§1.3 Regularization

UV divergence is from (many) high-energy states.

But we do not know for sure whether they are really "there".

- the Standard model : not sure beyond TeV
  certainly not above $M_{pl} \sim 2\times 10^{18}$ GeV.
- QED : not above 80 GeV ($m_{W^\pm}$)
- $\pi^\pm$, N : not above 700 MeV ($\rho^\pm$, ρ^0)
- condensed matter system : not $|\vec{k}| > (1/a_B)$
  Brillouin zone boundary
  (lattice spacing
  not continuous)

$\boxed{\text{Do we need to know the right UV modification to those theories??}}$

No! $\boxed{\text{regularization + renormalization}}$

Modify QED Lagrangian at high-energy

eg. $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{4\Lambda^2} F_{\mu\nu} D^\kappa D_\kappa F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$

- higher covariant derivative
- Pauli-Villars
- string theory .....

their combination ....

Now

$$[\not{p}A + B]^{(1)} \Rightarrow -i(eQ)^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\kappa[(\not{p}+\not{k})+m]\gamma_\kappa}{[(p+k)^2 - m^2 + i\varepsilon]\left[k^2 - \frac{k^4}{\Lambda^2} + i\varepsilon\right]}$$

$\rightarrow (-k^2)(k^2-\Lambda^2)/\Lambda^2$

$$= -i(eQ)^2 \int \frac{d^4k}{(2\pi)^4} \int_\triangle dx\,dy \frac{[-2(\not{p}+\not{k})+4m]\quad(-2\Lambda^2)}{[k^2 + 2x k\cdot p - x m^2 - y\Lambda^2 + x p^2 + i\varepsilon]^3}$$

$$= -i(eQ)^2 \int_\triangle dx\,dy \int \frac{d^4k'}{(2\pi)^4} \frac{[-2(1-x)\not{p}+4m](-2\Lambda^2)}{[(k')^2 - x m^2 - y\Lambda^2 + x(1-x)p^2 + i\varepsilon]^3}$$

Wick rotation : then.

$$\left[\not{p}A+B\right]^{(1)} = -i(eQ)^2\int_\Delta dx\,dy\; i\int\frac{d^4k_E}{(2\pi)^4}\frac{(-2\Lambda^2)}{(-)}\frac{\left[-2(1-x)\not{p}+4m\right]}{\left[k_E^2+xm^2+y\Lambda^2-x(1-x)p^2\right]^3}$$

$$= \frac{(eQ)^2}{(2\pi)^4}\left[2\pi^2 = vol(S^3)\right]\frac{1}{2}\int_0^{+\infty} d(k_E^2)(k_E^2)\,(2\Lambda^2)\frac{\left[-2(1-x)\not{p}+4m\right]}{\left[(k_E^2)+xm^2+y\Lambda^2-x(1-x)p^2\right]^3}$$

$$= \frac{(eQ)^2}{16\pi^2}\int_0^{+\infty}\frac{dK\,K}{\left[K+xm^2+y\Lambda^2-x(1-x)p^2\right]^3}(2\Lambda^2)\left[-2(1-x)\not{p}+4m\right]$$

Now $\displaystyle\int_0^{+\infty}\frac{dK\,K}{\left[K+(xxx)\right]^3} = \frac{1}{(xxx)}\int_0^{+\infty}dk\frac{k+1-1}{(k+1)^3} = \frac{1}{(xxx)}\left(1-\frac{1}{2}\right) = \frac{1}{2(xxx)}$ converges

$$= \frac{\alpha Q^2}{4\pi}\int_0^1 dx\int_0^{(1-x)} dy\frac{\Lambda^2\left[-2(1-x)\not{p}+4m\right]}{\left[xm^2+y\Lambda^2-x(1-x)p^2\right]}$$

y-integral then

$$= \frac{\alpha Q^2}{4\pi}\int_0^1 dx\left[-2(1-x)\not{p}+4m\right]\ln\left(\frac{(1-x)\Lambda^2+xm^2-x(1-x)p^2}{xm^2-x(1-x)p^2}\right)$$

We found that

$$\boxed{\begin{aligned} A^{(1)}(p^2,m^2) &= \frac{\alpha Q^2}{4\pi}\int_0^1 dx\; -2(1-x)\ln\left(\frac{(1-x)\Lambda^2+xm^2-x(1-x)p^2}{xm^2-x(1-x)p^2}\right)\\ B^{(1)}(p^2,m^2) &= \frac{\alpha Q^2}{4\pi}\int_0^1 dx\; 4m\;\ln\left(\frac{(1-x)\Lambda^2+xm^2-x(1-x)p^2}{xm^2-x(1-x)p^2}\right)\end{aligned}}$$

A and B remain finite so long as Λ is finite (not infinite)

The UV divergence is under control.

