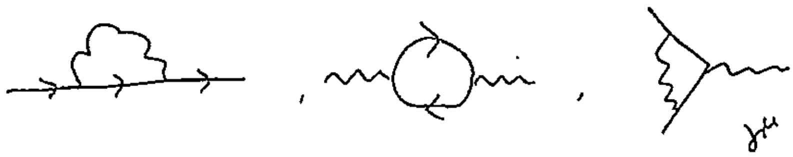
§ ~~5.3~~ 2.3 Superficial Degree of Divergence

In QED (at 1-loop level)

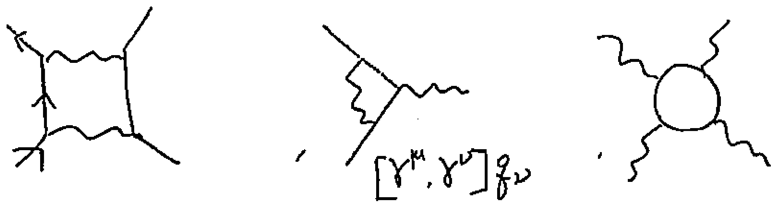
4 renormalization conditions.

- e^- (and e^+) (pole) mass. : m
- electric charge (at $q^\mu = 0$): e_r.
- Ψ_e field normalization : 1
- A_μ field : : 1.

to rewrite $(M, e; \Lambda)$.



divergence appear. (taken care of).



+ ... any other amplitudes with divergence?

How do we know?

In QED, we've done all possible "renormalization".

fermion propagator

$$\frac{i[\not{p}+M]}{[p^2-M^2+i\varepsilon]} \Rightarrow -1$$

scalar propagator

$$\frac{i}{[p^2-M^2+i\varepsilon]} \Rightarrow -2.$$

vector boson propagator (Feynman gauge)

$$\frac{-i\eta_{\mu\nu}}{(p^2+i\varepsilon)} \Rightarrow -2.$$

(scalar)^2 – vector vertex

$$\propto (p_1-p_2)^\mu \Rightarrow +1.$$

(fermion)^2 – vector vertex

$$\gamma^\mu \Rightarrow 0.$$

(vector)^3 vertex (non-Abelian)

$$\propto p^\mu \Rightarrow +1$$

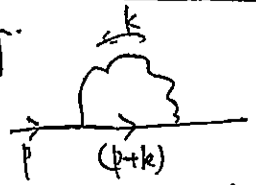
(vector)^4 vertex

$$\propto (\eta^{\mu\nu}\eta^{\rho\sigma} + \dots) \Rightarrow 0.$$

1-loop momentum $\dfrac{d^4 p}{(2\pi)^4} \Rightarrow +4$

$\boxed{\text{UV divergence : mass, external momenta don't matter.}}$

eg.

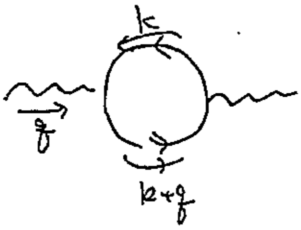


$\Rightarrow D = \underset{\gamma}{(-2)} + \underset{\psi}{(-1)} + 4 = +1$

truncate $\xrightarrow{\text{odd part.}}$ 0

$\int d^4k \; \gamma^\mu (\cancel{p} + \cancel{k}) \gamma_\mu$ $\Rightarrow$ use $\cancel{p}$ not $\cancel{k}$

Logarithmic div.



$\Rightarrow D = \overset{\psi}{(-1)} \times 2 + 4 = +2$ for $(\ \ )_{\mu\nu}$.

but gauge symmetry :

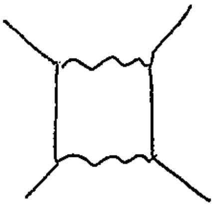
$(\ \ \ )_{\mu\nu} = i(q^2 \eta_{\mu\nu} - q_\mu q_\nu)\,\Pi(q^2)$

use q^2. or $q_\mu \cdot q_\nu$ not k^2 $\Rightarrow$ 0

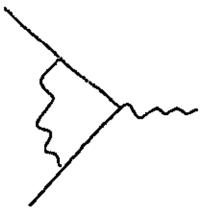
Logarithmic. div

$\Rightarrow D = 0$. Logarithmic.

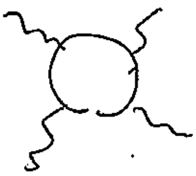
eg.



$\Rightarrow D = \overset{\psi}{(-1)} \times 2 + \overset{\gamma}{(-2)} \times 2 + 4 = -2$

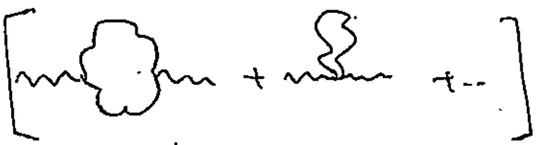


$\Rightarrow D = 0$ .. but. $[\gamma^\mu, \gamma^\nu]\, q_\nu$. $\Rightarrow$ -1 ..



