Theory of Elementary Particles

homework III (Apr. 26, '21)

- submission via ITC-LMS of U Tokyo. Multiple files can be uploaded multiple times until the deadline (in early August).
- We request that the file name includes the problem number, II-1***.pdf or ****-IV-2-IX-1.jpeg. The ITC-LMS shows who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number.
- Reports do not have to be neatly written or type-set just for the reason that the reports have to be readable for me.
- Pick up any problems that are suitable for your study. **You are not expected to work on all of them!**
- A sample solution has been prepared and is made available in the form of a PDF file on the problems with "⋆" (e.g., III-1, III-5). The PDF is posted to you through the ITC-LMS in return for an early submission of a report on that problem during the semester.
- Keep your own copy, if you need one. Reports will not be returned.

1. **Quantum Correction I: fermion propagator** [B] ⋆
In the lecture, we have seen that the propagator of a Dirac fermion becomes

$$\frac{i}{(\not{p} - m_p + i\epsilon) - [A^{(1)}(p^2, m_p^2)\not{p} + B^{(1)}(p^2, m_p^2)] + \delta_{Z2}(m_p^2, \Lambda, e)\not{p} - [\delta_{Z2} m_p + \delta m(m_p, \Lambda, e)]} \tag{1}$$

at the 1-loop level. Here, the higher covariant derivative regularization was used to obtain

$$A^{(1)}(p^2, m^2) = \frac{\alpha Q^2}{4\pi} \int_0^1 dx \, [-2(1-x)] \ln \left(\frac{(1-x)\Lambda^2 + xm^2 - x(1-x)p^2}{xm^2 - x(1-x)p^2} \right), \tag{2}$$

$$B^{(1)}(p^2, m^2) = \frac{\alpha Q^2}{4\pi} \int_0^1 dx \, 4m \ln \left(\frac{(1-x)\Lambda^2 + xm^2 - x(1-x)p^2}{xm^2 - x(1-x)p^2} \right). \tag{3}$$

The denominator can be grouped into two; one is proportional to $\not{p}$, and the other to the unit 4×4 matrix.

(a) Verify that the terms proportional to $\not{p}$ as a whole is free from UV divergence as we take the limit $\Lambda \to +\infty$. Furthermore, verify that the limit is

$$\lim_{\Lambda \to +\infty} \left[1 - A^{(1)}(p^2, m_p^2) + \delta_{Z2}\right] \tag{4}$$
$$= 1 - 2\frac{\alpha Q^2}{4\pi} \int_0^1 dx \, \left[(1-x)\ln\left(\frac{m_p^2 - (1-x)p^2}{x m_p^2}\right) + \frac{2(1-x)^2 - 4(1-x)}{x}\right].$$

Note that this is a non-trivial function of p^2 and m_p^2, but free from UV divergence. [remark: If you are careful enough, however, you will also notice that the integral is divergent at $x \simeq 0$. This divergence is associated with IR degrees of freedom, not UV.]

(b) Verify that the remaining terms (those proportional to $\mathbf{1}_{4\times 4}$) also have a finite limit,

$$\lim_{\Lambda \to +\infty} \left[m_p + B^{(1)}(p^2, m_p^2) + (\delta m)(m_p, \Lambda, e) + \delta_{Z2}(m_p, \Lambda, e) m_p\right] \tag{5}$$
$$= m_p \left[1 + \frac{\alpha Q^2}{\pi} \int_0^1 dx \, \left\{\ln\left(\frac{x m_p^2}{m_p^2 - (1-x)p^2}\right) + \frac{-(1-x)^2 + 2(1-x)}{x}\right\}\right].$$