(regularization)

[Regularization methods may be different from the true mechanism adopted by the nature in rendering amplitudes UV finite.

This is not a problem (as we will see later), because the choice of regularization method does not matter after completing the procedure of renormalization.]

§2 Renormalized Perturbation Theory

§2.1 Idea

We have seen that

$$\begin{cases} \bullet \displaystyle\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\, e^{ip\cdot x}\, e^{-iq\cdot y}\, \langle 0|T\{\Psi_I(x)\,\bar{\Psi}_I(y)\exp\left[i\int d^4z\,\mathcal{L}_{int,I}(z)\right]\}|0\rangle_{\text{connected}} \\ \quad = (2\pi)^4\delta^4(p-q)\,\dfrac{i}{[\not p - m - (A\not p + B) + i\varepsilon]} =: (2\pi)^4\delta^4(p-q)\left(\dfrac{i\not p\cdot Z_2}{p^2-m_{pole}^2+i\varepsilon} + \cdots\right) \\ \bullet\ \Psi_I(x) = \displaystyle\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\left(u(\vec p)_s\, \underline{\underline{e^{-ip\cdot x}}}\, \underset{(\text{free})}{a_{\vec p,s}} + v(\vec p)_r\, \underline{\underline{e^{ip\cdot x}}}\, \underset{(\text{free})}{b^\dagger_{\vec p,s}}\right) \end{cases}$$

$\Rightarrow$ Renormalized fields $\frac{1}{\sqrt{Z_2}}\Psi_I(x) =: [\Psi_I]_r(x)$ and $\frac{1}{\sqrt{Z_2}}\bar{\Psi}_I = \overline{[\Psi_I]}_r$

have the right normalization.

also a theory parameter "m" should be chosen

so that the "m_{pole}" $\left(\begin{array}{l}(1-A(p^2,m^2))p^2-(m+B(p^2,m^2))^2=0 \\ \qquad\text{has a solution } p^2 = m_{pole}^2\end{array}\right)$

has the right value.

QED action

$$\mathcal{L} = \bar{\Psi}\left[i\gamma^\mu(\partial_\mu + ieQA_\mu) - m\right]\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

(For now, keep e and A_μ untouched $\Rightarrow$ allowed in QED)

PLUS

$$\mathcal{L}_{free} = \bar{\Psi}\left[i\gamma^\mu \partial_\mu - m\right]\Psi$$

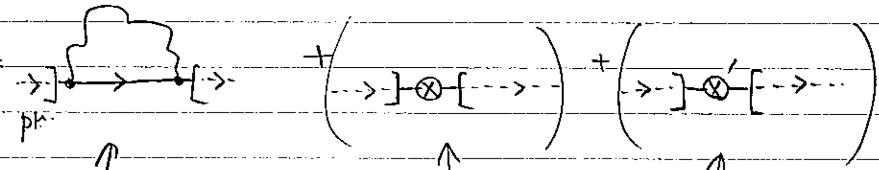
$$= Z_2 \bar{\Psi}_r\left[i\gamma^\mu\partial_\mu - m\right]\Psi_r$$

$$=: (1+\delta_{Z2})\bar{\Psi}_r\left[i\gamma^\mu\partial_\mu - (m_{pole}+\delta m)\right]\Psi_r$$

$$= \underbrace{\bar{\Psi}_r\left[i\gamma^\mu\partial_\mu - m_{pole}\right]\Psi_r}_{= \mathcal{L}_0} + \underbrace{\delta_{Z2}\bar{\Psi}_r i\gamma^\mu\partial_\mu\Psi_r - \left\{\delta_{Z2}m_{pole} + (1+\delta_{Z2})\delta m\right\}\bar{\Psi}_r\Psi_r}_{\text{treat as a part of } \mathcal{L}_{int}}$$

(along with $(1+\delta_{Z2})\bar{\Psi}_r\gamma^\mu\Psi_r A_\mu(-eQ)$)

Then $-i(A\not{p}+B)^{\text{1-loop}} =$



1-Loop $\Rightarrow \mathcal{O}(e^2)$ def. now

use $(-ieQ\gamma^\mu)$ rather than $(-i(1+\delta_{Z2})eQ\gamma^\mu)$

$i\delta_{Z2}\gamma^\mu p_\mu$

$-i\left\{\delta_{Z2}m_{pole} + \delta m\right\}$

We know already that $\delta_{Z2} \sim \mathcal{O}(e^2)$

To be more explicit,

$$\begin{cases} A^{(1)}(p^2, m^2) = \dfrac{\alpha Q^2}{4\pi}\displaystyle\int_0^1 dx \; -2(1-x)\ln\left(\dfrac{(1-x)\Lambda^2 + x m^2 - x(1-x)p^2}{+x m^2 - x(1-x)p^2}\right) \\ B^{(1)}(p^2, m^2) = \dfrac{\alpha Q^2}{4\pi}\displaystyle\int_0^1 dx \quad 4m \quad \ln\left(\dfrac{(1-x)\Lambda^2 + x m^2 - x(1-x)p^2}{+x m^2 - x(1-x)p^2}\right) \end{cases}$$

