§2.4 Renormalized Perturbation Theory of "Non-renormalizable" Theories

Historically (70's)

renormalized theories: (correlat'n fcns of renormalized fields do not diverge in the large regulator scale limit when expressed in terms of the ~~regulator scale & the renormalized coupling~~)

renormalizable QFT's which need only a finite number of renormalized coupling constant (coefficients)

(Week 05 lecture note)

$$\boxed{D = 4 - \left(\tfrac{3}{2}E_\psi + E_\phi + E_{vect}\right) + \Sigma_i(\Delta_i - 4)V_i}$$

(Δ_i: naive operator (mass) dimension, V_i: # of the "i-th type" vertex in a graph.)

◎ $\Delta_i \leq 4$ for all the interact'n terms
good criterion for "renormalizable theories in the historical sense."

◎ If $\Delta_i > 4$ in one of interaction terms
- for a fixed set of external lines: D can be arbitrarily large.
- $D \geq 0$ graph exists for infinitely many choices of $(E_\psi, E_\phi, E_{vect})$.

called

$\mathcal{L}_{int} \supset m^{(4-\Delta_j)}\, \mathcal{O}_j$ renormalizable operators $(4-\Delta_i \geq 0)$

$\mathcal{L}_{int} \supset \frac{1}{M^{(\Delta_i-4)}}\, \mathcal{O}_i$ non-renormalizable operators $(4-\Delta_i < 0)$

If the neutrino mass is due to

$$\mathcal{L}_{int} \supset \frac{1}{M}(l^A_\alpha \phi_A)(l^B_\beta \phi_B)\,\varepsilon^{\alpha\beta} + h.c.$$

(B, A = 1,2 $SU(2)_L \subset SU(3)_C \times SU(2)_L \times U(1)_Y$; $\alpha\beta = 1,2$. $SO(3,1)$ 2-component spinor.)

rather than $\mathcal{L}_{int} \supset y\, l^A_\alpha (\nu^c)_\beta \phi_A\, \varepsilon^{\alpha\beta} + h.c.$ $\quad (y \sim O(10^{-12}))$

it is really a question of practical importance whether/how renormalized perturbation theory works when non-renormalizable operators are present.

(infinitely many operators where the coefficients in $\mathcal{L}_{int}$ can be quite different from the observed ones.)

$\mathcal{L}_{int} \supset$ infinitely many non-renormalizable operators.

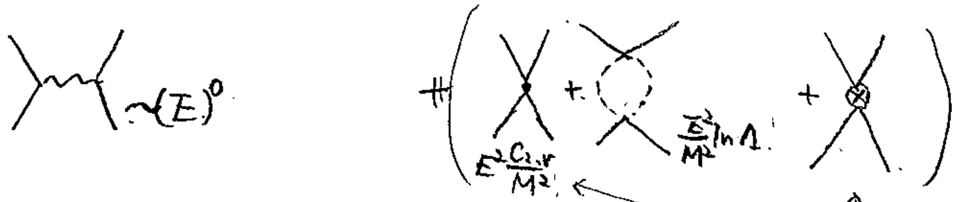
possible to use infinitely many measured coupling constants (scattering amplitudes) to set the infinitely many Lagrangian coefficients?

Doable? (how?) any predictability left?

In reality, don't forget that we have to live with limited precision.

processes.

$ll \to ll$

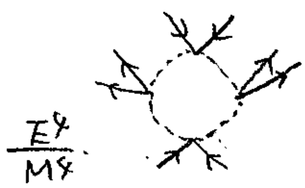


$ll \to \phi\phi$

$$\mathcal{L}_{int} \supset (l\phi\, l\phi)\left(\frac{(C_1)r}{M} + \text{counter term}\right) + ll\,\bar{l}\bar{l}\left(\frac{C_2 r}{M^2} + \text{counter term}\right)$$

$ll \to ll\bar{l}\bar{l}ll$

$\vdots$



⊙ Data / precision are often available only up to finite order in $1/M$ for high energy scale M.

$\Rightarrow$ only finite # of terms in $\mathcal{L}_{int}$ are relevant.

(under an expectat'n that all the renormalized coupling constants scale by M. (or even less.))

