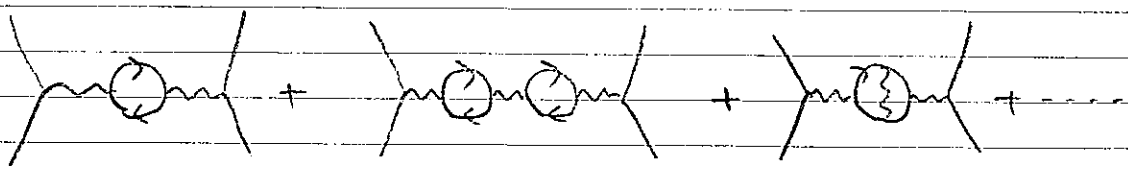
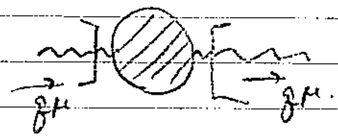
§2.2 QED in Renormalized Perturbation Theory

beyond electron self-energy



Σ(1 particle irreducible graphs) = 

$$=: i\left(q^2\eta_{\mu\nu} - q_\mu q_\nu\right)\Pi(q^2)$$

(~~non-trivial that the $q^2\eta_{\mu\nu}$ term and the $q_\mu q_\nu$ term~~ are the same except sign. $\Leftarrow$ consequence of gauge sym)

Then. (in the Feynman gauge)

$$\frac{-i\eta_{\mu\nu}}{(q^2+i\varepsilon)} + \frac{-i\delta_\mu{}^{\kappa_1}}{q^2}\, i\Pi(q^2)\left(q^2\eta_{\kappa_1\kappa_2} - q_{\kappa_1}q_{\kappa_2}\right)\frac{-i\delta^{\kappa_2}{}_\nu}{q^2}$$

$$+\frac{-i\delta_\mu{}^{\kappa_1}}{q^2}\, i\Pi\left(q^2\eta_{\kappa_1\kappa_2} - q_{\kappa_1}q_{\kappa_2}\right)\frac{-i\eta^{\kappa_2\kappa_3}}{q^2}\, i\Pi\left(q^2\eta_{\kappa_3\kappa_4} - q_{\kappa_3}q_{\kappa_4}\right)\frac{-i\delta^{\kappa_4}{}_\nu}{q^2}$$

$$+ \cdots$$

$$= \frac{-i\eta_{\mu\nu}}{(q^2+i\varepsilon)(1-\Pi(q^2))}.$$ (the geometric series is summed up.) hw III-3

~~At 1-loop level, just one graph contributes:~~ .

~~Using the Pauli-Villars regularization~~ (in unrenormalized fields & couplings)

$$\boxed{\Pi^{(1)}(q^2) = \frac{(eQ)^2}{2\pi^2}\int_0^1 dx\; x(1-x)\,\ln\left(\frac{m_p^2 - x(1-x)q^2}{M_{reg}^2}\right)}$$ $\Leftarrow$ ~~homework III-4~~

- ~~divergence under control (if $M_{reg}^2 < \infty$)~~
- no mass shift; the pole remains @ $q^2=0$.
- only the normalization is affected.

- residue $\dfrac{-i\eta_{\mu\nu}Z_3}{q^2+i\varepsilon}$ @ $q^2=0$ $\Rightarrow$ $Z_3 = \dfrac{1}{1-\Pi(q^2)\big|_{q^2=0}} \simeq 1+\Pi^{(1)}(q^2)\big|_{q^2=0}$

$\Rightarrow$ The renormalized field $[A_\mu]_r := \dfrac{1}{\sqrt{Z_3}} A_\mu$

has the properly normalized two point correlation fcn.

$$\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \Rightarrow -\frac{1}{4}Z_3[F_{\mu\nu}]_r[F^{\mu\nu}]_r \cong \underbrace{-\frac{1}{4}(F_{\mu\nu}F^{\mu\nu})_r}_{\Downarrow\ \mathcal{L}_0} \underbrace{-\frac{1}{4}\delta_{23}^{(1)}(F_{\mu\nu}F^{\mu\nu})_r}_{\Downarrow\ \mathcal{L}_{int}}$$

