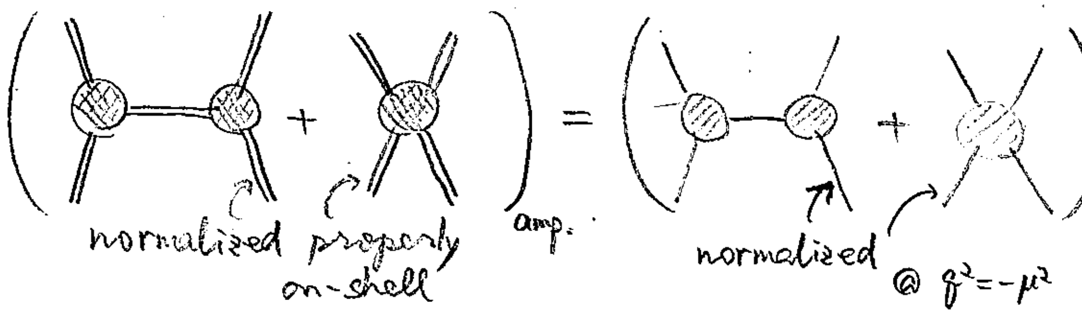
## §3.3 Meaning of Running Coupling Constants I

☆ Observables (e.g. $|\mathcal{M}|^2$ to compute cross section at a given kinematics.) should not depend on the choice of a renormalization scale.




$$\times \prod_{i=1}^{n} \frac{1}{\sqrt{Z_{r,i}(\mu)}}$$

☆ better approximation <u>at fixed order perturbation</u>

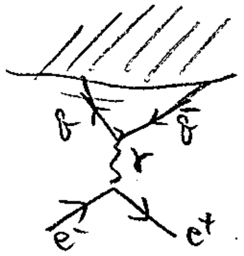
e.g. QED scattering amplitude

$$i\mathcal{M} \sim i\frac{(e_r(\mu))^2}{q^2+i\epsilon}\eta_{\mu\nu} \times \begin{pmatrix}\text{spinor}\\ \text{polarization}\end{pmatrix} \quad \text{at tree level.}$$

Corrections of order $\times \frac{\alpha_e}{\pi}\ln\left(-q^2/\mu^2\right)$ remain.

($\Leftrightarrow$ The $(i\mathcal{M}^{\text{tree+1-loop}})$ can be approximated as above by using $e_r(\mu)$ rather than $e_r$.)

e.g. total hadron production cross section.




$$\sigma_{tot} = \frac{4\pi\alpha_e^2}{3s}(Q_q)^2 \times 3 \times \left[ 1 + \frac{\alpha_s(\mu^2)}{\pi} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 \left\{c_2 - \pi b \ln\left(\frac{s}{\mu^2}\right)\right\} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^3 \left[ \left\{c_3 + \pi b \ln\left(\frac{s}{\mu^2}\right)\right\}^2 - \cdots \right] + \cdots \right]$$

$$\left( \frac{\partial}{\partial \ln \mu^2}\left(\frac{1}{\alpha_s(\mu^2)}\right) = b + O(\alpha_s) \right)$$

↑ $\mu$-indep. in the full [ ⋯ ]
when truncated at a fixed order, use $\alpha_s(\mu^2 = s)$ for a better approximation

☆ $$\frac{1}{\alpha(\mu_1)} = \frac{1}{\alpha(\mu_0)} + \frac{b'}{2\pi}\ln\left(\frac{\mu_1}{\mu_0}\right) \cdots$$

$$\Longrightarrow \alpha(\mu_1) = \frac{\alpha(\mu_0)}{1 + \frac{b'}{2\pi}\alpha(\mu_0)\ln\left(\frac{\mu_1}{\mu_0}\right)} = \sum_{n=0}^{\infty} (\alpha(\mu_0))^{n+1}\left(\frac{-b'}{2\pi}\right)^n \left(\ln\left(\frac{\mu_1}{\mu_0}\right)\right)^n.$$

(leading log resummation)

## §3.4 Wilson's interpretation of renormalization group

(meaning of running coupling constants II)

Remember that the procedure of renormalization is technically ...

eg. quantum correction $\Pi^{(1)}_{\mathrm{ren}(\mu)}(q^2)$

counter term

$$= \frac{(eQ)^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left\{\ln\left(\frac{m_e^2 - x(1-x)q^2}{M_{\mathrm{reg}}^2}\right) - \ln\left(\frac{m_e^2 + x(1-x)\mu^2}{M_{\mathrm{reg}}^2}\right)\right\}$$

to render a divergent (large log) integral finite by removing the near-$M_{\mathrm{reg}}$ contribution.

