.

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The thermal contribution to $\frac{\partial F}{\partial(\phi_{cl}^2)}$

e.g. $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda}{4}\phi^4$ model

$$-\beta\, \mathrm{vol}_d \left(F^{1\text{-loop}}\right) = \Sigma_{n\in\mathbb{Z}}\, \mathrm{vol}_d \int \frac{d^dp}{(2\pi)^d} \ln\left((2\pi T n)^2 + \vec{p}^{\,2} + m^2(\phi_{cl})\right)^{-\frac{1}{2}}$$

$$m^2(\phi) = (m_0^2 + 3\lambda\phi_{cl}^2).$$

At the quadratic order

$$-\beta\, \mathrm{vol}_d \left(\Delta F^{1\text{-loop}}\right) = \Sigma_{n\in\mathbb{Z}}\, \mathrm{vol}_d \int \frac{d^dp}{(2\pi)^d} \frac{3\lambda\phi_{cl}^2}{(2\pi T n)^2 + \vec{p}^{\,2} + m_0^2}\left(\frac{-1}{2}\right).$$

The above relation is also understood diagramatically:

vertex $e^{-\frac{\lambda}{4}\times 6\,\phi_{cl}^2\phi_{fl}^2} \Rightarrow (-3\lambda\phi_{cl}^2)$ coupling for the operator $\frac{1}{2}\phi_{fl}^2$

× 1-loop propagator $\left(\Sigma_n \int \frac{d^dp}{(2\pi)^d} \frac{+1}{(2\pi T n)^2 + \vec{p}^{\,2} + m_0^2}\right)$ connecting the same point

× symmetry factor ½ × (trivially satisfied momentum conservat'n delta fcn. ⇒ vol_d)

$$\beta\left(\Delta F^{1\text{-loop}}\right) = \frac{3\lambda\phi_{cl}^2}{2} \times \int \frac{d^dp}{(2\pi)^d}\left(\Sigma_{n\in\mathbb{Z}} \frac{1}{(2\pi T n)^2 + E_{\vec{p}}^2} = \coth(\beta E/2)\frac{1}{2TE_{\vec{p}}}\right)$$

$$\left(\Delta F^{1\text{-loop}}\right) = \frac{3\lambda\phi_{cl}^2}{2}\int \frac{d^dp}{(2\pi)^d}\frac{1}{2E_p}\left(1 + \frac{2}{e^{\beta E} - 1}\right)$$

↑quantum ↖ thermal

$$\left(\Delta F^{1\text{-loop}}\right)^{\text{thermal}} = \frac{\phi_{cl}^2}{2}3\lambda \int \frac{d^dp}{(2\pi)^d}\frac{1}{E_p}\frac{1}{e^{\beta E} - 1}$$

$$\underset{m_0^2 \ll T^2}{\sim} \frac{\phi_{cl}^2}{2}3\lambda \cdot \frac{\mathrm{vol}(S_{d-1})}{(2\pi)^d}\int_0^\infty dp\, p^{d-2}\left(\sum_{n=1}^{\infty} e^{-\beta p n}\right) = \frac{\phi_{cl}^2}{2}3\lambda\frac{\mathrm{vol}(S_{d-1})}{(2\pi)^d}T^{d-1}\Gamma(d-1) \times \zeta(d-1)$$

$$\boxed{(\text{thermal mass})^2 = 3\lambda \frac{\Gamma(d-1)\,\mathrm{vol}(S_{d-1})}{(2\pi)^d}\zeta(d-1)\cdot T^{d-1}}$$

to be added to the vac. m_0^2.

PLUS