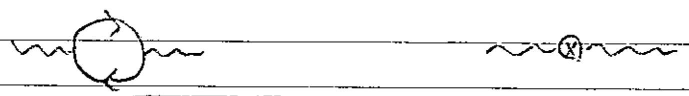
Feynman rule : $-i(q^2\eta_{\mu\nu}-q_\mu q_\nu)\delta_{23}^{(1)}$

$= -i(q^2\eta_{\mu\nu}-q_\mu q_\nu)\Pi^{(1)}(q^2=0)$.

So, at the 1-loop level,

the 1PI graphs for the renormalized photon field is sum of

$$i(q^2\eta_{\mu\nu}-q_\mu q_\nu)\Pi^{(1)}(q^2) \quad + \quad -i(q^2\eta_{\mu\nu}-q_\mu q_\nu)\Pi^{(1)}(0)$$



$$\Rightarrow \boxed{\Pi^{(1)}(q^2)-\Pi^{(1)}(0) = \frac{(eQ)^2}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left(\frac{m_f^2-x(1-x)q^2}{m_f^2}\right)}$$

$\qquad \| \qquad\qquad\qquad\qquad\qquad\qquad \overline{M}_{reg}^2$: disappeared.

$\Pi^{(1)}_{ren}(q^2)$

$\Pi(q^2)$ or $\Pi_{ren}(q^2)$ is called vacuum polarization.

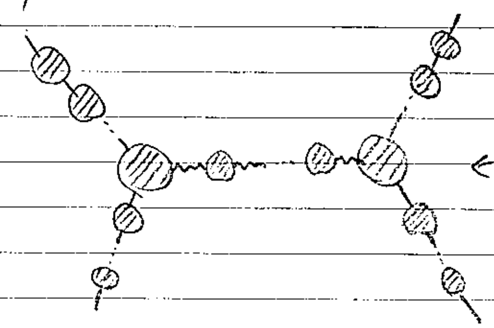
(memo: $\dfrac{Qe}{|\vec{q}|^2} \xrightarrow{\text{Fourier transform}} \dfrac{Qe}{4\pi}\dfrac{1}{r}$ Coulomb potential

$\dfrac{Qe}{(1-\Pi)|\vec{q}|^2} \Rightarrow \dfrac{Qe}{4\pi(1-\Pi)}\dfrac{1}{r}$

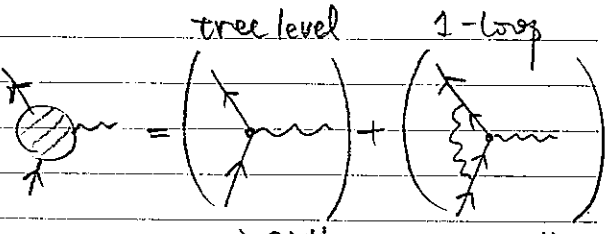
$\qquad\qquad (\varepsilon = 1+\delta\varepsilon)$ $\vec{P}=\delta\varepsilon\vec{E}$.

polarizatn.)

Finally, $\mathcal{L}_{QED} \supset -eQ(\bar{\Psi}\gamma^\mu A_\mu \Psi)$.



1PI 3pt.



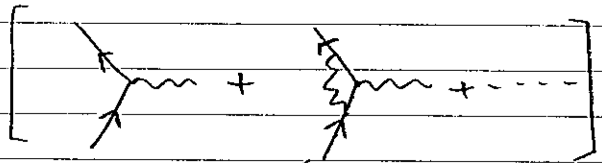
divergent

(finite log if Λ remains finite)

<u>mass</u>

the pole of the quantum corrected 2-point function $\Rightarrow$ observed mass $M \neq m_p$

<u>coupling strength</u>



the coupling strength of the cloud of $e^- + e^-(e^+e^-) + e^-(e^+e^-)^2 + \cdots$