⊙ still. (loop correct'n + counter term)'s are non-trivial functions of kinematical variables. (they are not just numbers.)

§3. Renormalization Group

§3.1. Variations of Renormalization Conditions.

For QED, we chose in §2.

- $\boxed{}$ $[\psi]_r$: normalized so $\langle [\psi]_r \overline{[\psi]}_r \rangle$ has residue $=1$ on pole.
- $[A_\mu]_r$: " " $\langle [A_\mu]_r [A_\nu]_r \rangle$ has " on pole.
- fermion mass in $(\mathcal{L} - \mathcal{L}_{int})$: set equal to $\underline{m_{pole}}$
- $\bar{\psi}\psi A$ coupling in $(\mathcal{L} - \mathcal{L}_{int})$: set equal to e @ $q^\mu = 0$ (static experiment)

Was it the only way to do renormalized perturbat'n?

eg.1 Electroweak theory. $SU(2)_L \times U(1)_Y \to U(1)_{QED}$.

symmetry breaking by scalar field condensat'n.

<u>parameters</u> g. g' (coupling constants of $SU(2)_L \times U(1)_Y$)

$\langle \phi \rangle = v/\sqrt{2}$.

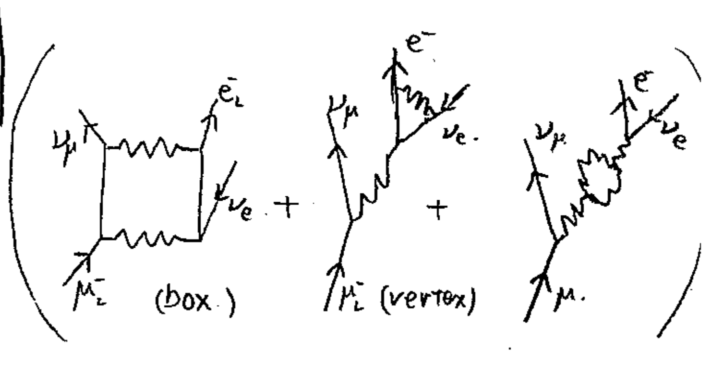
<u>observables</u> $\alpha_{QED} = e^2/(4\pi)$ m_Z, m_W, $G_\mu (\mu \to \nu_\mu + e + \bar{\nu}_e)$

<u>at tree level</u> $m_W = g\frac{v}{2}$, $m_Z = \sqrt{g^2 + (g')^2}\,\frac{v}{2}$, $e = \frac{g g'}{\sqrt{g^2 + (g')^2}}$, $\frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8 m_W^2}$.

$$\Rightarrow \boxed{\frac{G_\mu}{\sqrt{2}} = \frac{\pi \alpha_{QED}}{2 m_W^2 \left(1 - \frac{m_W^2}{m_Z^2}\right)}}$$

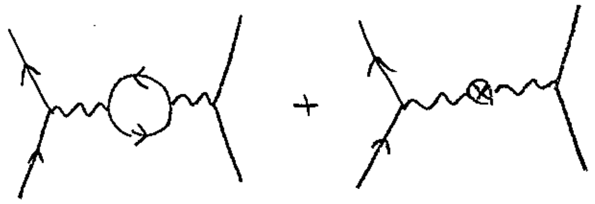
<u>at 1-loop level</u>

$$\frac{\pi \alpha_{QED}}{2 m_W^2 \left(1 - \frac{m_W^2}{m_Z^2}\right)} \neq \frac{G_\mu}{\sqrt{2}} =$$



We may use any three of (α_{QED}, m_Z, m_W, G_μ) to set g, g', v in $(\mathcal{L} - \mathcal{L}_{int})$.

eg.2 scattering at large momentum transfer.



$$\frac{-i\eta_{\mu\nu}}{(q^2+i\varepsilon)(1-\Pi_{ren}(q^2))}$$

$$\Pi^{(1)}_{ren}(q^2) = \Pi^{(1)}(q^2) - \Pi^{(1)}(q^2=0) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left\{\ln\left(\frac{m_e^2 - x(1-x)q^2}{M_{reg}^2}\right) - \ln\left(\frac{m_e^2}{M_{reg}^2}\right)\right\} \quad q^2=0$$

when $\frac{\alpha_{QED}}{\pi}\ln\left(-q^2/m_e^2\right)$ is not $\ll 1$,

☆ $\frac{1}{1-\Pi^{(1)}_{ren}(q^2)}$ is a better approx'n. than $(1+\Pi^{(1)}_{ren}(q^2))$

☆ at certain energy scale ($q^2 \sim q_*^2$) $1/(1-\Pi^{(1)}_{ren}(q^2))$ is still approximately a number than a fcn.