(c) (this is not intended as a part of the report problem, but you can work on it, if you like) If you wish to be convinced that the divergence in the x-integration at $x \simeq 0$ is due to IR degrees of freedom, rather than UV degrees of freedom, you can repeat the same computation by replacing the photon propagator as follows:

$$\frac{-i\eta_{\kappa\lambda}}{k^2 + i\epsilon} \Rightarrow \frac{-i\eta_{\kappa\lambda}}{k^2 - k^4/\Lambda^2} \Rightarrow \frac{-i\eta_{\kappa\lambda}}{k^2 - k^4/\Lambda^4 - \mu^2}, \tag{6}$$

where the k^4/Λ^2 term is due to the higher covariant derivative regularization, and now the μ^2 term is introduced in order to modify the IR (small k^2) behavior of the photon propagator; keep in mind that $\mu^2 \ll m_p^2, p^2 \ll \Lambda^2$. You will see at the end of calculations that the result of (4) becomes

$$1 - 2\frac{\alpha Q^2}{4\pi} \int_0^1 dx \, \left[(1-x)\ln\left(\frac{(1-x)\mu^2 + x m_p^2 - x(1-x)p^2}{(1-x)\mu^2 + x^2 m_p^2}\right) + \frac{\{2x(1-x)^2 - 4x(1-x)\} m_p^2}{(1-x)\mu^2 + x^2 m_p^2}\right]; \tag{7}$$

the earlier result would be recovered by simply setting $\mu = 0$. Now, the x integration is like $\int dx/x$ for $\mu/m_p \lesssim x$, and hence is like $\ln(m_p/\mu)$, but remains finite and well-defined. The divergence is now under control, so long as we keep μ to be small but non-zero. This can be taken as an indication that this divergence is due to IR degrees of freedom.

2. **1-Loop Calculation I, Pauli–Villars Regularization, Unitarity** [C]

Let us consider a theory where a complex scalar field φ and a 4-component (Dirac) fermion Ψ have an interaction (called Yukawa interaction);

$$\mathcal{L}_{\text{kin}} = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - M_\varphi^2 |\varphi|^2 + \overline{\Psi}\left(i\gamma^\mu \partial_\mu - m\right)\Psi, \tag{8}$$

$$\mathcal{L}_{\text{int}} = \lambda \varphi \overline{\Psi}\left(\frac{1-\gamma_5}{2}\right)\Psi + \lambda^* \varphi^* \overline{\Psi}\left(\frac{1+\gamma_5}{2}\right)\Psi. \tag{9}$$

(a) Compute the 1-loop contribution (Fig. 1 (a)) to the scalar self-energy $-i\Sigma(p^2, m^2) = i\mathcal{M}(p^2, m^2)$ (which does not include the external line propagators or a momentum conservation delta function), and show that it is

$$\mathcal{M} = \frac{2|\lambda|^2}{16\pi^2} \int_0^1 dx \int_0^\infty dK_E \, \frac{K_E(K_E + x(1-x)p^2)}{\left[K_E + m^2 - x(1-x)p^2\right]^2}; \tag{10}$$

here, p^μ is the momentum of the scalar field coming from the left, and K_E corresesponds to the invariant momentum square $(k' \cdot k')$ in the Euclidean signature of shifted momentum k'. [Did you remember to include the extra (-1) factor for a fermion loop?] Confirm that this integral is approximately $\propto \int dK_E$ (unlike $\int dK_E K_E^{-1}$ or $\int dK_E 1/K_E^2$) for $K_E \gg m^2, |p^2|$. We say in this situation that the integral is quadratically divergent (remember that K_E corresponds to momentum-square).

(b) (**momentum cut-off regularization**) When the divergent integral $\mathcal{M}$ above is made well-defined (finite) by replacing the integral over $K_E \in [0, \infty]$ with a finite range integral over $K_E \in [0, \Lambda_0^2]$, the 1-loop scalar self-energy is denoted by $-i\Sigma^{\text{mom. cutoff}} = i\mathcal{M}^{\text{mom. cutoff}}$. Determine $\mathcal{M}^{\text{mom. cutoff}}$ by carring out the integration.