$\Longrightarrow$ measured strength $e_r \neq e$

$$-ieQ\Gamma^\mu_{(1)} =: -ieQ\left(\gamma^\mu V_1^{(1)}(q^2) + [\gamma^\mu, \gamma^\nu] q_\nu V_2^{(1)}(q^2) + \cdots\right)$$

~~Then the measured coupling strength should be ....~~

the three point function $\Psi_r - \bar{\Psi}_r - (A_\mu)_r$

$$= \underset{\Psi_r, \bar{\Psi}_r, A_r}{\Pi}\left[\text{quantum corrected propagator}\right] \times \underbrace{Z_2\sqrt{Z_3} \times \left(-ieQ\gamma^\mu + (-ieQV_1^{(1)})\gamma^\mu + \cdots\right)}_{\parallel}$$

$$(-iQ\gamma^\mu)\left\{e + \left(eV_1^{(1)} + e\delta_{Z_2}^{(1)} + \tfrac{1}{2}e\delta_{Z_3}^{(1)}\right) + \cdots\right\}$$

So, $\boxed{e + \left(eV_1^{(1)}(q^2) + e\delta_{Z_2}^{(1)} + \tfrac{1}{2}e\delta_{Z_3}^{(1)}\right) + \mathcal{O}(e^5) = e_r(q^2)}$ $\Leftarrow$ (eg. use the one for $q^2 = 0$)

just like $\boxed{M^2 + \left(2M^2 A^{(1)}\big|_{p^2=M^2} + 2M B^{(1)}\big|_{p^2=M^2}\right) + \mathcal{O}(e^4) = m^2_{pole}}$

Lagrangian parameters $\longleftrightarrow$ observed parameters

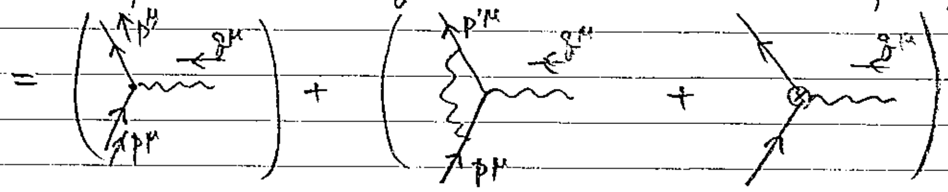
Renormalized perturbation theory for QED :

use $\{\Psi_r,\ A_\mu^{(r)},\ m_{pole},\ e_r\}$ for perturbation

$$\mathcal{L}_{QED} = \bar{\Psi}_r\left[i\gamma^\mu(\partial_\mu + ie_rQA_{\mu,r}) - m_p\right]\Psi_r - \frac{1}{4}F^{(r)}_{\mu\nu}F^{(r)\mu\nu} \Rightarrow \mathcal{L}_0$$

$$\left.\begin{aligned} &+\delta_{Z2}\,\bar{\Psi}_r\, i\gamma^\mu\partial_\mu\Psi_r \qquad -\frac{1}{4}\delta_{Z3}F^{(r)}_{\mu\nu}F^{(r)\mu\nu} \\ &-(Z_2M - m_p)\bar{\Psi}_r\Psi_r \\ &-Q(Z_2\sqrt{Z_3}\,e - e_r)(\bar{\Psi}_r\gamma^\mu A^{(r)}_\mu\Psi_r) \end{aligned}\right\} \Rightarrow \mathcal{L}_{int}$$

At 1-loop level, the quantum corrected 3pt fcn /(propagators)



$-ie_rQ\gamma^\mu \qquad -ie_rQ\gamma^\mu V_2^{(1)}(q^2) + \cdots \qquad -iQ\{e - e_r + e_r(\delta_{Z2}^{(1)} + \frac{1}{2}\delta_{Z3}^{(1)})\}$

$$= -iQ\gamma^\mu\Big[e_r + e_r\big(V_1^{(1)}(q^2) - V_1^{(1)}(0)\big) + \underbrace{\{e - e_r + e_r(V_1^{(1)}(0) + \delta_{Z2}^{(1)} + \frac{1}{2}\delta_{Z3}^{(1)})\}}_{\text{inserted}} + \mathcal{O}(e^5)\Big].$$

