

§9.6 Effective theory and coarse graining

When a system (theory, model) has both heavy particles and light particles (ϕ and ψ), it is possible to integrate out ϕ at the very beginning, if you are interested only in correlation functions involving ψ 's, (not ϕ 's).

$$Z[\chi_H, \chi_L] := \int \mathcal{D}\phi \mathcal{D}\psi e^{i \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} M^2 \phi^2 + \frac{1}{2} (\partial_\mu \psi)^2 - g \phi \psi \psi + \phi \chi_H + \psi \chi_L \right]}$$

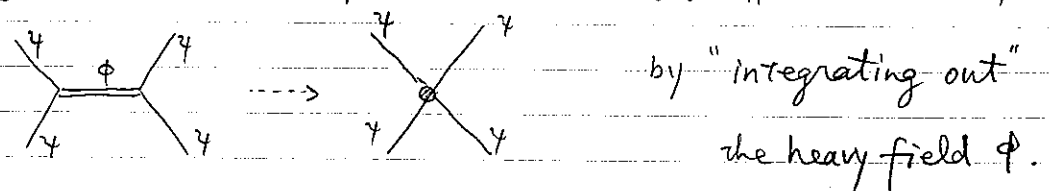
↑ ↗
external fields (so Z is a generating functional) of all correlation fns,

★ complete a square in the exponent

★ carry out the Gaussian integral.

$$\Rightarrow Z[\chi_H, \chi_L] \propto \int \mathcal{D}\psi e^{i \int d^d x \left[\frac{1}{2} (\partial_\mu \psi)^2 + \psi \chi_L + (g \psi \psi - \chi_H) \frac{1}{\partial^2 + M^2} (g \psi \psi - \chi_H) \right]}$$

to get the action of the low-energy effective theory.



Also possible (and useful) to integrate out

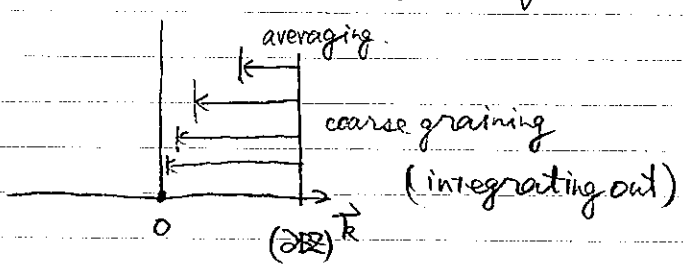
- only two components $[(\pm 1)\text{-eg space of } \gamma^0]$ in the four component Dirac fermion
- only negative frequency modes of a complex scalar

so we obtain a non-relativistic effective theory of $\left\{ \begin{array}{l} \bar{e} \\ \text{cold atom} \end{array} \right\}$.

The idea dates back to Born-Oppenheimer approximation.

Consider a spin system turned into a field theory (eg. ϕ^4 -theory)

- spacial momentum: $|\vec{k}| \leq$ bdry of the Brillouin zone. $\sim \pi/a$
- scalar field(s): local average of magnetization
(already a length scale of smearing is in place)
 $> a$.
- coarse graining (粗視化) / block spin transformation.
 \Leftrightarrow integrating out $\phi(\vec{k})$'s with larger \vec{k} 's



\Rightarrow In the effective action,
new terms may show up,
and coupling constants may be
modified accordingly.

- coupling constants: fns of $|\vec{k}|_{\max}$ $g(|\vec{k}|_{\max})$
- free energy / effective action: depend on $g(|\vec{k}|_{\max})$ & $|\vec{k}|_{\max}$
(GL potential) apparently

but should not dep. on $|\vec{k}|_{\max}$ in the end.

$\frac{\partial g(|\vec{k}|_{\max})}{\partial \ln(|\vec{k}|_{\max})}$ is related to the $|\vec{k}|_{\max}$ dependence
of the computation of the effective action.

renormalization (group)

§ 9.7 Brief introduction to $\left\{ \begin{array}{l} \text{In-In} \\ \text{Schwinger-Keldysh} \\ \text{real time} \end{array} \right\}$ formalism

Consider a system with

$$\left[\begin{array}{l} H = H_0 + H_{\text{int.}} \\ \tilde{H} = H + H' \end{array} \right]$$

$H_{\text{int.}}$ as in QED...

H' : coupling with time-varying background fields.

Start from a system in an equilibrium under the Hamiltonian H .

H' is turned on after time t_0 .

so the system deviates from the original equilibrium.

- AC electric field. (conductance)
- magnetic field switched $\left\{ \begin{array}{l} \text{on} \\ \text{off} \end{array} \right\}$ (NMR)
- time-varying $\left\{ \begin{array}{l} \text{inflaton} \\ \text{metric} \end{array} \right\}$ order parameter.

What happens then?