(c) (**Pauli–Villars regularization**) As an alternative to the momentum cut-off regularization, one can make the 1-loop divergent integral $\mathcal{M}$ well-defined, by introducing other spiecies of "fermions" Ψ_j ($j = 1, 2 \cdots$) that have exactly the same interaction with φ as $\Psi_0 := \Psi$. Those "fermions" are assumed to have mass M_j and signature of the 1-loop diagram (+ for ordinary bosons and − for ordinary fermions) that are either the same ($\gamma_j = +1$) as or opposite ($\gamma_j = -1$) from that

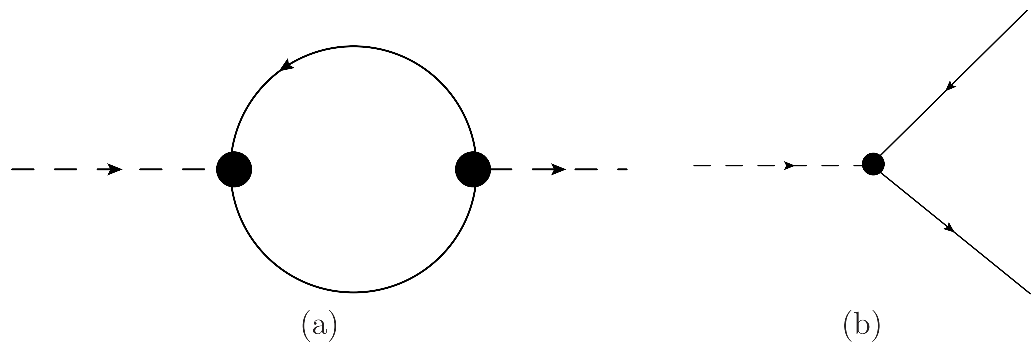

Figure 1: Scalar self-energy 1-loop diagram (a) and scalar decay diagram (b).

of Ψ_0 for each j. This regularization is called Pauli–Villars regularization. To see how this work, let us first consider introducing just $\Psi_{j=1}$ whose mass is M_1 and the signature opposite ($\gamma_1 = -1$). Show that the K_E integral of

$$\mathcal{M}(p^2, m^2) - \mathcal{M}(p^2, M_1^2) = \sum_{j=0}^{1} \gamma_j \mathcal{M}(p^2, M_j^2) \tag{11}$$

is still approximately $\propto dK_E$ for $m^2, |p^2| \ll K_E \ll M_1^2$, but the integral becomes $\propto dK_E M_1^2/K_E$ approximately in the region $M_1^2 \ll K_E$. This means that the Pauli–Villars regularization cannot render the divergent 1-loop integral $\mathcal{M}$ finite, if we are to introduce only one spiecies of "fermion" $\Psi_{j=1}$.

(d) This 1-loop integral for the scalar self-energy diagram can be made finite, by introducing three "fermions" $\Psi_{j=1,2,3}$. The signature of $\Psi_{j=1,2}$ are set to be opposite from that of the original fermion Ψ_0 (that is, $\gamma_{1,2} = -1$), and the signature of $\Psi_{j=3}$ to be the same as that of Ψ_0 (that is, $\gamma_3 = +1$). The 1-loop integral (including the contributions from these "fermions") become finite, if we take their masses, M_1, M_2, M_3, in such a way that the following relation is satisfied:

$$m^2 + M_3^2 = M_1^2 + M_2^2. \tag{12}$$

Compute $\mathcal{M}^{\text{P.V.}}(p^2, m^2; M_1^2, M_2^2, M_3^2)$

$$\lim_{\Lambda_0 \to \infty} \left[\sum_{j=0}^{3} \gamma_j \mathcal{M}^{\text{mom. cutoff}}(p^2, M_j^2) \right]$$
$$= \lim_{\Lambda_0 \to \infty} \left[\mathcal{M}^{\text{mom. cut}}(p^2, m^2) - \mathcal{M}^{\text{mom. cut}}(p^2, M_1^2) - \cdots \right].$$

In this context of Pauli–Villars regularization, the momentum cutoff scale Λ_0 plays the role of preregulator.