$\hookrightarrow \mathcal{O}(e^5)$

$\left(+[\gamma^\mu, \gamma^\nu]\text{ term}\right)$

~~Using the result~~

Peskin-Schroeder §6.3 (or QFT II week 8-9)

$$\Gamma^\mu_{(1)} = \frac{(eQ)^2}{16\pi^2}\int_\Delta dx\,dy\left[\gamma^\mu\cdot 2\left\{\ln\left(\frac{(1-w)\Lambda^2 + w^2M^2 - xyq^2}{w^2M^2 - xyq^2}\right) + \frac{M^2\{1 - 4(1-w) + (1-w)^2\} + (1-x)(1-y)q^2}{w^2M^2 - xyq^2}\right\} - \frac{[\gamma^\mu,\gamma^\nu]q_\nu}{4M}\,\frac{4M^2(1-w)w}{w^2M^2 - xyq^2}\right],$$

$w := (x+y)$

~~we see that~~

$$V_1^{(1)}(q^2) - V_1^{(1)}(0) = \frac{\alpha Q^2}{2\pi}\int_\Delta dx\,dy\left[\ln\left(\frac{w^2m_p^2}{w^2m_p^2 - xyq^2}\right) + \frac{m_p^2\{1 - 4(1-w) + (1-w)^2\} + (1-x)(1-y)q^2}{w^2m_p^2 - xyq^2} - \frac{1 - 4(1-w) + (1-w)^2}{w^2}\right]$$

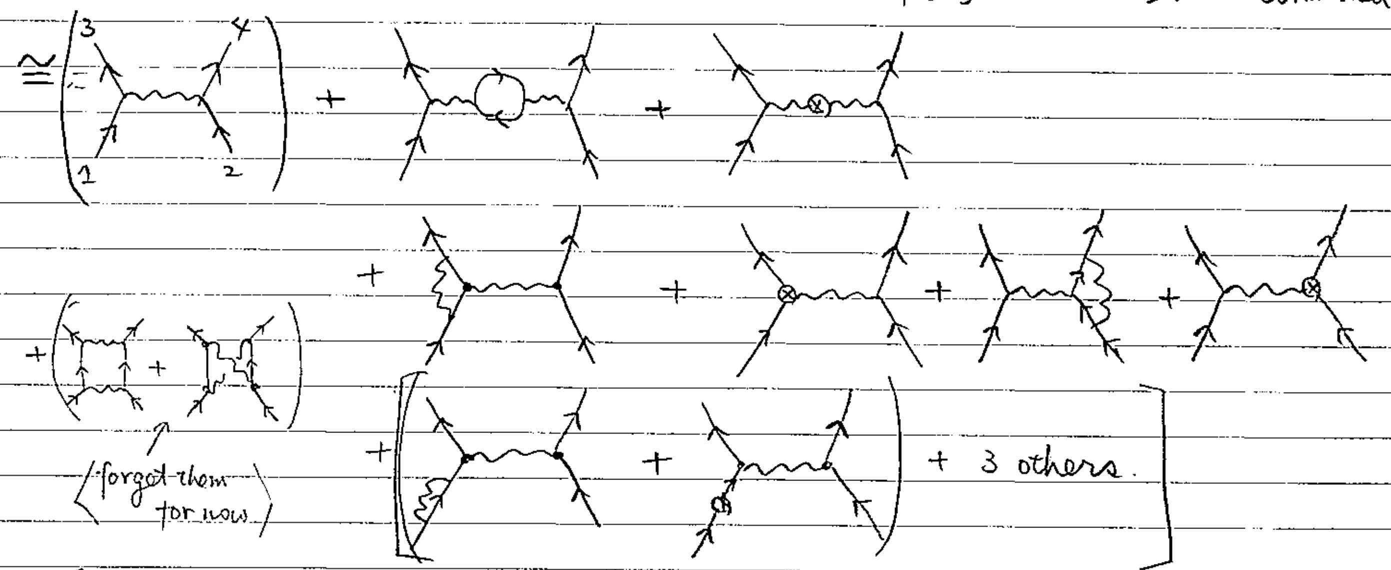
~~is free from the UV divergence (regulator scale Λ),~~

but with quantum corrections. (not without IR divergence from $x+y \simeq 0$) PLUS