The easiest question to ask: $\frac{\text{Tr}[\mathcal{O}(t) \rho]}{\text{Tr}[\rho]}$

$$\rho = e^{-\beta H} = e^{-\beta H_0} \mathcal{P} \left[\exp \left(- \int_0^\beta dt' H_{\text{int.}H_0}(-i\tau + t_0) \right) \right]$$

$$\mathcal{O}(t) = \left(\mathcal{P} \left[\exp \left(-i \int_{t_0}^t dt' \tilde{H}(t') \right) \right] \right)^{-1} \mathcal{O}(t_0) \left(\mathcal{P} \left[\exp \left(-i \int_{t_0}^t dt' \tilde{H}(t') \right) \right] \right)$$

$$= \left(\mathcal{P} \left[\exp \left(-i \int_{t_0}^t dt' H_0 \right) \right] \right)^{-1} \mathcal{P} \left[\exp \left(-i \int_{t_0}^t dt' H_0 \right) \right]^{-1} \mathcal{O}(t_0) \left(\mathcal{P} \left[\exp \left(-i \int_{t_0}^t dt' H_0 \right) \right] \right)$$

$$\left(\mathcal{P} \left[\exp \left(-i \int_{t_0}^t dt' H_0 \right) \right] \right)^{-1} \cdot \mathcal{P} \left[\exp \left(-i \int_{t_0}^t dt' \tilde{H}(t') \right) \right]$$

$$= \mathcal{P} \left[\exp \left(-i \int_{t_0}^t dt' (H + H_{\text{int.}})_{H_0}(t') \right) \right]^{-1} \cdot \mathcal{O}_{H_0}(t) \cdot \mathcal{P} \left[\exp \left(-i \int_{t_0}^t dt' (H + H_{\text{int.}})_{H_0}(t') \right) \right]$$

using the interaction picture operators.

$$\Rightarrow \frac{\text{Tr}[\mathcal{O}(t) \rho]}{\text{Tr}[\rho]} = \frac{\text{Tr} \left[e^{-\beta H_0} \cdot \mathcal{P} \left[\mathcal{O}(t) \exp \left(-i \int_{t_0}^t dt' (H + H_{\text{int.}})_I(t') \right) \right] \right]}{\text{Tr} \left[e^{-\beta H_0} \cdot \mathcal{P} \left[\exp \left(-i \int_{t_0}^t dt' (H + H_{\text{int.}})_I(t') \right) \right] \right]}$$

an example: AC electric field on a metal.

$$\vec{O} = \left(\psi^\dagger \frac{-i\vec{\partial}}{m} \psi \right) \leftarrow \text{spacial component of the Noether current.}$$

↑
2 component fermion

$$H' = \psi^\dagger \frac{(-ieQ)}{2m} (\vec{A} \cdot \vec{\partial} + \vec{\partial} \cdot \vec{A}) \psi \quad \vec{A} = \frac{\vec{E}_0}{i\omega} e^{-i\omega t} \quad \text{classical background turned on after } t=0$$

The AC conductance is defined by $\langle \vec{O} \rangle = \sigma \vec{E}_0$ (linear response)

$$\left\langle \frac{1}{\omega} \int dt' \left[\left(\psi^\dagger \frac{-i\vec{\partial}}{m} \psi \right)(t), \left(\psi^\dagger \frac{-i\vec{\partial}}{m} \psi \right)(t') \right] e^{-i\omega t'} \right\rangle_{\beta} \sim \sigma$$

$$\begin{aligned} \psi_{\vec{r}} &\sim e^{-i(E(\vec{r})-\mu)t} a_{\vec{r}} + e^{i(\mu-E(\vec{r}))t} b_{\vec{r}}^\dagger \\ \psi_{\vec{r}}^\dagger &\sim e^{i(E(\vec{r})-\mu)t} a_{\vec{r}}^\dagger + e^{-i(\mu-E(\vec{r}))t} b_{\vec{r}} \end{aligned} \quad \left(\begin{array}{l} \text{electron } (E(\vec{r}) > \mu) \Rightarrow a, a^\dagger \\ \text{hole } (E(\vec{r}) < \mu) \Rightarrow b, b^\dagger \end{array} \right)$$

t' -dep background: possible to excite $a_{\vec{r}}^\dagger b_{\vec{r}}^\dagger$ (tilt the Fermi surface)

This is not a time-ordered correlation function, but a contour-ordered correlation function.