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2 - m^2 + x(1-x)q^2]^2} \simeq \frac{i}{16\pi^2}\int_0^{\Lambda^2}\frac{dK\,K}{[K + m^2 - x(1-x)q^2]^2}$$

$$\approx$$

$$\boxed{\frac{i}{16\pi^2}\int_{m^2,-q^2}^{\Lambda^2}\frac{dK}{K} \Longrightarrow \frac{i}{16\pi^2}\left\{\int_{m^2,-q^2}^{\Lambda^2}\frac{dK}{K} - \int_{\mu^2}^{\Lambda^2}\frac{dK}{K}\right\}}$$

When renormalization conditions at the energy scale $\mu$ are employed, degrees of freedom with $\mu^2 \lesssim K$ are virtually not integrated over.

Set $\mu^2 \gtrsim$ (kinematical invariants of interest).

$$\boxed{\boxed{\text{Theory}^{\text{org}} \text{ regularized at } \Lambda^2, M_{\mathrm{reg}}^2 \text{ with } M, e} \Longleftrightarrow \boxed{\text{Theory}^{\text{eff}} \text{ loop momentum integrated only } K \lesssim \mu^2 \text{ with } m(\mu),\ e_r(\mu)\ (+\cdots)}}$$

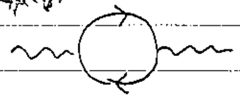
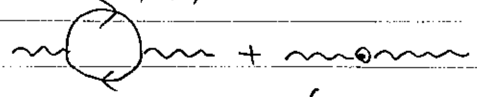
In path-integral language

$$\boxed{Z[J,K] = \int_{|k|\lesssim\Lambda,\, M_{\mathrm{reg}}} \mathcal{D}A_\mu \mathcal{D}\psi\, e^{iS[A,\psi;M,e] + i\int(A\cdot J + \psi\cdot K)}} \Rightarrow \boxed{Z[J,K] = \int_{|k|<\mu}\mathcal{D}A\mathcal{D}\psi\, e^{iS_{(\mu)} + i\int(A\cdot J + K\cdot\psi)}}$$

If interested only in $\langle A(p)\text{'s}\ \psi(p)\text{'s}\rangle$ with $p<\mu$ (low-energy physics)
$Z[J(p), K(p)]$ with $p<\mu$

$\boxed{\text{OK. to integrate over } A(k),\ \psi(k)\text{'s with } k>\mu \text{ first.}}$

$S_{(\mu)}[A, \psi, \cdots]$ cannot be the same as $S[A, \psi, M, e]$.

☆ $A_\mu(q)$  $= i\left(q^2\eta_{\mu\nu} - q_\mu q_\nu\right)\underset{\text{full. un-ren.}}{\Pi(q^2)} =$ 

$(\mu \lesssim k)$ & $(k \lesssim \mu)$ (available in theory@$\mu$) $(k \lesssim \mu)$

$i\left(q^2\eta_{\mu\nu} - q_\mu q_\nu\right)\left(\Pi_{\mu\lesssim k}\right)$ $\Uparrow$ short distance contrib.