☆ $p^2 = m_{pole}^2$ must be a solution to

$$\boxed{\left(1 - A^{(1)}(p^2, m^2)\right)^2 p^2 = \left(m + B^{(1)}(p^2, m^2)\right)^2.}$$

Keeping terms of $\mathcal{O}(1)$ and $\mathcal{O}(\alpha)$ but not $\mathcal{O}(\alpha^2)$,

$$\boxed{m_{pole}^2 = m^2 + \left\{2m B^{(1)}(m^2, m^2) + 2m^2 A^{(1)}(m^2, m^2)\right\} + \mathcal{O}(\alpha^2)}$$

$$\boxed{m^2 = m_{pole}^2 - \left\{2m_{pole} B^{(1)}(m_{pole}^2, m_{pole}^2) + 2m_{pole}^2 A^{(1)}(m_{pole}^2, m_{pole}^2)\right\} + \mathcal{O}(\alpha^2)}$$

- When we keep track of the difference $(m^2 - m_{pole}^2)$ of order α, we can use m^2 and m_{pole}^2 interchangablly for the arguments of $A^{(1)}$ and $B^{(1)}$.
- $\begin{cases} m_{pole}^2 \text{ is determined by } m^2, \Lambda^2 \text{ and } e^2 \\ m^2 \text{ is determined by } m_{pole}^2, \Lambda^2 \text{ and } e^2 \end{cases}$

$\Rightarrow m = m_{pole} + (\delta m)$ expressed in terms of $m_{pole}^2, \Lambda^2, e^2$.

☆ The residue at the pole is determined by

$$\frac{\left(1 - A(p^2, m^2)\right)\cancel{p} + \cdots}{\left(1 - A(p^2, m^2)\right)^2 p^2 - \left(m + B(p^2, m^2)\right)^2} \overset{\text{residue}}{\Longrightarrow} \frac{(1 - A^{(1)})\Big|}{(1 - A^{(1)})^2\Big| - 2\dfrac{\partial A^{(1)}}{\partial p^2}\Big|(m_{pole}^2) - 2m\dfrac{\partial B^{(1)}}{\partial p^2}\Big|}$$

$$Z_2^{(1)} = \left(1 + \delta_{z2}^{(1)}\right) = 1 + A^{(1)}(m_{pole}^2, m_{pole}^2) + 2m_{pole}^2\left(\frac{\partial A^{(1)}}{\partial p^2}\right)(m_{pole}^2, m_{pole}^2) + 2m_{pole}\frac{\partial B^{(1)}}{\partial p^2}(m_{pole}^2, m_{pole}^2)$$

We can use m^2 and m_{pole}^2 interchangablly in the arguments of $A^{(1)}$, $B^{(1)}$ etc., because we ignore $\mathcal{O}(\alpha^2)$ terms in δ_{z2} in this calculation.

Then at $\mathcal{O}(\alpha)$, the 2-point function (correlator) becomes

$$\frac{i}{(\not{p} - m_{pole} + i\varepsilon) - \left[\not{p} A^{(1)}(p^2, m_{pole}) + B^{(1)}(p^2, m_{pole})\right] + \delta_{Z2}(m_{pole}, \Lambda, e)\not{p} - (\delta_{Z2} \cdot m_{pole} + \delta m)}$$

The coefficients of $\not{p}$ in the denominator : $\left[1 - A^{(1)}(p^2, m_{pole}^2) + \delta_{Z2}(m_{pole}, \Lambda, e)\right]$

$$- A^{(1)}(p^2, m_p^2) + A^{(1)}(m_p^2, m_p^2) = \frac{\alpha Q^2}{4\pi}\int_0^1 dx \; 2(1-x)\left\{\ln\left(\frac{(1-x)\Lambda^2 + x m_p^2 - x(1-x)p^2}{+x m_p^2 - x(1-x)p^2}\right) - \ln\left(\frac{(1-x)\Lambda^2 + x m_p^2 - x(1-x)m_p^2}{+x m_p^2 - x(1-x)m_p^2}\right)\right\}$$

$\rightarrow$ $\boxed{\text{finite in the } \Lambda \to +\infty \text{ limit}}$

The quantum corrected correlator remains finite
when expressed in terms of observed (physical) parameter m_{pole}.

✓ The resulting expression does not depend on the choice of
the regularization method ....

(strictly speaking, this is a desire/crietrion than a theorem.)

insensitive to the high-energy theory.

⟨ "divergence from infinitely many UV DOFs
has been swept under the carpet (rug)"
(in "m_{pole}") ⟩