$D = 0$. but eventually $\Rightarrow$ -4

no divergence



$\Rightarrow$ Logarithmic

$$D = 4\cdot(\#L) - I_\psi - 2I_\gamma$$

$$2\times(\#V) = 2I_\psi + E_\psi$$

$$(\#V) = 2I_\gamma + E_\gamma$$

$$D = 4\left[(\#C) + (\#I_{tot}) - (\#V)\right] - I_\psi - 2I_\gamma$$

$$= 4 + (3I_\psi + 2I_\gamma) - \cancel{\left[3(\#V) + (\#V)\right]}^{\,4(\#V)}$$

$$= \cancel{4 + (3I_\psi + 2I_\gamma) - \left[\left(3I_\psi + \tfrac{3}{2}E_\psi\right) + (2I_\gamma + E_\gamma)\right]}$$

$$= 4 + \left(3(\#V) - \tfrac{3}{2}E_\psi\right) + \left((\#V) - E_\gamma\right) - 4(\#V) = \boxed{4 - \tfrac{3}{2}E_\psi - E_\gamma}$$

for any higher loop amplitudes.

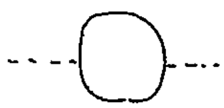
$D \le 4$, $E_\psi, E_\gamma > 0$: only finite amplitudes w/ $D > 0$.

All the $D \ge 0$ amplitudes exist in $\mathcal{L}_{QED}$. (w counter terms)

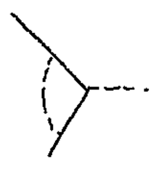
Yukawa theory

eg. $\mathcal{L} = \bar{\Psi}[i\gamma^\mu \partial_\mu - m]\Psi + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{M^2}{2}\phi^2 - g\bar{\Psi}\Psi\phi$. ϕ: real scalar.

$$\boxed{D = 4 - \tfrac{3}{2}E_\psi - E_\phi}$$

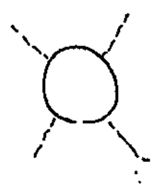


$D = 2$.
mass, wavefn ren. of ϕ.



$D = 0$
$g + g_r$.

but



$D = 0$.

need $\frac{\lambda}{4!}\phi^4$ term in $\mathcal{L}$.

otherwise, regulator scale remains.

(λ_r may just happen to be 0 though)

To recap

- particle species a : propagator p^{-k_a}
- vertex (interaction) type i : (field "a") $\times n_{ai}$ and $d_i \times (\partial_\mu)$

$$\boxed{D = 4L - \sum_a (k_a \cdot I_a) + \sum_i (d_i \cdot V_i)}$$

- for particle species "a" : $\boxed{(2I_a + E_a) = \sum_i n_{ai} V_i}$
- $\boxed{L - (C=1) = \left(\sum_a I_a\right) - \left(\sum_i V_i\right)}$

$$\Rightarrow D = 4 + \sum_a (4-k_a) I_a + \sum_i (d_i - 4) V_i$$

$$= 4 + \sum_i \Big(\underbrace{\sum_a \left(\tfrac{4-k_a}{2}\right) n_{ai} + d_i}_{\Delta_i} - 4 \Big) V_i - \sum_a \left(\tfrac{4-k_a}{2}\right) E_a .$$

- $\Delta_i = \sum_a \left(\frac{4-k_a}{2}\right) n_{ai} + d_i$ (naive) (eg) operator dim.

$\boxed{(4-\Delta_i) : \text{mass-dimension of the coefficient.}}$ of the vertex "i"

~~scalar~~ $\left(\frac{4-k_a}{2}\right)$: (naive) mass-dim. of field "a"

scalar : $\frac{i}{p^2-m^2+i\varepsilon}$ $\Rightarrow 1$

vector : $\frac{-i\eta_{\mu\nu}}{p^2+i\varepsilon}$ $\Rightarrow 1$. Ψ : $\frac{i(\not{p}+m)}{p^2-m^2} \Rightarrow 3/2$.

QED, (Yukawa + ϕ^4) theory. Both have only interactions where $\Delta_i = 4$ $(4-\Delta_i = 0)$

$\Rightarrow$ limited variety of amplitudes ($\{E_a\}$) where $D \geq 0$ (divergent).