☆ quantum effects: $e_r^2 \Longrightarrow \frac{e_r^2}{[1-\Pi^{(1)}_{ren}(q_*^2)]}$

Why don't we use $[A_\mu]_r^{(q_*)}$ instead of $[A_\mu]_r$.

$$\left([A_\mu]_r^{(q_*)} = [A_\mu]_r\left(1-\Pi_{ren}(q_*^2)\right)^{1/2} = A_\mu\sqrt{\frac{1-\Pi_{ren}(q_*^2)}{Z_3}} = A_\mu\sqrt{1-\Pi(q_*^2)}\right)$$

so $\langle [A_\mu]_r^{(q_*)} [A_\nu]_r^{(q_*)}\rangle \sim \frac{-i\eta_{\mu\nu}}{q^2+i\varepsilon}$ at $q^2 \sim q_*^2$. ?

we only need the "eA_μ" combinat'n. if γ is being exchanged

<u>an idea</u>

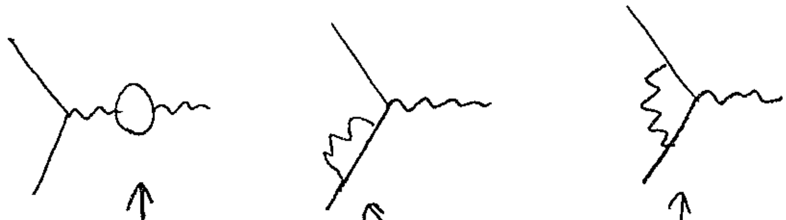
normalize all the fields so that $\langle[\phi]_{r(q_*)}[\phi]_{r(q_*)}\rangle \sim \frac{i}{q^2}$ @ $q^2 \sim q_*^2$

$\langle[\psi]_{r(q_*)}[\bar\psi]_{r(q_*)}\rangle \sim \frac{i\not{q}}{q^2}$

and just remember to multiply $\prod_i (\text{residue}_{(i)} \text{ on pole})^{1/2}$ to the scattering amplitude calculated by using $[\phi]_{r(q_*)}$ & $[\psi]_{r(q_*)}$?

§3.2 Renormalization at Energy-scale E_* and Renormalization Group.

In QED.



log correction   log correction also here.

$$ie\Gamma^\mu_{(1)} \cong ie\frac{e^2}{16\pi^2}\int_\Delta dx\,dy\; 2\gamma^\mu\left[\ln\left(\frac{(1-x-y)\Lambda^2+(x+y)^2 m_e^2 - xy\,q^2}{(x+y)^2 m_e^2 - xy\,q^2}\right) + \frac{[m_e^2\{1-4(1-x-y)+(1-x-y)^2\}+(1-x)(1-y)q^2]}{\{(x+y)^2 m_e^2 - xy\,q^2\}}\right] - [\gamma^\mu,\gamma^\nu]q_\nu \times (\cdots)$$

$$\Gamma^\mu_{(1).\mathrm{ren}}(q^2) = \Gamma^\mu_{(1)}(q^2,\Lambda^2) - \Gamma^\mu_{(1)}(q^2=0,\Lambda^2) \propto \ln\left(\frac{m_e^2}{-q^2}\right)$$

$$e_r + e_r \times \frac{e^2}{\pi^2}\ln\left(\frac{m_e^2}{-q^2}\right)$$

- corrections in a power series of $\left(\alpha_{QED}\ln\left(m_e^2/-q^2\right)\right)$
  what if $m_e^2 \ll (-q^2)$ large momentum transfer.
- cannot expect a structure like a geometric series.
- there must be a clever idea than to always go back to electrostatic measurements!