(e) In the case of $4m^2 \leq p^2 \ll M^2_{j=1,2,3}$, the logarithm appearing in $\mathcal{M}^{\text{mom. cutoff}}$ and $\mathcal{M}^{\text{P.V.}}$ means that a branch cut has to be introduced along the real positive axis of the p^2 complex plane. Show that

$$\frac{1}{i}\left[\mathcal{M}(p^2+i\epsilon, m^2) - \mathcal{M}(p^2-i\epsilon, m^2)\right] = \frac{2\pi|\lambda|^2}{16\pi^2}\sqrt{\frac{p^2-4m^2}{p^2}}(p^2-2m^2). \tag{13}$$

Note that this result does not depend on the choice of regularization schemes.

(f) (If you are not tired yet...) Compute the decay rate of φ (Feynman diagram Figure 1 (b)), $\Gamma(\varphi \to \Psi + \overline{\Psi})$, and confirm that $(2M_\varphi)\times\Gamma$ is the same as (13). [This is one of consequesnces of the optical theorem.] Here, we assume that $M_\varphi \geq 2m$, so that the scalar field can decay into the fermion pair.

(g) Because of this branch cut, we need to be a little more careful in phrasing how to compute the scalar self-energy 1-loop diagram. We define, for $p^2 > 4m^2$, the scalar self-energy $\Sigma(p^2, m^2)$ to be the amplitude $-\mathcal{M}(p^2, m^2)$ for p^2 in the upper half complex plane; $\Sigma(p^2, m^2)$ for p^2 in the lower complex half plane is defined by the analytic continuation through the $\text{Re}(p^2) < 4m^2$ region, where the branch cut is absent. Show that the propagator with 1-loop 1PI correction,

$$\frac{i}{p^2 - M_\varphi^2 - \Sigma(p^2, m^2) + i\epsilon} \tag{14}$$

has a pole at

$$p^0 \simeq M_\varphi - \frac{1}{2M_\varphi}\text{Re}\mathcal{M}(M_\varphi^2, m^2) - i\frac{\Gamma}{2} \tag{15}$$

for the $\vec{p} = \vec{0}$ case for simplicity. [This means that the propagator in the spacetime picture exhibits the time dependence $e^{-iM_\varphi t} \times e^{-\Gamma t/2}$. After taking its absolute value square of this quantum mechanical amplitude, we obtain the $e^{-\Gamma t}$ dependence of an unstable particle.]

3. **Summing up Geometric Series for Photon Propagator** [A]

Photon propagator is

$$\frac{-i}{q^2+i\epsilon}\left[\eta_{\mu\nu} + (\xi - 1)\frac{q_\mu q_\nu}{q^2}\right], \tag{16}$$

where ξ is a gauge parameter, and $\xi = 1$ [$\xi = 0$] corresponds to the Feynman gauge [Landau gauge], respectively. When the photon "self-energy" (sum of 1 particle irreducible diagrams: better known as vacuum polarization in this case) is given by

$$i\left(q^2\eta_{\mu\nu} - q_\mu q_\nu\right)\Pi(q^2) \tag{17}$$

for some function $\Pi(q^2)$ of q^2, the quantum corrected photon propagator is of the form

$$\begin{aligned}
&\frac{-i}{q^2+i\epsilon}\left[\eta_{\mu\nu} + (\xi-1)\frac{q_\mu q_\nu}{q^2}\right] \\
+ \; &\frac{-i}{q^2+i\epsilon}\left[\eta_{\mu\kappa} + (\xi-1)\frac{q_\mu q_\kappa}{q^2}\right] i(q^2\eta^{\kappa\lambda} - q^\kappa q^\lambda)\Pi(q^2)\frac{-i}{q^2+i\epsilon}\left[\eta_{\lambda\nu} + (\xi-1)\frac{q_\lambda q_\nu}{q^2}\right] + \cdots .
\end{aligned}$$

Sum up this geometric series to show that it is the same as

$$\frac{-i}{(q^2+i\epsilon)(1-\Pi(q^2))}\left[\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] + \xi\frac{-iq_\mu q_\nu}{q^2 q^2}. \tag{18}$$