$\boxed{\text{Recap}}$ At 1-loop level, the correlator of the renormalized fields

$$\int d^4x_3 \int d^4x_4 \, e^{ip_3\cdot x_3} e^{ip_4\cdot x_4} \int d^4x_1 \int d^4x_2 \, e^{-ip_1\cdot x_1} e^{-ip_2\cdot x_2}$$

$$\langle 0 | T\{ \Psi_{I,r}(x_4) \, \Psi_{I,r}(x_3) \, \bar{\Psi}_{I,r}(x_1) \, \bar{\Psi}_{I,r}(x_2) \exp\left[i \int d^4z \, \mathcal{L}_{int,I}(z) \right] \} | 0 \rangle_{\text{connected}}$$

$\cong$ 

$$= \Big[iQ^2 (e_r\gamma^\mu) \frac{\eta_{\mu\nu}}{q^2} (e_r\gamma^\nu) + iQ^2 (e_r\gamma^\mu) \frac{\eta_{\mu\nu}}{q^2} \left(\Pi^{(1)}(q^2) - \Pi^{(1)}(0) \right) (e_r\gamma^\nu)$$
$$+ iQ^2 \, e_r \left(\Gamma^\mu_{(1)} - V_1^{(1)}(0)\gamma^\mu \right) \frac{\eta_{\mu\nu}}{q^2} (e_r\gamma^\nu)$$
$$+ iQ^2 (e_r\gamma^\mu) \frac{\eta_{\mu\nu}}{q^2} \, e_r \left(\Gamma^\nu_{(1)} - V_1^{(1)}(0)\gamma^\nu \right) \Big]$$

multiplied by $(2\pi)^4 \delta^4(p_3+p_4-p_1-p_2)$ and quantum corrected propagators.

☆ Regularization parameters disappeared $(\Lambda, \bar{M}^2_{reg})$ when the correlator is expressed in terms of (e_r, m_{pole}) and we take the limit $\Lambda, \bar{M}^2_{reg} \to +\infty$.

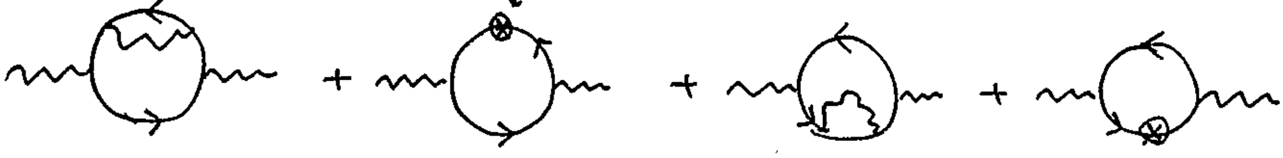
☆ Measurements look at "things" that include all sorts of quantum effects. Lagrangian parameters (e, M) are not the same as (e_r, m_{pole})

☆ The difference $(e - e_r)$ and $(M^2 - m^2_{pole})$ diverge if $\Lambda, \bar{M}^2_{reg}$ are literally ∞.

But the regulators are not more than an easy going version of some physical UV modification : the difference is finite. not scary.

