§ 1. Ultraviolet Divergence and Regularization

(2-point functions as examples.)

§ 1.1 Self-energy

Consider a two-point function in QED:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\left[i\gamma^\mu(\partial_\mu + ieQ_e A_\mu) - m\right]\Psi$$

$$\langle\Omega|T\{\Psi(y)\,\bar{\Psi}(x)\}|\Omega\rangle.$$

Due to the translational symmetry of $\mathbb{R}^{1,3}$, it should depend only on $(y^\mu - x^\mu)$.

Think of

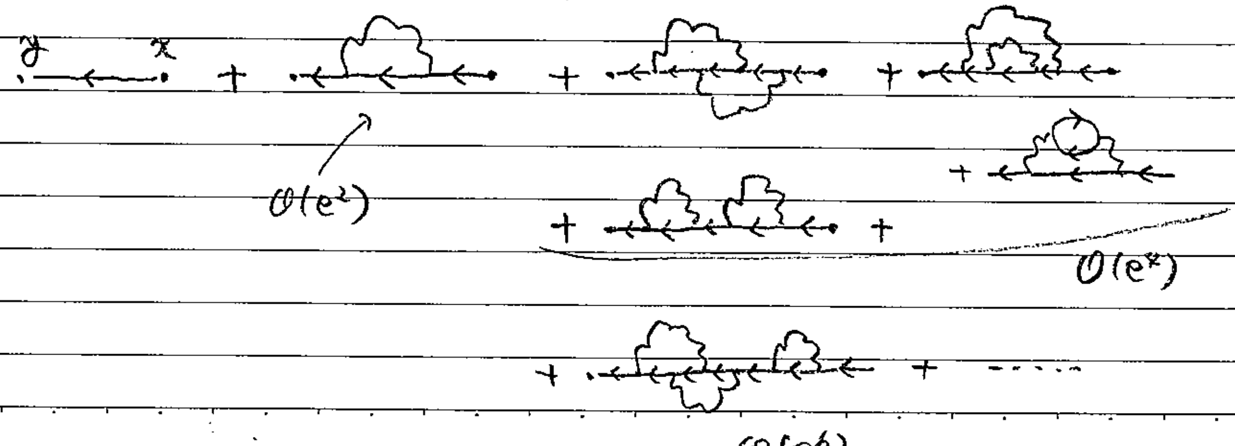
$$\int d^4(y-x)\,\langle\Omega|T\{\Psi(y)\,\bar{\Psi}(x)\}|\Omega\rangle\; e^{ip\cdot(y-x)} =: G(p).$$

The leading order contribution to $G(p)$ is

$$\int d^4(y-x)\, e^{ip\cdot(x-y)} \times \left(\text{propagator} = \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q}+m)\, e^{-iq\cdot(y-x)}}{(q^2-m^2+i\varepsilon)}\right)$$

$$= \int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta^4(p-q) \frac{i(\not{q}+m)}{(q^2-m^2+i\varepsilon)} = \frac{i(\not{p}+m)}{(p^2-m^2+i\varepsilon)}$$

But there are other contributions (graphs).

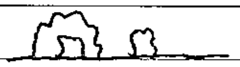


Here is a better way of organizing all those contributions than ~~just~~ going order by order in (e^{2n}).

$$= \frac{i(\not p+m)}{(p^2-m^2+i\varepsilon)} \times (\text{something}_1) \times \frac{i(\not p+m)}{(p^2-m^2+i\varepsilon)}$$

$$= \frac{i(\not p+m)}{(p^2-m^2+i\varepsilon)} \times \left\{ \begin{matrix} (\text{something}_{2.2}) \\ (\text{something}_{2.1}) \\ (\text{something}_{2.3}) \end{matrix} \right\} \times \frac{i(\not p+m)}{(p^2-m^2+i\varepsilon)}$$

$$= \frac{i(\not p+m)}{(p^2-m^2+i\varepsilon)} (\text{something}_1) \frac{i(\not p+m)}{(p^2-m^2+i\varepsilon)} (\text{something}_1) \frac{i(\not p+m)}{p^2-m^2+i\varepsilon}$$



$$= \frac{i(\not p+m)}{p^2-m^2+i\varepsilon} (\text{something}_{2.1}) \frac{i(\not p+m)}{p^2-m^2+i\varepsilon} (\text{something}_1) \frac{i(\not p+m)}{p^2-m^2+i\varepsilon}$$

$$= \frac{i(\not p+m)}{p^2-m^2+i\varepsilon} (\text{something}_1) \frac{i(\not p+m)}{p^2-m^2+i\varepsilon} (\text{something}_{2.1}) \frac{i(\not p+m)}{p^2-m^2+i\varepsilon}$$