A case study

$$\text{Tr} \left(e^{-\beta H_0} \left[\left(\psi^\dagger \psi \right)(t, \vec{x}), \left(\psi^\dagger \psi \right)(0, \vec{y}) \right] \right) \quad \text{with } t > 0 \quad \left(\begin{array}{l} \text{forget about } \frac{-i\vec{\partial}}{m} \text{ etc.} \\ \text{not essential} \end{array} \right)$$

Note that

$$\begin{aligned} \Theta(t) [\psi_1^\dagger \psi_1, \psi_2^\dagger \psi_2] &= \Theta(t) \left(\psi_1^\dagger \psi_1 \psi_2^\dagger \psi_2 - \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 - \psi_2^\dagger \psi_2 \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_1^\dagger \psi_2 \psi_1 \right) \\ &= \Theta(t) \left(\{ \psi_1, \psi_2^\dagger \} \psi_1^\dagger \psi_2 - \{ \psi_2, \psi_1^\dagger \} \psi_2^\dagger \psi_1 \right). \end{aligned}$$

So, we need three propagators.

$$\Delta^R(t, \vec{x}) := \Theta(t) \frac{\text{Tr}[e^{-\beta H_0} \{ \psi(t, \vec{x}), \psi^\dagger(0, \vec{0}) \}]}{\text{Tr}[e^{-\beta H_0}]} = \Theta(t) \langle 0 | \{ \psi(t, \vec{x}), \psi^\dagger(0, \vec{0}) \} | 0 \rangle$$

$$\Delta^A(t, \vec{x}) := -\Theta(-t) \frac{\text{Tr}[e^{-\beta H_0} \{ \psi(t, \vec{x}), \psi^\dagger(0, \vec{0}) \}]}{\text{Tr}[e^{-\beta H_0}]} = -\Theta(-t) \langle 0 | \{ \psi(0, \vec{0}), \psi^\dagger(t, \vec{x}) \} | 0 \rangle$$

$$\Delta^<(t, \vec{x}) := -\frac{\text{Tr}[e^{-\beta H_0} \psi^\dagger(0, \vec{0}) \psi(t, \vec{x})]}{\text{Tr}[e^{-\beta H_0}]}$$

$$\Delta^>(t, \vec{x}) := \frac{\text{Tr}[e^{-\beta H_0} \psi(t, \vec{x}) \psi^\dagger(0, \vec{0})]}{\text{Tr}[e^{-\beta H_0}]}$$

(There is one relation: $(\Delta^> - \Delta^<) = (\Delta^R - \Delta^A)$.)

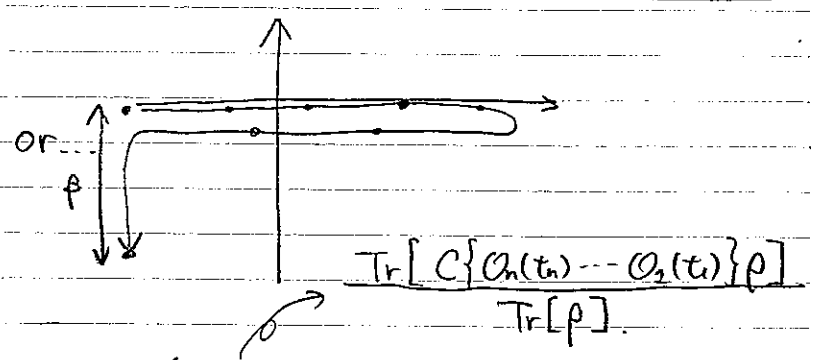
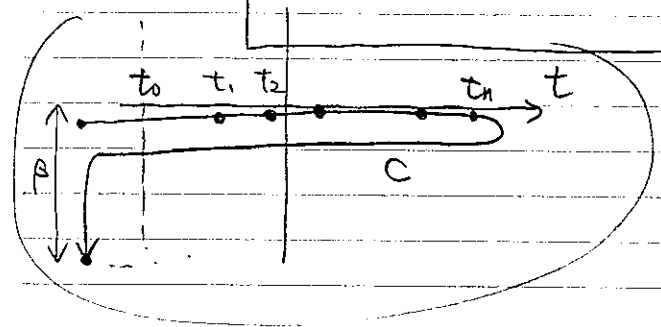
$$\frac{\text{Tr}[e^{-\beta H_0} [(\psi^\dagger \psi)(t, \vec{x}), (\psi^\dagger \psi)(0, \vec{0})]]}{\text{Tr}[e^{-\beta H_0}]} \Theta(t) = -(\Delta^R(t, \vec{x}-\vec{0}) \Delta^<(-t, \vec{0}-\vec{x}) + \Delta^<(t, \vec{x}-\vec{0}) \Delta^A(-t, \vec{0}-\vec{x})).$$

It is known that any of

$$\frac{\text{Tr}[T\{O_n(t_n) \dots O_1(t_1)\} \rho]}{\text{Tr}[\rho]}$$

can be computed perturbatively

by using the 3 propagators above.



(def) (of any use??)
 out-of-time-order(ed) correlation functions:
 not just going up and down just once.

like $\langle |[A(t), B(0)]|^2 \rangle = \langle [B(0), A(t)] [A(t), B(0)] \rangle$