$\boxed{\text{at 2-Loop}}$

$\delta_{22}^{(1)}, \delta_M^{(1)}$



$\cancel{\delta}$ $e - e_r + e_r\left(V_1^{(1)}(0) + \delta_{22}^{(1)} + \frac{1}{2}\delta_{23}^{(1)}\right)$

$\boxed{\cancel{\delta_{23}^{(2)}}}$

$\Uparrow$

$\boxed{-i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\delta_{23}^{(2)}}$

$$= i\left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\Pi^{(2)}_{ren}(q^2).$$

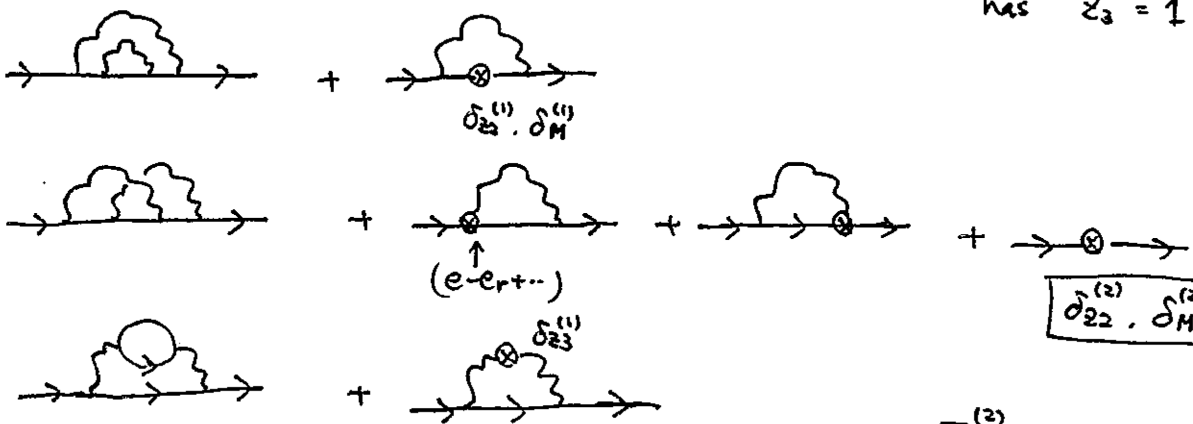
$\delta_{23}^{(2)}$: determined so that $\Pi^{(2)}_{ren}(q^2) = 0$

$\Updownarrow$

The relative normalization of A_μ and $[A_\mu]_r$ should be such.

$$\frac{-i\eta^{\mu\nu}}{[q^2+i\varepsilon]\left(1 - (\Pi^{(1)}_{ren} + \Pi^{(2)}_{ren})(\ldots\right.}$$

has "Z_3" $= 1$.