$\vdots$

So, we have.

$$G(p) = \frac{i(\not p+m)}{p^2-m^2+i\varepsilon} + \frac{i(\not p+m)}{p^2-m^2+i\varepsilon} \begin{pmatrix} \text{some}_1 \\ + \\ \text{some}_{2.1} \\ \text{some}_{2.2} \\ \text{some}_{2.3} \\ +\text{some}_{3.*} \\ + \cdots \end{pmatrix} \frac{i(\not p+m)}{p^2-m^2+i\varepsilon}$$

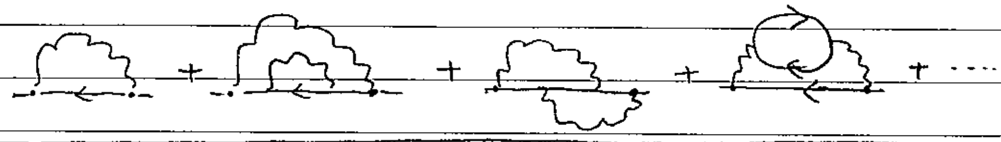
$$+ \frac{i(\not p+m)}{(p^2-m^2+i\varepsilon)} \left(\Sigma\ \text{something}\right) \frac{i(\not p+m)}{p^2-m^2+i\varepsilon} \left(\Sigma\ \text{something}\right) \frac{i(\not p+m)}{p^2-m^2+i\varepsilon}$$

$+ \cdots$

forming a geometric series.

(等比級数)

$$\Sigma(p) :=$$

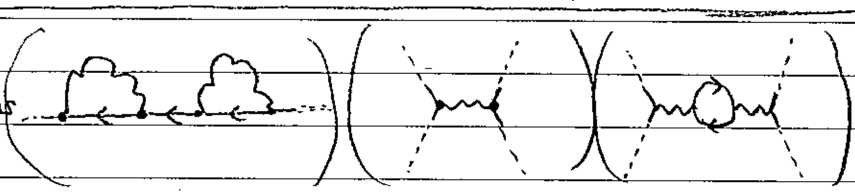


(external lines, $(2\pi)^4\delta^4(p_{in}-p_{out})$ are not included in $\Sigma(p)$)

Def 1PI graphs (1-particle irreducible)

connected graphs with all the external lines removed, that remain connected when any one of internal lines is cut

These graphs



are not.

In the case of the two-point function of a scalar field.

$$G(p) = \frac{i}{p^2-m^2+i\varepsilon} + \frac{i}{p^2-m^2+i\varepsilon}\,\Sigma\,\frac{i}{p^2-m^2+i\varepsilon} + \cdots$$

$$= \frac{i}{p^2-m^2-i\Sigma(p)+i\varepsilon}\ .$$

In the case of a Dirac spinor field.

note that $(\not p+m)(\not p-m) = p^2-m^2$. so $\dfrac{i(\not p+m)}{p^2-m^2+i\varepsilon} = \dfrac{i}{\not p-m+i\varepsilon}$

$$G(p) = \frac{i}{\not p-m+i\varepsilon} + \frac{i}{\not p-m+i\varepsilon}\,\Sigma(p)\,\frac{i}{\not p-m+i\varepsilon} + \cdots$$

$$= \frac{i}{\not p-m-i\Sigma(p)+i\varepsilon}\ .$$

Remember that what we regard (perceive) as particles

are the poles in two-point functions. (incl. hadrons.)
(note: field redef. allowed in path integral.)

$$\int \langle \Omega | T\{\phi(y-x)\,\phi^\dagger(0)\} | \Omega \rangle e^{ip\cdot(y-x)} d^4(y-x) = \frac{i}{p^2 - m^2 - i\Sigma(p) + i\varepsilon}$$

$$\int \langle \Omega | T\{\Psi(y-x)\,\bar{\Psi}(0)\} | \Omega \rangle e^{ip\cdot(y-x)} d^4(y-x) = \frac{i}{\not{p} - m - i\Sigma(p) + i\varepsilon}$$

☆ The particle mass (:= the location of the pole in p^2-plane)
may be shifted from the "mass parameter" in the Lagrangian.
(= free theory mass)
because of the interaction.

