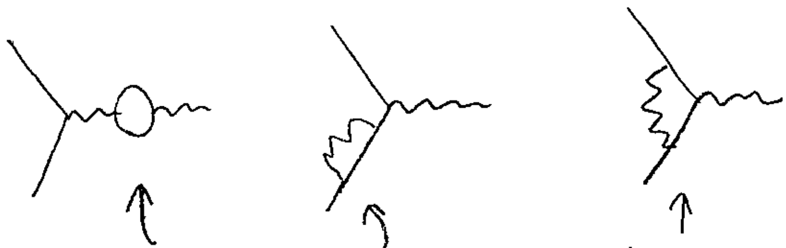
§3.2 Renormalization at Energy-scale E_* and Renormalization Group.

In QED.



log correction

log correction also here.

$$ie\Gamma^{\mu}_{(1)} \cong ie\frac{e^2}{16\pi^2}\int_{\triangle} dx\,dy\; 2\gamma^{\mu}\left[\ln\left(\frac{(1-x-y)\Lambda^2+(x+y)^2 m_e^2 - xyq^2}{(x+y)^2 m_e^2 - xy\,q^2}\right) + \frac{[m_e^2\{1-4(1-x-y)+(1-x-y)^2\}+(1-x)(1-y)q^2]}{\{(x+y)^2 m_e^2 - xy\,q^2\}}\right] - [\gamma^{\mu}, \gamma^{\nu}]q_{\nu}\times(\cdots)$$

$$\Gamma^{\mu}_{(1).\mathrm{ren}}(q^2) = \Gamma^{\mu}_{(1)}(q^2, \Lambda^2) - \Gamma^{\mu}_{(1)}(q^2=0, \Lambda^2) \propto \ln\left(\frac{m_e^2}{-q^2}\right) - \cdots$$

$$e_r + e_r \times \frac{e^2}{\pi^2}\ln\left(\frac{m_e^2}{-q^2}\right)$$

— corrections in a power series of $\left(\alpha_{QED}\ln\left(m_e^2/-q^2\right)\right)$

what if $m_e^2 \ll (-q^2)$ large momentum transfer.

— cannot expect a structure like a geometric series.

— there must be a clever idea than to always go back to electrostatic measurements!

> use the effective value of $\langle [\psi]_{r(q_*)} [\bar{\psi}]_{r(q_*)} [A_\mu]_{r(q_*)} \rangle$ at $q_\mu \sim q_*$ for the non-counter term part $e_{r(q_*)}[A_\mu]_{r(q_*)}$ to compute processes with momentum transfer of order q_*.

denoted by $e(\mu)$

What is the relation between $e_{r(q=0)}$ and $e_{r(q^2)}$?

$$\boxed{\left(i e_r \gamma^\mu + i e_r \underbrace{\Gamma^{\mu}_{(1)\,\mathrm{ren}}}_{\gamma^\mu\ \text{part of}}\right)\frac{1}{\sqrt{1-\Pi^{(1)}_{\mathrm{ren}}(-\mu^2)}\ \sqrt{1-A^{(1)}_{\mathrm{ren}}(-\mu^2)}^{\,2}} = i e(\mu)\gamma^\mu}$$

(space-like)

by ignoring the non-log parts....

$$i e(\mu) = i e_r \left\{1 + \frac{e^2}{16\pi^2}\int dx\,dy\ 2\ln\left(\frac{(x+y)^2 m_e^2}{(x+y)^2 m_e^2 + xy\mu^2}\right)\right\}$$

$$\times\left\{1 - \frac{e^2}{2\pi^2}\int dx\, x(1-x)\ln\left(\frac{m_e^2 + x(1-x)\mu^2}{m_e^2}\right)\right\}^{-1/2}$$

$$\times\left\{1 - \frac{e^2}{16\pi^2}\int dx\,(-2)(1-x)\ln\left(\frac{x^2 m_e^2}{x m_e^2 + x(1-x)\mu^2}\right)\right\}^{-\frac{2}{2}}.$$

For $\mu^2 \ll m_e^2$, $\quad e(\mu) \cong e_r + \mathcal{O}\left(e^3\frac{\mu^2}{m_e^2}\right) \cong e_r$