$\delta_{22}^{(1)}, \delta_M^{(1)}$

$(e - e_r + \cdots)$

$\boxed{\delta_{22}^{(2)}, \delta_M^{(2)}}$

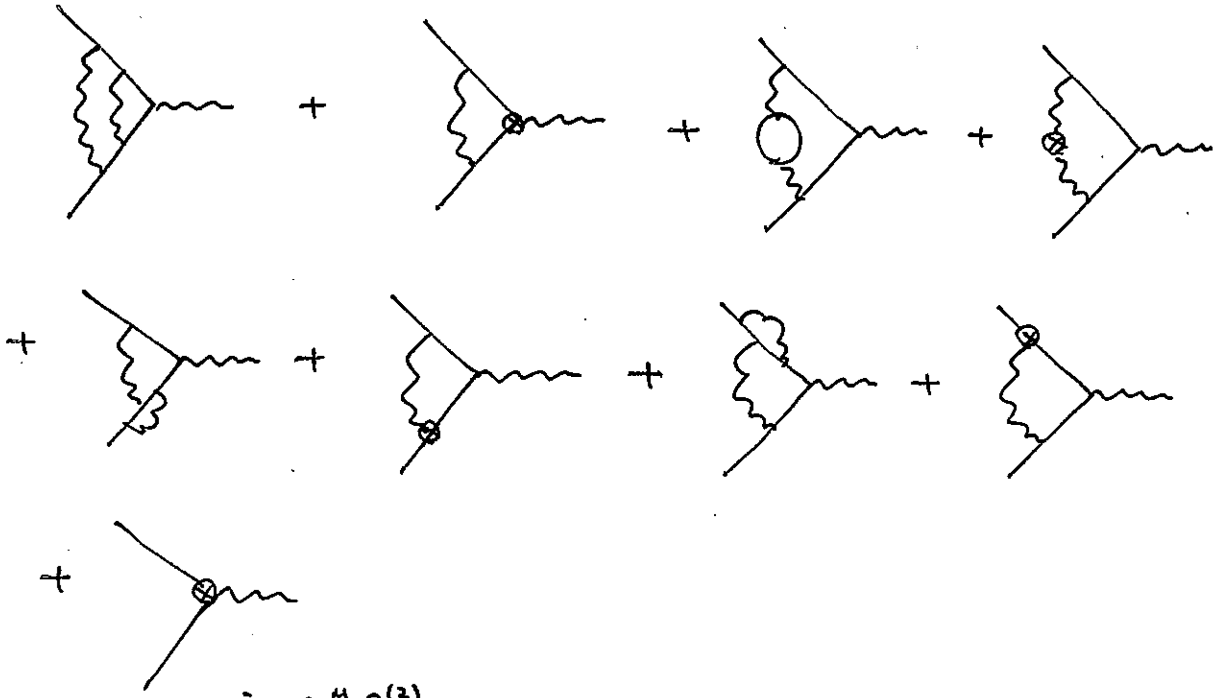
$\delta_{23}^{(1)}$

$$= -i\,\Sigma^{(2)}_{ren}(p, m)$$

$$= -i\left(\cancel{\Sigma^{(2)}} A^{(2)}_{ren}\not p + B^{(2)}_{ren}\right)$$

$\delta_{22}^{(2)}$, $\delta_M^{(2)}$ determined so that

- $\boxed{\left(1 - A^{(1)}_{ren} - A^{(2)}_{ren}\right)^2 m^2 = \left(m + B^{(1)}_{ren} + B^{(2)}_{ren}\right)^2\Big|_{p^2=m^2} \quad \text{mod } O(e^6)}$
- $\boxed{\left[\left(1 - A^{(1)}_{ren} - A^{(2)}_{ren}\right)^2 + 2\left(A^{(1)}_{ren} + A^{(2)}_{ren} - 1\right)\frac{\partial A^{(1)+(2)}_{ren}}{\partial p^2} - 2\left(m + B^{(1)+(2)}_{ren}\right)\frac{\partial B^{(1)+(2)}}{\partial p^2}\right]\Bigg|_{p^2=m^2} = \left(1 - A^{(1)+(2)}_{ren}\right)\Big|_{p^2=\ldots}}$



$-Qie_r\gamma^\mu \underline{\underline{\delta_{Z1}^{(2)}}}$

determine so that $F_1(q^2)\big|_{q^2=0} = 1$

$$-Qie_r\,\Gamma^\mu_{ren} = -Qie_r\left(F_1(q^2)\gamma^\mu - \frac{F_2(q^2)}{4m}[\gamma^\mu,\gamma^\nu]q_\nu\right.$$

$+ \bmod\ [\times(\not p - m)]$

$+ \bmod\ [(\not p - m)\times]$

F_1 : ~~Pauli~~ Dirac
F_2 : Pauli

$\boxed{\text{Renormalization conditions. (on-shell)}}$

- ✓ Ψ_r canonically normalized. ('Z_2' = 1)
- ✓ fermion pole mass = m.
- ✓ $A_{r\mu}$ canonically normalized ('Z_3' = 1)
- ✓ $F_1(q^2=0) = 1$

$\Rightarrow$ determine. $\underbrace{\delta_{Z2}^{(n)},\ \delta_{Z3}^{(n)},\ (M-m),\ (e-e_r)\ \ (\text{or } \delta_{Z1}^{(n)})}_{\Uparrow}$

order by order.

$\boxed{\text{counter terms.}}$