4. **1-Loop Calculation III: Photon Vacuum Polarization in Pauli–Villars** [C]

Photon 1-loop "self-energy" (or vacuum polarization) in QED

$$\begin{aligned}
&(2\pi)^4\delta^4(q'-q)\; i\mathcal{M}^{\mu\nu} := \\
&\qquad \int d^4x d^4y\; e^{iq'\cdot x}e^{-iq\cdot y}\langle 0|T\left\{\left(-ieQ\overline{\Psi}_I\gamma^\nu\Psi_I\right)(x)\left(-ieQ\overline{\Psi}_I\gamma^\mu\Psi_I\right)(y)\right\}|0\rangle
\end{aligned} \tag{19}$$

corresponds to the Feynman diagram in Figure 2 (a). Let us calculate this by using the Pauli–Villars regularization, and show that $i\mathcal{M}^{\mu\nu}$ is indeed of the form (17). To do this,

(a) show that, for a Dirac fermion with mass M,

$$\begin{aligned}
&i\mathcal{M}^{\mu\nu}(q^2, M^2) \\
&= (-4i(eQ)^2)\int_0^1 dx\int\frac{d^4k_E}{(2\pi)^4}\frac{\left[\frac{1}{2}(k_E^2)\eta^{\mu\nu}\right] + \left[x(1-x)(q^2\eta^{\mu\nu} - 2q^\mu q^\nu)\right] + \left[M^2\eta^{\mu\nu}\right]}{\left[k_E^2 + M^2 - x(1-x)q^2\right]^2}
\end{aligned} \tag{20}$$

after Wick rotation. k_E^2 indicates that the 4-dim Euclidean metric is used in determining $k\cdot k$.

(b) Carry out angle and radial integration of 4-dimensional d^4k_E space; as a pre-regulator, introduce a cut-off in the range of integration, $k_E^2 \le \Lambda_0^2$. Note that this integral in the momentum cut-off regularization $i\mathcal{M}^{\mu\nu}_{\text{mom. cut}}(q^2, M^2; \Lambda_0^2)$ does not have a form of (17) at all.

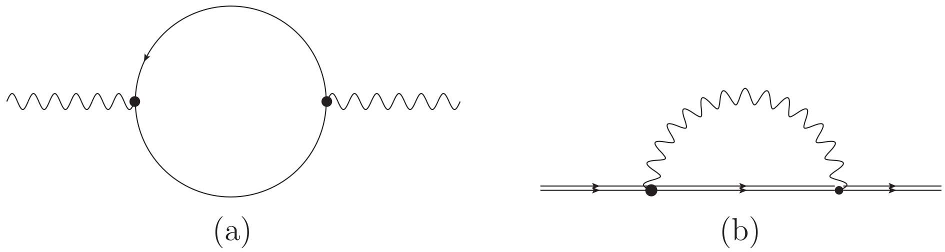

Figure 2: Self-energy graph of photon (a) and heavy fermion (b).

(c) The photon 1-loop "self-energy" (vacuum polarization) in the Pauli–Villars regularization is given by

$$i\mathcal{M}^{\mu\nu}_{\rm P.V}(p^2, M^2) = \lim_{\Lambda_0^2\to\infty}\left[\sum_{j=0}^{3}\gamma_j \mathcal{M}^{\mu\nu}_{\text{mom. cut}}(q^2, M_j^2; \Lambda_0^2)\right], \tag{21}$$

just like in homework III-3. $\gamma_0 = +1$ and $M_0^2 = M_2$ by definition. We should take $\gamma_{1,2} = -1$ and $\gamma_3 = +1$, and $M_0^2 + M_3^2 = M_1^2 + M_2^2$ so that the integral remains finite, when the pre-regulator (momentum cutoff) is removed ($\Lambda_0^2 \to \infty$). Show that

$$i\mathcal{M}^{\mu\nu}_{\rm P.V.}(p^2, M^2) = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\frac{(eQ)^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(\prod_j \left[M_j^2 - x(1-x)q^2\right]^{\gamma_j}\right). \tag{22}$$