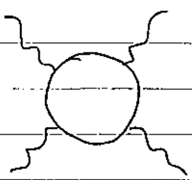
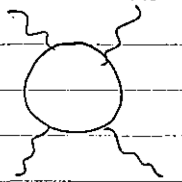
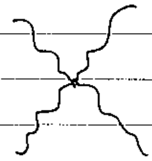
$$S_{(\mu)} \sim \int \frac{d^4 q}{(2\pi)^4} \overbrace{\left(\frac{1}{2} A_\mu \partial^2 A^\mu\right)}^{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}} - \frac{1}{2} A_\mu \partial^2 A^\mu \left(\Pi_{(\mu\lesssim k)}\right) + \cdots\cdots$$

$$=: \int \frac{d^4 q}{(2\pi)^4} \frac{1}{2} [A_\mu]_{r(\mu)} \partial^2 [A^\mu]_{r(\mu)} + \cdots\cdots$$

This definition of $[A_\mu]_{r(\mu)}$ reproduces the relation

$$\boxed{\underset{\text{renormalized field}}{[A_\mu]_r} = A_\mu \times \sqrt{(1-\Pi_{\mu\lesssim k})}_{\text{un-ren.}} \simeq A_\mu \cdot \sqrt{1-\Pi(\mu^2)}_{\text{unren.}} = A_\mu \Big/ \sqrt{Z_3(\mu)}}$$

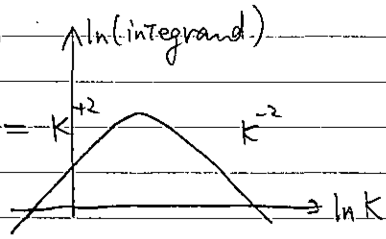
as expected. (no counter term instead)

☆  $\Rightarrow$  $+$  $\Longleftrightarrow \Delta\mathcal{L}(\mu) = \frac{e^4}{(\mu^4)} F..F..F..F..$

$(\mu \lesssim k)$ & $(k \lesssim \mu)$ $(k \lesssim \mu)$ short distance contrib.

In the end, though,

$$\int \frac{dK \quad K}{[K + m^2 - q^2]^4}$$



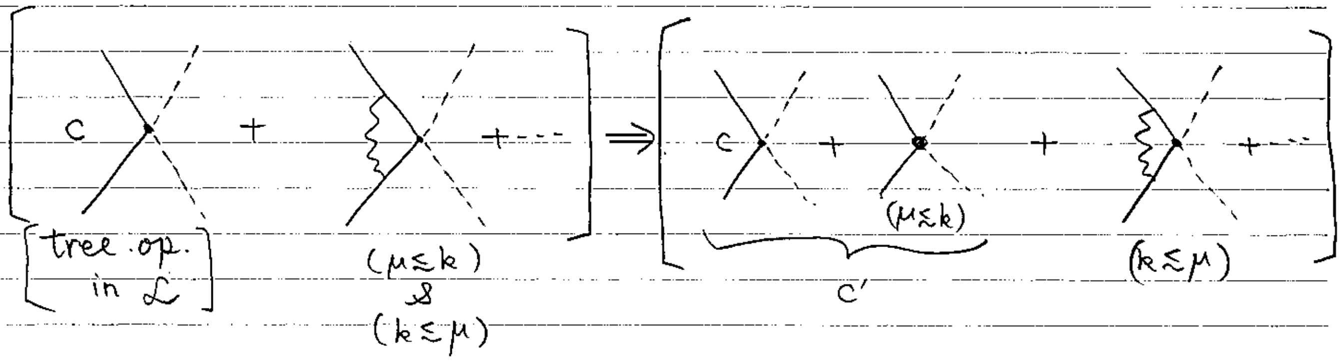

is dominated by the contribution from $K \sim (m^2 - q^2)$.

So, the effective operator $\Delta\mathcal{L}(\mu)$ is not important when $m^2 - q^2 \ll \mu^2$.

★ non-renormalizable operator

(eg  Majorana neutrino mass  $\mathcal{L} = \frac{1}{M}(l\phi)(l\phi)$

The four-fermi op.  $\mathcal{L} = G_F(\bar{\psi}\Gamma^{\mu}\psi)(\bar{\psi}\Gamma_{\mu}\psi)$ )




$$\mathcal{L} \supset c(\psi\psi\phi\phi) \Rightarrow \mathcal{L}_{(\mu)} \supset c'(\psi\psi\phi\phi)$$

$$= \underbrace{c' Z_{\psi}^{(\mu)} Z_{\phi}^{(\mu)}}_{\overset{\text{!!}}{c(\mu)}} (\psi_r \psi_r \phi_r \phi_r)$$

$c(\mu)$ : logarithmic running

# § ~~II~~ 4. Low-energy Effective Theory

## § 4.1 matching

Wilson's interpretation on "renormalization at scale $\mu$."

