convention

$c = \hbar = 1$

$\eta^{\mu\nu} = \mathrm{diag}(+,---)$

$e > 0.\quad Q_e = -1 \qquad \alpha_e = \frac{e^2}{4\pi}$

§ 0. Feynman rule.

When a Lagrangian is given in terms of fields,

how do we quantize the fields (Quantum mechanical DOF's associated with points in space)

and how do we compute correlation functions?

Example : complex scalar field ϕ.

$$\boxed{\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 |\phi|^2 - \frac{\lambda}{4} |\phi|^4}$$

☆ Canonical quantization

① focus on the bilinear part.

$$\mathcal{L}_{bilin} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - M^2 |\phi|^2.$$

② use the Fourier modes.

$$\phi(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} e^{i\vec{p}\cdot\vec{x}} \phi_{\vec{p}}(t)$$

$$\phi^\dagger(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} e^{-i\vec{p}\cdot\vec{x}} \phi_{\vec{p}}^\dagger(t)$$

$$L_{bilin} = \int d^3\vec{x}\, \mathcal{L}_{bilin} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{(2E_p)} \left\{ (\partial_t \phi_{\vec{p}}^\dagger)(\partial_t \phi_{\vec{p}}) - (M^2 + \vec{p}^2)(\phi_{\vec{p}}^\dagger \phi_{\vec{p}}) \right\}$$

⇒ Infinitely many harmonic oscillators

for each $\vec{p}$ 3dim. $(\mathrm{Re}\,\phi_{\vec{p}})$ & $\mathrm{Im}(\phi_{\vec{p}})$

frequency $\omega = \sqrt{M^2 + \vec{p}^2} = E_{\vec{p}}$.

⇒⇒ Heisenberg picture operators

$$\phi(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(e^{-ip\cdot x} a_{\vec{p}} + e^{ip\cdot x} b_{\vec{p}}^\dagger \right)$$

$$\phi^\dagger(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(e^{ip\cdot x} a_{\vec{p}}^\dagger + e^{-ip\cdot x} b_{\vec{p}} \right)$$

$p^\mu = (E_{\vec{p}}, \vec{p})$ positive on-shell.

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{q})$$

$$[b_{\vec{p}}, b_{\vec{q}}^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{q})$$

☆ Time-ordered correlation functions

constitute a class of observables we can define in a QFT

$$\langle \Omega | T\{ \mathcal{O}_1(x_1^\mu)\, \mathcal{O}_2(x_2^\mu) \cdots \mathcal{O}_n(x_n^\mu) \} | \Omega \rangle$$

- local operators $\mathcal{O}_i$ placed at $x_i^\mu \in \mathbb{R}^{1,3}$ Minkowski space-time.

  sorted out in the order of $(x_i^{\mu=0})$.

  $\mathcal{O}(x^0)\,\mathcal{O}'(y^0)$ if $x^0 > y^0$

  ~~$\mathcal{O}'(y^0)\,\mathcal{O}(x^0)$ if $y^0 > x^0$~~

- $|\Omega\rangle$: vacuum state (the lowest energy state of a QFT. w/ time-translatn invariance)

  normalized so that $\langle \Omega | \Omega \rangle = 1$.

It is possible to extract such things as scattering amplitudes, decay rates, cross sections. etc.

See Peskin-Schroeder §4.5 + 4.6 (for example) for how to do that.

An important short-cut in calculations (usefull)

$$\boxed{\langle \Omega | T\{ \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \} | \Omega \rangle = \left(\langle 0 | T\{ \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \exp\left(i \int dt\, d^3\vec{x}\, \mathcal{L}_{int} \right) \} | 0 \rangle \right)^{\text{free theory}}_{\text{no bubbles}}}$$

$\mathcal{L}_{int} := \mathcal{L} - \mathcal{L}_{bilin}$

"free theory" on the RHS : compute the RHS by

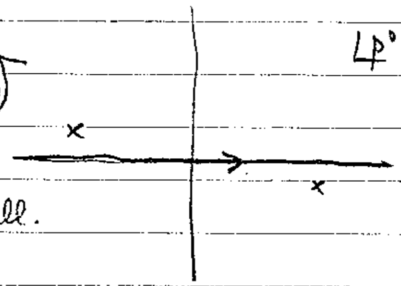
- substituting the free theory $\phi(\vec{x},t)$ into $\mathcal{O}_i(x)$'s & $\mathcal{L}_{int}$ and
- using ~~creation / annihilation operators applied to the~~ the algebra of free theory vacuum state $|0\rangle$,