(d) (not a problem) If we take the Pauli–Villars regulator masses M_1^2, M_2^2 and M_3^2 much larger than the original Dirac fermion mass M^2 and momentum flow q^2, the last logarithmic factor is approximately

$$\ln\left(\frac{M^2 - x(1-x)q^2}{\overline{M}^2}\right), \qquad \overline{M}^2 := M_1^2 M_2^2/M_3^2. \tag{23}$$

In the Pauli–Villars regularization, $i\mathcal{M}^{\mu\nu}$ is in the form of (17) as expected from the gauge invariance of QED, and (at 1-loop)

$$\Pi^{(1)}(q^2) = \frac{(eQ)^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(\frac{M^2 - x(1-x)q^2}{\overline{M}^2}\right). \tag{24}$$

5. **Mass Correction of Non-relativistic Fermion (Heavy Quark Effective Theory)** [C] $\star$

Consider a non-relativisitic fermion with a Q unit of electric charge.

(a) Show (understand) that the 1-particle irreducible diagram (Figure 2 (b)) for the non-relativistic fermion is given at the leading order in $1/M$ expansion by

$$i\mathcal{M} = -i\Sigma = \int \frac{d\omega}{(2\pi)} \int \frac{d^3\vec{k}}{(2\pi)^3} \; (-ieQ)\frac{i}{\omega^0+\omega}(-ieQ)\frac{-i}{\omega^2 - |\vec{k}|^2 + i\epsilon}, \tag{25}$$

where the spacial component of the external line momentum, $\vec{p}$, is set to $\vec{0}$ for simplicity, and $\omega^0 := p^0 - M$ is the energy flow of the external fermion field. The propagator of a non-relativistic two-component fermion is of the form

$$\frac{i\,\mathbf{1}_{2\times 2}}{\omega - \frac{\vec{p}^2}{2M} + i\epsilon}, \tag{26}$$

and the term proportional to $1/M$ has been dropped in the expression above.

Note that only the $A_0 = \varphi$ component of photon contributes at this leve of fermion mass non-relativistic expansion ($1/M$ expansion).

(b) (not a problem) It is necessary to regularize this integral, or otherwise the self-energy correction is divergent and not well-defined. So, we use the higher covariant derivativee regularization for the photon propagator, which is to modify the photon propagator in the following way:

$$\frac{-i}{k^2+i\epsilon} \Longrightarrow \frac{-i}{k^2 - k^4/\Lambda^2} \Longrightarrow \frac{i\Lambda^2}{(k^2+i\epsilon)(k^2-\Lambda^2+i\epsilon)}. \tag{27}$$

Here, we have in mind a situation characterized by $\omega^0 \ll \Lambda \ll M$.

(c) As the first step of evaluating the 1-loop contribution to the self-energy $-i\Sigma$, we wish to introduce a trick (similar to the Feynman parameters) in combining the denominator of the integrand together for non-relativisitc cases that. Verify some of the following relations:

$$\frac{1}{a}\frac{1}{b} = \int_0^\infty d\lambda \frac{2}{(a+2b\lambda)^2}, \tag{28}$$

$$\frac{\Gamma(m)}{a^m}\frac{\Gamma(n)}{b^n} = \int_0^\infty d\lambda \frac{2\Gamma(m+n)(2\lambda)^{n-1}}{(a+2b\lambda)^{m+n}}, \tag{29}$$

$$\begin{aligned}\frac{1}{a_1a_2}\frac{1}{b} &= \frac{1}{b}\int_0^1 dx \frac{1}{(xa_1+(1-x)a_2)^2} \\ &= \int_0^1 dx \int_0^\infty d\lambda \frac{2\Gamma(3)}{(xa_1+(1-x)a_2)+2b\lambda)^3},\end{aligned} \tag{30}$$

$$\frac{1}{\prod_{i=1}^{m} a_i}\frac{1}{\prod_{j=1}^{n} b_j} = \int_0^1 d^m x \delta(1-\sum_i x_i) \int_0^1 d^n y \delta(1-\sum_j y_j) \int_0^\infty d\lambda \frac{2\Gamma(m+n)(2\lambda)^{n-1}}{(\sum_i x_i a_i + 2\lambda \sum_j y_j b_j)^{m+n}}. \quad (31)$$