$\boxed{\text{If all the interactions of a theory satisfy } \Delta_i \leq 4. \ (4-\Delta_i \geq 0).}$

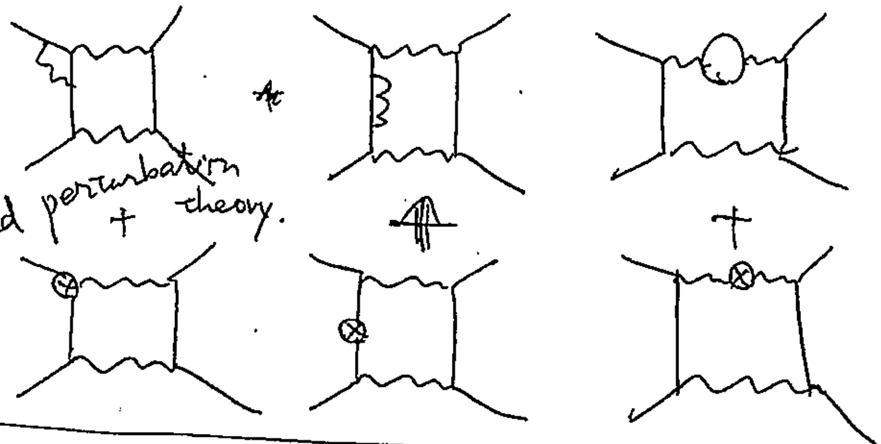
Renormalizable QFT

If all possibly divergent amplitudes have corresponding terms in $\mathcal{L}$ ~~so that~~ ($\Leftrightarrow$ $\exists$ counter terms), such amplitudes can be written in terms of observed (renormalized) coupling coefficients (and kinematical variables) w/o referring to the regulator scale.

subtlety 1

subdiagram may diverge even when $D<0$.

eg.



but in renormalized perturbation theory.

subtlety 2

$\exists$ counter term alone. is not enough.

$$\text{eg.}\quad \Lambda^2 \ln\left(\frac{\Lambda^2}{p^2}\right) - \underbrace{\Lambda^2 \ln\left(\frac{\Lambda^2}{M^2}\right)}_{\text{c.t.}} \Rightarrow \Lambda^2 \ln\left(\frac{M^2}{p^2}\right)$$

finite at the kinematics for the renormalization condition.

Bogoliubov – Parasiuk, Hepp, Zimmermann.

renormalizable

§ ~~5.4~~ 2.4 Renormalized Perturbation Theory of "Non-Renormalizable" Theories.

Historically

Renormalizable theories :

$\boxed{\text{Renormalizable QFT's. which need only a finite \# of renormalized coefficients.}}$

$$\boxed{D = 4 - \left(\frac{3}{2}E_\psi + E_\phi + E_A\right) + \sum_i (\Delta_i - 4) V_i}$$

Δ_i : (naive) operator dim.

✓ If $(\Delta_i - 4) > 0$ in one of interaction terms..

D > 0 unlimitedly.

✓ ~~set~~ infinite # of renormalization conditions.

what's wrong? divergent ⇒ subtract by counter terms.

$$\underset{\uparrow\ \text{many}}{(g_i)_r} = \text{fn.}(\underset{\uparrow\ \text{many.}}{\{g_k\text{'s}\}} ; \Lambda).$$

doable? well-defined?

✓ In the real world...

ν mass may be due to $\mathcal{L}_{int} = \frac{1}{M}$ "$\underbrace{\Psi\Psi}_{\text{lepton}}\underbrace{\phi\phi}_{\text{Higgs field.}}$"

$\Delta = 5$ dimension-5 operator.

☆ matrix elements $\mathcal{M}$

$\mathcal{S}$-matrix : mass-dim $= -$ #(Ext) $\Leftarrow$ $\langle \vec{p} | \vec{q} \rangle = \underbrace{(2\pi)^3 \delta^3(\vec{p}-\vec{q})(2E_{\vec{p}})}_{\text{mass-dim}=-2}$

$|\vec{q}\rangle^{\text{in}}$: mass-dim $= -1$.

$\mathcal{S} = (2\pi)^4 \delta^4(p_{in} - p_{out})\, i\mathcal{M} + 1.$

$\Rightarrow$ $\mathcal{M}$: mass-dim $= 4 -$ #(Ext)

☆ Coefficients w/ mass dimensions

$$\begin{cases} \mathcal{L}_{int} \supset m^{(4-\Delta_j)} \mathcal{O}_j & \text{renormalizable operators} \quad (4-\Delta_j \geq 0) \\ \mathcal{L}_{int} \supset \dfrac{1}{M^{\Delta_i - 4}} \mathcal{O}_i & \text{non-renormalizable operators} \quad (\Delta_i - 4 > 0) \end{cases}$$

Think of a theory whose non-ren. operators come with coefficients scaled by a common energy scale M.

Require a precision (for a fixed δ)

$$\mathcal{M} \sim (\text{Energy})^{4-\#(\text{Ext})} \left[1 + \ldots \left(\frac{\text{Energy}}{M}\right) + \cdots + \underbrace{\ldots \left(\frac{\text{Energy}}{M}\right)^{\delta}}_{(*)} + \mathcal{O}\left(\left(\frac{\text{Energy}}{M}\right)^{\delta+1}\right) \right]$$

for various processes.

Then only finitely many operators contribute. (for a fixed δ)

The renormalized coupling constants of those operators are set so that the ~~renormalized~~ renormalization conditions at some energy scale are statisfied.

$\boxed{\text{Log corrections appearing in } (*) \text{ provide non-trivial predictions.}}$

example: Lagrangian of pions.