☆ propagators

$$\langle 0|T\{\phi(\vec{x},t)\,\phi^\dagger(\vec{y},t)\}|0\rangle = \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}\,2E_{\vec{q}}}}\times\begin{cases} t>t' & \langle 0|\left(a_{\vec{p}}e^{-ip\cdot x}+b_{\vec{p}}^\dagger e^{ip\cdot x}\right)\left(a_{\vec{q}}^\dagger e^{iq\cdot y}+b_{\vec{q}}e^{-iq\cdot y}\right)|0\rangle \\ t'>t & \langle 0|\left(a_{\vec{q}}^\dagger e^{iq\cdot y}+b_{\vec{q}}e^{-iq\cdot y}\right)\left(a_{\vec{p}}e^{-ip\cdot x}+b_{\vec{p}}^\dagger e^{ip\cdot x}\right)|0\rangle \end{cases}$$

$$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\begin{cases} t>t' & e^{-iE_{\vec{p}}(t-t')}\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \quad (\Leftarrow \langle 0|a\,a^\dagger|0\rangle) \\ t'>t & e^{-iE_{\vec{p}}(t'-t)}\,e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \quad (\Leftarrow \langle 0|b\,b^\dagger|0\rangle) \end{cases}$$

$$=\int\frac{dp^0}{2\pi}\int\frac{d^3p}{(2\pi)^3}\frac{i\,e^{-ip^0(t-t')}\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{(p^2-M^2+i\varepsilon)}$$



$\varepsilon>0$ infinitesimally small.

$$=\int\frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot(x-y)}}{(p^2-M^2+i\varepsilon)}$$

It is a partial contribution to the 2-point function

$$\langle\Omega|T\{\phi(\vec{x},t)\,\phi^\dagger(\vec{y},t')\}|\Omega\rangle = \left(\langle 0|T\{\phi(x)\,\phi^\dagger(y)\,\exp\left(-\frac{i\lambda}{4}\int d^4z\,\phi(z)^4\right)\}|0\rangle\right)_{\text{no bubble}}^{\text{free}}$$

$$\exp\left(-\frac{i\lambda}{4}\int d^4z\,\phi^4\right) = 1 + \cdots\cdots$$

☆ more contributions to the 2-pt function.

☆☆ $\exp\left(-\frac{i\lambda}{4}\int d^4z\, \phi(z)^4\right) = 1 - \frac{i\lambda}{4}\int d^4z\, \phi^4(z) + \frac{1}{2}\left(-\frac{i\lambda}{4}\right)^2 \int d^4z_1 \int d^4z_2\, \phi^4(z_1)\, \phi^4(z_2)$

(1 ↑ done.)

$\boxed{x^0 > y^0}$

$\langle\Omega| T\{\phi(x)\phi^\dagger(y)\}|\Omega\rangle = \text{propagator}(x^\mu, y^\mu) +$

$$\frac{1}{2}\left(-\frac{i\lambda}{4}\right)\int_{z_1^0 > z_2^0} d^4z_1\, d^4z_2 \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}$$

$$\langle 0|\left(a_{\vec p}e^{-ip\cdot x} + b^\dagger_{\vec p}e^{ip\cdot x}\right)\left(a_{\vec k_1}e^{-ik_1\cdot z_1} + b^\dagger_{\vec k_1}e^{ik_1\cdot z_1}\right)\left(a_{\vec k_2}e^{-ik_2\cdot z_1} + b^\dagger_{\vec k_2}e^{ik_2\cdot z_1}\right)$$
$$\left(a^\dagger_{\vec k_3}e^{ik_3\cdot z_1} + b_{\vec k_3}e^{-ik_3\cdot z_1}\right)\left(a^\dagger_{\vec k_4}e^{ik_4\cdot z} + b_{\vec k_4}e^{-ik_4\cdot z}\right)$$
$$\left(a_{\vec q_1}e^{-iq_1\cdot z_2} + b^\dagger_{\vec q_1}e^{iq_1\cdot z_2}\right)\left(a_{\vec q_2}e^{-iq_2\cdot z_2} + b^\dagger_{\vec q_2}e^{iq_2\cdot z_2}\right)$$
$$\left(a^\dagger_{\vec q_3}e^{iq_3\cdot z_2} + b_{\vec q_3}e^{-iq_3\cdot z_2}\right)\left(a^\dagger_{\vec q_4}e^{iq_4\cdot z_2} + b_{\vec q_4}e^{-iq_4\cdot z_2}\right)$$
$$\left(a^\dagger_{\vec p'}e^{ip'\cdot y} + b_{\vec p'}e^{-ip'\cdot y}\right)|0\rangle$$

$+ \cdots$

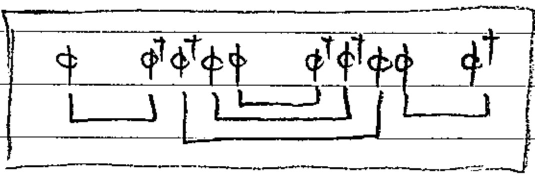
To evaluate this matrix element. $\langle 0|$ creat'n & annihil op's $|0\rangle$

use the algebra $[a_{\vec p}, a^\dagger_{\vec q}] = (2\pi)^3\delta^3(\vec p - \vec q)$ and $a_{\vec p}|0\rangle = 0$