[Ref: §3.1 of A. Manohar and M. Wise, "*Heavy Quark Physics*," Cambridge U. Press]

(d) A factor linear in loop energy ω in the denominator (such as $(\omega+\omega^0)$) is treated as one of b_j's, and a factor quadratic in ω (such as $(\omega^2-|\vec{k}|^2)$ and $(\omega^2-|\vec{k}|^2-\Lambda^2)$) are treated as one of a_i's in using the trick, so we can complete the square in the combined single factor in the denominator. Verify that

$$\frac{1}{\omega+\omega^0}\frac{1}{\omega^2-|\vec{k}|^2}\frac{1}{\omega^2-|\vec{k}^2-\Lambda^2} = \int_0^\infty d\lambda \int_0^1 dx \frac{2}{[(\omega+\lambda)^2-|\vec{k}|^2-x\Lambda^2+2\lambda\omega^0-\lambda^2]^3} \quad (32)$$

(e) Do the Wick rotation, which is to change the contour of integration in $\omega' := (\omega+\lambda)$ from the real axis to the imaginary axis, and carry out the integration in the loop energy and momentum. One will find that

$$-i\Sigma = \frac{+i(eQ)^2}{16\pi^2}\int_0^\infty d\lambda \int_0^1 dx \int_0^\infty dK_E \frac{K_E 2\Lambda^2}{[K_E+x\Lambda^2-2\lambda\omega^0+\lambda^2]^3} \quad (33)$$

$$= \frac{i(eQ)^2}{16\pi^2}\int_0^\infty d\lambda \int_0^1 dx \frac{\Lambda^2}{x\Lambda^2-2\lambda\omega^0+\lambda^2}, \quad (34)$$

$$= \frac{i(eQ)^2}{16\pi^2}\Lambda^2 \int_0^1 dx \frac{\mathrm{ArcCos}(-\omega^0/(\sqrt{x}\Lambda))}{\sqrt{x\Lambda^2-(\omega^0)^2}}. \quad (35)$$

(f) Expand the self-energy $\Sigma(\omega^0;\Lambda)$ in ω^0/Λ before carrying out the x-integral, and keep only the terms that are in a non-negative power of the regulator energy scale Λ. Show, if the range of dx integration is limited to $[(\mu/\Lambda)^2, 1]$, that (don't worry if the you do not get the coefficients right. maybe there is a typo here, or elsewhere.)

$$\Sigma \simeq -\frac{(eQ)^2}{16\pi^2}\left(\pi\Lambda + 2\omega^0 \ln(\Lambda/\mu)\right). \quad (36)$$

(g) (not a problem) These two terms corresponds to the decomposision of the fermion self-energy $\Sigma(p^\mu;\Lambda) = B + A\not{p}$ in the relativistic case. The mass correction is linearly divergent in the regulator energy scale $\Lambda \ll M$, while the wavefunction renormalization is logarithmically divergent. For the mass correction (self energy) $-\alpha\Lambda/4$ to be below the electron mass M, the cut-off energy scale Λ (where apparent contradiction seems inevitable) should be $4M/\alpha$ or less than that; this is how the "classical electron radius" was derived in the classical electromagnetism course. The linear divergence in the mass correction becomes logarithmic divergence in QED, in fact, because the non-relativisitic approximation is not valid at energy scale around or above the electron mass, and the positron also contributes to the self-energy at energy scale above the electron mass. We see in the renormalized perturbation theory that physical correlation functions can be expressed without divergence when written down as functions of observable parameters such as pole masses, rather than theoretical parameters in the microscopic lagrangian. So, regardless of whether we have a linear divergence or logarithmic divergence, it does not matter in the end.