When all the external lines (probes) have momenta $\ll \mu$.

integrate out all the D.O.F. w/. $k \gtrsim \mu$ first!
$\parallel$
(take care of)

$\Longrightarrow$ change coupling constant values. (renormalize coupling constants

additional operators.

(effectively do the job of high mom. mode

Take one step further.

✓ for a massive particle $\phi$ with $\mu \ll M_\phi$.

some $\mathcal{D}\phi(k)$ left. but no on-shell modes for $k^\mu \ll \mu$.

$\longrightarrow$ does not come out as an external state.

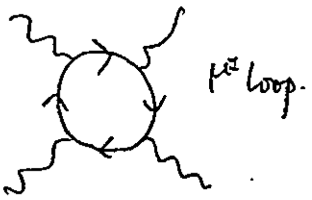
$\Longrightarrow$ integrate out $\phi$ completely! get it done!

## e.g. 1. QED w/ $e^+ . e^-$, $\mu^+ . \mu^-$

$$m_e \simeq 0.511 \text{ MeV}, \qquad m_\mu \simeq 105 \text{ MeV}.$$

what if we take. $\boxed{m_e \ll \mu \ll m_\mu}$ ??

(i)



$\mu^\pm$ loop. carry out this integral completely.

$$\Rightarrow \Delta\mathcal{L} = \frac{2\alpha_{QED}^2}{45 m_\mu^4}\left\{\frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{7}{16}\left(F_{\mu\nu}F_{\kappa\lambda}\frac{\epsilon^{\mu\nu\kappa\lambda}}{2}\right)^2\right\}$$

(ii) phoron propagator. $\dfrac{e^2}{q^2\left(1-\Pi_{ren}\right)}$

$$\boxed{\left(\frac{1-\Pi_{ren}(q^2)}{e^2}\right) = \frac{1}{e^2_{*(e\mu)}(E)} - \frac{1}{2\pi^2}\int_0^1 dx\; x(1-x)\sum_i \ln\left(\frac{m_i^2 - x(1-x)q^2}{m_i^2 + x(1-x)E^2}\right).}$$

$$\left(\text{c.f.} \quad = \frac{1}{e^2_{\overline{MS}}(E)} - \frac{1}{2\pi^2}\int_0^1 dx\; x(1-x)\sum_i \ln\left(\frac{m_i^2 - x(1-x)q^2}{E^2}\right)\right).$$

scheme dependence.

☆ $$\frac{1}{e^2_{*(e\mu)}(E)} = \frac{1}{e^2_{*(e\mu)}(E_0)} - \frac{1}{2\pi^2}\int_0^1 dx\; x(1-x)\sum_i \ln\left(\frac{m_i^2 + x(1-x)E^2}{m_i^2 + x(1-x)E_0^2}\right)$$

$$\boxed{\frac{\partial\left(1/e^2_{*(e\mu)}\right)}{\partial \ln(E^2)} = -\frac{1}{2\pi^2}\sum_i \int_0^1 dx\; x(1-x)\left(\frac{x(1-x)E^2}{m_i^2 + x(1-x)E^2}\right)} \to \begin{cases} 1/6 & (m_i^2 \ll E^2) \\ \frac{1}{30}\left(\frac{E^2}{m_i^2}\right) & (E^2 \ll m_i^2) \\ \ll 1. & \end{cases}$$

☆ when $E \ll m_\mu$.

$$\frac{1}{e^2_{*(e)}(E)} \equiv \frac{1}{e^2_{*(e\mu)}(E)} - \frac{1}{2\pi^2}\int dx\; x(1-x)\ln\left(\frac{m_\mu^2}{m_\mu^2 + x(1-x)E^2}\right) \qquad \int_0^1 dx\; x^2(1-x)^2 = \frac{2^2}{5!}$$