$\langle 0|a^\dagger_{\vec q} = 0.$

$$\langle 0|\underbrace{a_{\vec p}}_{@x\ \ \phi}\ \underbrace{\left(a^\dagger_{\vec k_3} b_{\vec k_4}\right.}_{\phi^\dagger\phi^\dagger}\ \underbrace{\left.a_{\vec k_1} a_{\vec k_2}\right)}_{@z_1\ \ \phi\,\phi}\ \underbrace{\left(a^\dagger_{\vec q_3}\, a^\dagger_{\vec q_4}\right.}_{\phi^\dagger\ \phi^\dagger}\ \underbrace{\left.b^\dagger_{\vec q_2}\, a_{\vec q_1}\right)}_{@z_2\ \ \phi\ \phi}\ \underbrace{a^\dagger_{\vec p'}}_{@y\ \ \phi^\dagger}|0\rangle \neq 0.$$

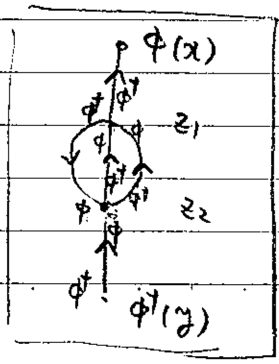
$$= [a_{\vec p}, a^\dagger_{\vec k_3}]\,[b_{\vec k_4}, b^\dagger_{\vec q_2}]\,[a_{\vec k_2}, a^\dagger_{\vec q_3}]\,[a_{\vec k_1}, a^\dagger_{\vec q_4}]\,[a_{\vec q_1}, a^\dagger_{\vec p'}] + \cdots$$



graphical presentation that

keeps track of the algebra

$\boxed{\text{Feynman graph}}$ (diagram)



PLUS

a Feynman graph $\Rightarrow$ one partial contribution to a time-ordered correlatn fcn.

- vertex : from the expansion of $\exp\left(i\int d^4z\, \mathcal{L}_{int}\right)$
  (and the inserted operators $\mathcal{O}_i(x_i^\mu)$)
- lines : joining a pair of free field operators (comm. relatn $\neq 0$)
  where $\langle 0|T\{\phi_i(x_i)\ \phi_j(x_j)\}|0\rangle \neq 0$

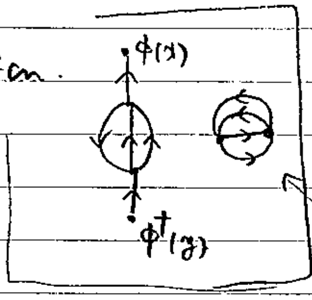
list up Feynman graphs for a given $\langle\Omega|T\{\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\}|\Omega\rangle$

and sum up those partial contributions.

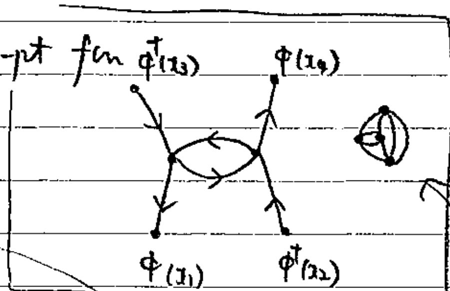
( ) no bubbles $\Leftrightarrow$ a prescription to include only the graphs where all the vertices are connected to some of the inserted operators.

example

2-pt fcn.



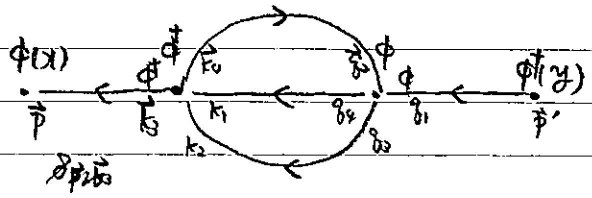
4-pt fcn



do not include them. bubbles.

This prescription is a part of the "useful short-cut" in p. I-2

★ Feynman rule in the momentum space



$$\left(\phi^\dagger(z_1) \Rightarrow \int \frac{d^3k_4}{(2\pi)^3} \frac{1}{\sqrt{2E_{k_4}}}\, b_{k_3}\, e^{-ik_3\cdot z_1}\right)\left(\phi(z_2) \Rightarrow \int \frac{d^3q_2}{(2\pi)^3} \frac{1}{\sqrt{2E_{q_2}}}\, b^\dagger_{q_2}\, e^{iq_2\cdot z_2}\right)$$

a part of the propagator. $(2\pi)^3\delta^3(\vec{k}_4 - \vec{q}_2)$

$$\int \frac{d^4q_2}{(2\pi)^4} \frac{i\, e^{-iq_2\cdot(z_1-z_2)}}{(q^2 - M^2 + i\varepsilon)}$$

◎ lines in the graph $\Rightarrow$ propagators.

◎ at an interaction vertex.

$$-\frac{i\lambda}{4} \times \int d^4z_1\, e^{ik_3\cdot z_1}\, e^{ik_4\cdot z_1}\, e^{-ik_1\cdot z_1}\, e^{-ik_2\cdot z_1}$$

$$= -\frac{i\lambda}{4} \cdot (2\pi)^4 \delta^4(k_3 + k_4 - k_1 - k_2)$$

vertex factor $\times$ momentum conservation δ-fcn