☆ $\Sigma(p)$: the sum of 1PI two-point graphs
is also called the self-energy (shift in mass (energy) due to the interaction with itself)

☆ What we see as a particle (a pole) is NOT an elementary field
with which we write a Lagrangian with, but it is
a part of $T\{\Psi \exp(i\int^{t_x} d^4z\, \mathcal{L}_{int})\}|\Omega\rangle$

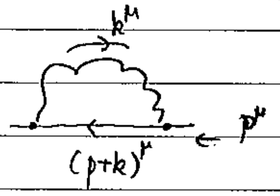
a linear combination of various states of multiple interacting fields.

"a cloud."

☆ Do we still have consistency if
"the correct mass" is different from "what we use to do computations"?

⇛ We think about in in this TEP course.

§1.2 Evaluation of 1-Loop Self-energy



$$(-ieQ)\left(A_\kappa \bar{\Psi}\gamma^\kappa \Psi\right)_{(z_2)}\ (-ieQ)\left(\bar{\Psi}\gamma^\lambda\Psi A_\lambda\right)_{(z_1)}$$

$$\Delta\Sigma(p) = \int\frac{d^4k}{(2\pi)^4}\left[(-ieQ\gamma^\kappa)\frac{i[(\not p+\not k)+m]}{(p+k)^2-m^2+i\varepsilon}(-ieQ\gamma^\lambda)\right]\frac{-i\eta_{\kappa\lambda}}{(k^2+i\varepsilon)}$$

4×4 matrix.

$$=: -i\left[A^{(1)}(p,m)\,\not p + B^{(1)}(p,m)\right]$$

So,

$$\left(A^{(1)}\not p + B^{(1)}\right) = -i(eQ)^2\int\frac{d^4k}{(2\pi)^4}\frac{\left\{\gamma^\kappa[(\not p+\not k)+m]\gamma_\kappa = -2(\not p+\not k)+4m\right\}}{[(p+k)^2-m^2+i\varepsilon][k^2+i\varepsilon]}$$

use
$$\begin{cases}\gamma^\kappa\gamma_\kappa = \frac{1}{2}\{\gamma^\kappa,\gamma^\lambda\}\eta_{\kappa\lambda} = \eta^{\kappa\lambda}\eta_{\kappa\lambda}\mathbb{1}_{4\times4} = 4\,\mathbb{1}_{4\times4} \\ \gamma^\kappa\gamma^\nu\gamma_\kappa = \gamma^\kappa\{\gamma^\nu,\gamma_\kappa\} - \gamma^\kappa\gamma_\kappa\gamma^\nu \\ \qquad = \gamma^\kappa 2\delta^\nu{}_\kappa - 4\gamma^\nu = -2\gamma^\nu.\end{cases}$$

Calculation trick ① ~~(see also QFT II Week 08~09)~~

$$\frac{1}{\alpha\beta} = \int_0^1 dx\,\frac{1}{[x\alpha+(1-x)\beta]^2}$$

$$\left(\text{proof. } \text{RHS} = \int_0^1 dx\,\frac{1}{(\alpha-\beta)^2}\frac{1}{\left(x+\frac{\beta}{\alpha-\beta}\right)^2} = \frac{1}{(\alpha-\beta)^2}\left(\frac{\alpha-\beta}{\beta}-\frac{\alpha-\beta}{\alpha}\right) = \text{LHS}.\right)$$

$$\frac{1}{\alpha\beta\gamma} = \int_{\triangle} dx\,dy\,\frac{2}{[x\alpha+y\beta+(1-x-y)\gamma]^3}$$

$$\left(A^{(1)}\not p + B^{(1)}\right) = -i(eQ)^2\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\,\frac{-2(\not p+\not k)+4m}{[x(p+k)^2-xm^2+(1-x)k^2+i\varepsilon]^2}$$