$m_e^2 \ll \mu^2$, $\quad \dfrac{\partial e(\mu)}{\partial \ln(\mu^2)} = e^3\left[\frac{-1}{16\pi^2}(2\times 1) + \underbrace{\frac{1}{4\pi^2}\left(\int dx\, x(1-x) = \frac{1}{6}\right) + \frac{-1}{16\pi^2}(-2)\cdot\frac{1}{2}}_{\text{cancel}}\right]$

renormalization group

$$= \frac{e^3}{24\pi^2}.$$

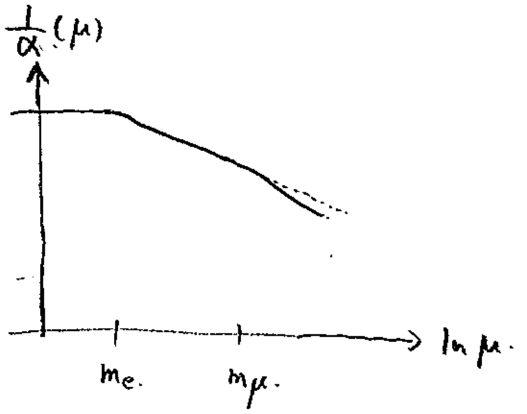
at 1-loop $\rightarrow$ $\dfrac{\partial\left(4\pi/e^2(\mu)\right)}{\partial \ln\mu} = \dfrac{4\pi}{e^3}(-2)\dfrac{\partial e(\mu)}{\partial\ln\mu} = \dfrac{4\pi}{e^3}\times(-2)\cdot 2\times\dfrac{e^3}{24\pi^2} = -\dfrac{2}{3\pi}.$

(for $m_e^2 \ll \mu^2 \ll m_\mu^2$.)

$$\frac{1}{\alpha_{QED}}(\mu) = \left(\frac{4\pi}{e^2}\right)(\mu) \simeq (\text{const}) - \frac{2}{3\pi}\ln\left(\frac{\mu}{m_e}\right).$$

$\wr$

137.



should also include quarks, though.

RG group equation

relation among renormalized coupling constants
for renormalized conditions
at different energy scales.

◎ $g_r(\mu) :=$ the n-point amputated amplitude ($\Leftarrow$ the n external line propagators factored out)
for the n-fields normalized at $q^2 = -\mu^2$
evaluated for kinematics $q^2 \sim -\mu^2$.

$$\psi = \sqrt{Z(q^2=-\mu^2)}\,\psi_{r(\mu)} \quad \text{instead of} \quad \psi = \sqrt{Z(q^2=m^2)}\,\psi_r. \quad \text{so } \psi_r = \sqrt{\frac{Z(-\mu^2)}{Z}}\,\psi_{r(\mu)} =: \sqrt{Z_r(\mu^2)}\,\psi_{r(\mu)}$$

Because

$$g_r(\mu) = \mathcal{M}^{amp}_{ren.}(q^2=-\mu^2) \times \prod_{i=1}^{n} \sqrt{Z_{r,i}(\mu)},$$

$$\frac{\partial g_r(\mu)}{\partial \ln\mu} = g_r(\mu)\left(\frac{\partial \ln \mathcal{M}^{amp}_{ren}(q^2=-\mu^2)}{\partial \ln\mu} + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial \ln(Z_{r,i}(\mu))}{\partial \ln\mu}\right).$$

When $m^2 \ll \mu^2$,

$$\mathcal{M}^{amp}_{ren}(q^2) \sim (\text{const}) + \circ\left\{\ln\left(\frac{\Lambda^2}{m^2-q^2}\right) - \ln\left(\frac{\Lambda^2}{m^2}\right)\right\}$$

$$Z_r(\mu^2) \sim (\text{const}) + \circ\left\{\ln\left(\frac{\Lambda^2}{m^2+\mu^2}\right) - \ln\left(\frac{\Lambda^2}{m^2}\right)\right\}$$

$\frac{\partial}{\partial \ln\mu}$ on $\ln(\mathcal{M}^{amp}_{ren}(-\mu^2))$ and $\ln(Z_r(\mu))$ can be replaced by

$-\frac{\partial}{\partial \ln\Lambda}$ on $\ln(\mathcal{M}^{amp}(-\mu^2;\Lambda))$ and $\ln(Z(\mu;\Lambda))$.