> use the effective value of $\langle[\psi]_{r(q_*)}[\bar\psi]_{r(q_*)}[A_\mu]_{r(q_*)}\rangle$ at $q_\mu \sim q_*$ for the non-counter term part $e_{r(q_*)}[A_\mu]_{r(q_*)}$ to compute processes with momentum transfer of order q_*.

What is the relation between $e_{r(q=0)}$ and $e_{r(q_*)}$? denoted by $e(\mu)$

$$\boxed{\left(ie_r\gamma^\mu + ie_r \underbrace{\Gamma^\mu_{(1).\mathrm{ren}}}_{\gamma^\mu \text{ part of}}\right)\frac{1}{\sqrt{1-\Pi^{(1)}_{\mathrm{ren}}(-\mu^2)}\;\sqrt{1-A^{(1)}_{\mathrm{ren}}(-\mu^2)}^{\,2}} = ie(\mu)\gamma^\mu}$$

(space-like)

by ignoring the non-log parts....

$$ie(\mu) = ie_r\left\{1+\frac{e^2}{16\pi^2}\int dx\,dy\; 2\ln\left(\frac{(x+y)^2 m_e^2}{(x+y)^2 m_e^2 + xy\,\mu^2}\right)\right\}$$

$$\times\left\{1-\frac{e^2}{2\pi^2}\int dx\, x(1-x)\ln\left(\frac{m_e^2+x(1-x)\mu^2}{m_e^2}\right)\right\}^{-1/2}$$

$$\times\left\{1-\frac{e^2}{16\pi^2}\int dx\,(-2)(1-x)\ln\left(\frac{x^2 m_e^2}{x m_e^2 + x(1-x)\mu^2}\right)\right\}^{-\frac{2}{2}}.$$

For $\mu^2 \ll m_e^2$, $\quad e(\mu) \simeq e_r + \mathcal{O}\left(e^3\frac{\mu^2}{m_e^2}\right) \simeq e_r$

$m_e^2 \ll \mu^2$, $\quad \dfrac{\partial e(\mu)}{\partial \ln(\mu^2)} = e^3\left[\dfrac{-1}{16\pi^2}(2\times 1) + \dfrac{1}{4\pi^2}\left(\int dx\, x(1-x) = \dfrac{1}{6}\right) + \dfrac{-1}{16\pi^2}(-2)\cdot\dfrac{1}{2}\right]$

renormalization group

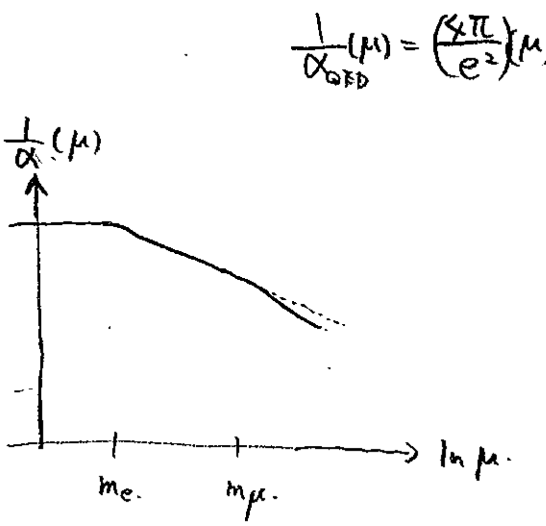
$$= \frac{e^3}{24\pi^2}.$$

at 1-loop $\rightarrow$ $\dfrac{\partial(4\pi/e^2(\mu))}{\partial\ln\mu} = \dfrac{4\pi}{e^3}(-2)\dfrac{\partial e(\mu)}{\partial\ln\mu} = \dfrac{4\pi}{e^3}\times(-2)\times 2\times\dfrac{e^3}{24\pi^2} = -\dfrac{2}{3\pi}.$

(for $m_e^2 \ll \mu^2 \ll m_\mu^2$)

$$\frac{1}{\alpha_{QED}}(\mu) = \left(\frac{4\pi}{e^2}\right)(\mu) \simeq (\mathrm{const}) - \frac{2}{3\pi}\ln\left(\frac{\mu}{m_e}\right)$$

$\mathrm{const} \simeq 137.$



should also include quarks, though.