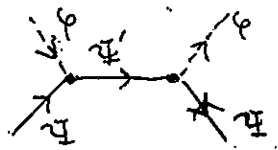
$$\Rightarrow \left(\frac{1-\Pi_{ren}(q^2)}{e^2}\right)\cdot q^2 = q^2\left\{\frac{1}{e^2_{*(e)}(E)} - \frac{1}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left(\frac{m_e^2 - x(1-x)q^2}{m_e^2 + x(1-x)E^2}\right)\right\} + \frac{1}{2\pi^2}\left(\frac{1}{30}\right)\frac{(q^2)^2}{m_\mu^2} + \cdots$$

$\Downarrow$ quantum correction in QED w/ $e^+ . e^-$ only.

$\Downarrow$ effective operator $\Delta\mathcal{L} \propto \left(\frac{e^2}{2\pi^2}\right)\frac{1}{30}\frac{1}{m_\mu^2}\left(F_{\mu\nu}\partial^2 F\ldots\right)$

eg. 2 "see-saw mechanism" $\Leftarrow$ [Weyl fermion (not Dirac fermion) should be used in reality]

$$\mathcal{L} = \bar{\Psi}(i\gamma\cdot D)\Psi + \bar{\Psi}'(i\gamma\cdot D)\Psi' - M\bar{\Psi}'\Psi'$$
$$+ (D^\mu\varphi)^*(D_\mu\varphi) \quad + \lambda\bar{\Psi}'\Psi\varphi + \lambda^*\bar{\Psi}\Psi'\varphi^*$$




at energy scale $E \ll M$.
(in ch momentum transfer)

propagator $\dfrac{i(\not{q}+M)}{q^2-M^2+i\varepsilon} \simeq -i\dfrac{M}{M^2} = \dfrac{-i}{M}.$

$$\cancel{\mathcal{L}_{eff}} = i\mathcal{M} \sim (i\lambda\cdot i\lambda^*)[\varphi^*\bar{\Psi}]\left(\frac{-i}{M}\right)[\varphi\Psi]$$

$\Uparrow$ reproduce.

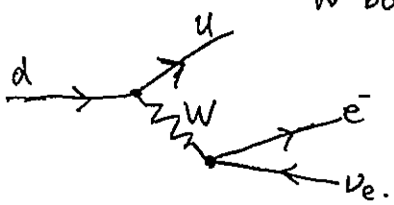
$$\boxed{\mathcal{L}_{eff} = \frac{|\lambda|^2}{M}(\varphi^*\bar{\Psi}\Psi\varphi)}$$

in theory w/o $\Psi'$

$\langle\varphi\varphi^*\rangle \sim v^2$
$\Rightarrow \Psi_{mass} \sim \dfrac{|\lambda|^2 v^2}{M}$
tiny if $v \ll M$.

eg. 3. 4-fermi operator. ($\beta$-decay)

at $E \ll m_W$ W-boson propagator $\dfrac{-i\eta_{\mu\nu}}{q^2-m_W^2} \Rightarrow \dfrac{i\eta_{\mu\nu}}{m_W^2}$




$$\boxed{\mathcal{L}_{eff.} = \left[\bar{u}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)d\right]\left[\bar{e}\gamma_\mu\left(\frac{1-\gamma_5}{2}\right)\nu\right]\left(\frac{-g^2}{2m_W^2}\right)}$$

$\uparrow$ in QCD × QED

$$i\mathcal{M} = (-ig)\left[\bar{e}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\nu\right]\frac{-i\eta_{\mu\nu}}{q^2-m_W^2}(-ig)\left[\bar{u}\gamma^\nu\left(\frac{1-\gamma_5}{2}\right)d\right]\frac{1}{2}$$
$$\hookrightarrow \simeq -i\frac{g^2}{2m_W^2}[\bar{e}\cdots\nu]^\mu[\bar{u}\cdots d]_\mu.$$

Low-Energy Effective Theory

heavy parcicles can be integrated out for low energy description.

{ couplings : renormalized.
effective operators generated.. no other footprints.

Standard Model is yet another effective theory of some more fund. theory