$\Rightarrow$ easier computation for

$$\boxed{\frac{\partial g_r(\mu)}{\partial \ln\mu} =: \beta_g}$$

$$\boxed{\frac{\partial \ln Z_{r,\mu}}{\partial \ln\mu} = \frac{\partial \ln Z(\Lambda^2, m^2, \mu^2)}{\partial \ln\mu} =: \gamma}$$

(anomalous dim.)

because we can focus on the Λ-dependence of $\mathcal{M}^{amp}(q^2, \Lambda^2)$ and $Z(\mu;\Lambda)$

Dimensional Regularization

An easy way to $\left\{ \begin{array}{l} \text{calculate } \beta\text{-fcn} \\ \text{renormalize. (regularize \& subtract)} \end{array} \right\}$

Loop momentum integration.

$$\frac{d^4k}{(2\pi)^4} \Rightarrow \frac{d^nk}{(2\pi)^n}(\mu)^{4-n} \Rightarrow i\,\frac{vol(S_{n-1})}{(2\pi)^n\cdot 2}\int dK\; K^{\frac{n}{2}-1}$$

~~after appropriate shift~~.

$vol(S_{n-1})$:

$$\int d^n x\; e^{-\sum_i (x_i)^2} = \left(\int_{-\infty}^{+\infty} dx\; e^{-x^2}\right)^n = \pi^{\frac{n}{2}}$$

$$\|$$

$$\int_0^{+\infty} dr\; r^{n-1}\, vol(S_{n-1})\cdot e^{-r^2} = \frac{vol(S_{n-1})}{2}\cdot\int_0^{+\infty} dR\; R^{\frac{n}{2}-1} e^{-R} = \frac{vol(S_{n-1})}{2}\,\Gamma\!\left(\frac{n}{2}\right)$$

$(R=r^2)$

$$\Rightarrow \boxed{\frac{vol(S_{n-1})}{2} = \frac{\pi^{n/2}}{\Gamma(n/2)}}$$

Idea: $$\int_0^{\Lambda^2} dK\, \frac{K}{[K+m^2-x(1-x)q^2]^2} \simeq \ln\left(\frac{\Lambda^2+m^2-x(1-x)q^2}{m^2-x(1-x)q^2}\right) - 1.$$

Logarithmically divergent.

but
$$\lim_{\Lambda\to+\infty}\int^{\Lambda^2} dK\,\frac{K^{\frac{n}{2}-1}\;\mu^{4-n}}{[K+m^2-x(1-x)q^2]^2} = \left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{2-\frac{n}{2}}\int_0^{+\infty} dy\,\frac{y^{\frac{n}{2}-1}}{(y+1)^2}$$

$$\int_0^1 d\xi\,(1-\xi)^{\frac{n}{2}-1}\,\xi^{1-\frac{n}{2}}$$

convergent if $\frac{n}{2}>0$ and $2-\frac{n}{2}>0 \Leftrightarrow \boxed{4>n}$

$$= \left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{2-\frac{n}{2}}\frac{\Gamma(\frac{n}{2})\,\Gamma(2-\frac{n}{2})}{\Gamma(2)}$$

(((((5) ~~2~~

$$\int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \mu^{4-n} \frac{1}{[k^2 - m^2 + x(1-x)q^2]^2} = \underset{\wedge \int_0^1 dx}{i} \frac{\pi^{n/2}}{(2\pi)^n \Gamma(n/2)} \Gamma\left(\frac{n}{2}\right) \Gamma\left(2-\frac{n}{2}\right) \left(\frac{\mu^2}{m^2 - x(1-x)q^2}\right)^{2-\frac{n}{2}}$$

$$= \int_0^1 dx \frac{i}{(4\pi)^2} \underbrace{\Gamma\left(2-\frac{n}{2}\right) \left(\frac{4\pi\mu^2}{m^2 - x(1-x)q^2}\right)^{2-\frac{n}{2}}}$$

small $\left(2-\frac{n}{2}\right)$ $(-\gamma = -0.5772)$

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{1}{z}\left(\Gamma(1) + \left.\frac{d\Gamma}{dz}\right|_{z=1} \times z + \cdots\right)$$

$$\left[\frac{1}{\left(2-\frac{n}{2}\right)} + (-\gamma) + \cdots\right]\left[1 + \left(2-\frac{n}{2}\right) \ln\left(\frac{\mu^2 4\pi}{m^2 - x(1-x)q^2}\right) + \cdots\right]$$

$$= \frac{1}{\left(2-\frac{n}{2}\right)} + (-\gamma) + \ln\left(\frac{\mu^2 4\pi}{m^2 - x(1-x)q^2}\right) + O\left(2-\frac{n}{2}\right).$$

- still divergent. when $n \to 4$.
- empirical rule: $\frac{1}{\left(2-\frac{n}{2}\right)} \Longleftrightarrow \ln(\Lambda^2)$

$\left(\nexists \ \frac{1}{(1-n/2)} \text{ pole} \Leftrightarrow \exists \text{ quadratic divergence.} \right)$

β-function as coefficients of $\underline{\ln(\Lambda^2)}$

$$\Rightarrow \underline{\frac{1}{(2-n/2)}}$$

- renormalization at scale μ.