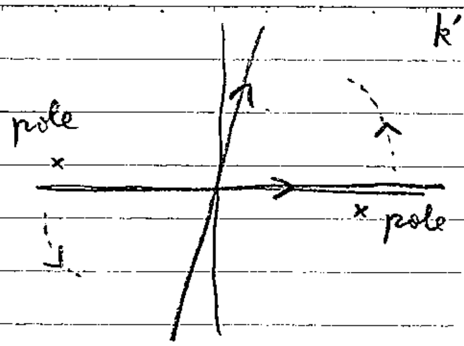
~~The denominator is~~ $k^2+2xp\cdot k+xp^2-xm^2+i\varepsilon = (\underbrace{k+xp}_{=k'})^2+(x-x^2)p^2-xm^2+i\varepsilon$

$$= -i(eQ)^2\int\frac{d^4k'}{(2\pi)^4}\int_0^1 dx\,\frac{-2(\not k'+(1-x)\not p)+4m}{[k'^2+x(1-x)p^2-xm^2+i\varepsilon]^2}.$$

The denominator is even in $k'\leftrightarrow -k'$ ⇒ Drop k' linear term in the numerator.

Calculation trick ②

$$\int_{-\infty}^{+\infty}\frac{dk'^0}{2\pi}\frac{1}{\left[(k'^0)^2-|\vec{k}'|^2+(\text{something})+i\varepsilon\right]^{\nu}}$$



OK to rotate the integrat'n contour (if $\nu \geq 1$)

$$\int_{-\infty}^{+\infty}dk'^0 \Rightarrow \int_{-i\infty}^{+i\infty}dk'^0 = i\int_{-\infty}^{+\infty}dk^4 \qquad (k^0 = ik^4)$$

$$\left[A\not{p}+B\right]^{(1)} = \underbrace{-i(eQ)^2\times i}_{=+(eQ)^2}\times\int_0^1 dx\int\frac{d^4k_E}{(2\pi)^4}\frac{-2(1-x)\not{p}+4m}{\left[-k_E^2-xm^2+x(1-x)p^2\right]^2}$$

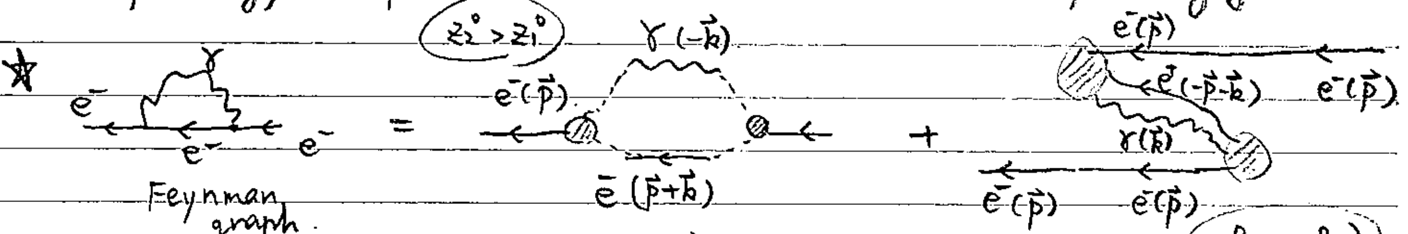
$k_E^2 := |\vec{k}|^2+(k^4)^2$ Euclidean signature

From the region $|k_E|^2 \gg m^2, |p|^2$, this integral has large contribut'n. $d|k_E|\, k_E^3/|k_E|^4 \sim$ log divergence.

Remember

☆ self-energy ⇒ perturbative correction to the mass/energy.

☆



$(e^-+\gamma)$ intermediate state ⇑ (off-shell) virtual states allowed in QM. ⟹ $(e^-e^+\gamma e^-)$ intermediate state

we just take a sum over $\vec{k}$.

Highly virtual (off-shell) intermediate states contribute less.

☆ UV divergence : there are too many DOF's to sum over.

QFT = QM w/ infinitely many DOF's.

☆ Log divergence of the QED e^- self-energy : benign than the linear divergence expected in the classical E&M. (or in the non-rela QED.)