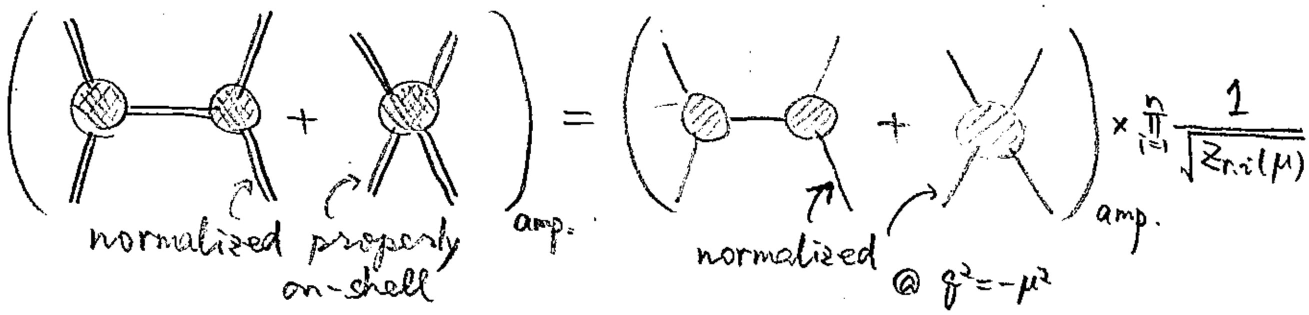
simply subtract $\frac{1}{\left(2-\frac{n}{2}\right)} \cancel{+} \left(-\gamma + \ln(4\pi)\right)$

renormalization scheme. : ~~mini~~ $\overline{MS}$.

(minimal subtraction or

§ 3.3 Meaning of Running Coupling Constants I

☆ Observables (e.g. $|\mathcal{M}|^2$ to compute cross section at a given kinematics) should not depend on the choice of a renormalization scale.



☆ better approximation <u>at fixed order perturbation</u>.

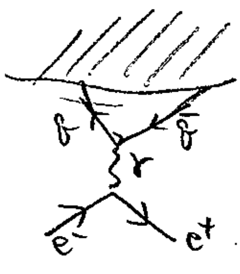
e.g. QED scattering amplitude.

$$i\mathcal{M} \sim i\frac{(e_r(\mu))^2}{q^2+i\varepsilon}\eta_{\mu\nu} \times \begin{pmatrix}\text{spinor}\\ \text{polarization}\end{pmatrix} \quad \text{at tree level.}$$

Corrections of order $\times \frac{\alpha_e}{\pi}\ln\left(\frac{-q^2}{\mu^2}\right)$ remain.

($\Leftrightarrow$ The $(i\mathcal{M}^{\text{tree+1-loop}})$ can be approximated as above by using $e_r(\mu)$ rather than e_r.)

eg. total hadron production cross section.



$$\sigma_{tot} = \frac{4\pi\alpha_e^2}{3S}(Q_q)^2 \times 3 \times \left[1 + \frac{\alpha_s(\mu^2)}{\pi} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\left\{c_2 - \pi b \ln\left(\frac{S}{\mu^2}\right)\right\} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^3\left[c_3 + \left\{\pi b \ln\left(\frac{S}{\mu^2}\right)\right\}^2 - \cdots\right] + \cdots\right]$$

$$\left(\frac{\partial}{\partial \ln\mu^2}\left(\frac{1}{\alpha_s(\mu^2)}\right) = b + \mathcal{O}(\alpha_s)\right)$$

$\Uparrow$ μ-indep. in the full $[\cdots]$
when truncated at a fixed order, use $\alpha_s(\mu^2 = S)$ for a better approximation

☆ $$\frac{1}{\alpha(\mu_1)} = \frac{1}{\alpha(\mu_0)} + \frac{b'}{2\pi}\ln\left(\frac{\mu_1}{\mu_0}\right) \cdots$$

$$\Leftrightarrow \alpha(\mu_1) = \frac{\alpha(\mu_0)}{1 + \frac{b'}{2\pi}\alpha(\mu_0)\ln\left(\frac{\mu_1}{\mu_0}\right)} = \sum_{n=0}^{\infty}(\alpha(\mu_0))^{n+1}\left(\frac{-b'}{2\pi}\right)^n\left(\ln\left(\frac{\mu_1}{\mu_0}\right)\right)^n.$$

(leading log resummation)