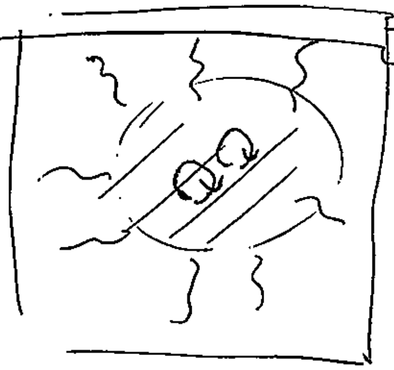
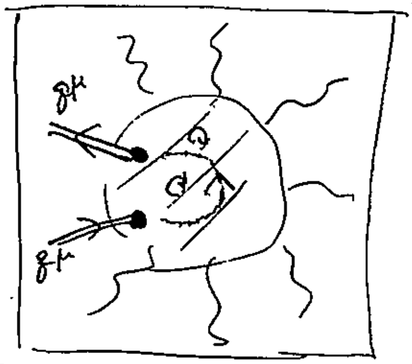
§8.$^{8.2}$ Operator Product Expansion.

Low-energy Eff. Theory
by carrying out high-~~loop~~ mom. loop first.



[all ext. mom. are $\underline{\underline{\text{soft}}}$.]



[all ext. mom. are soft except 2]

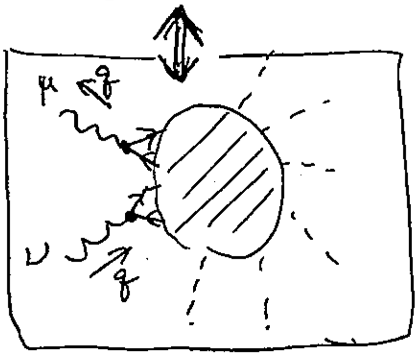
Two operators combined ⇒ no net momentum. flow outside.

Any effective description.

Consider.

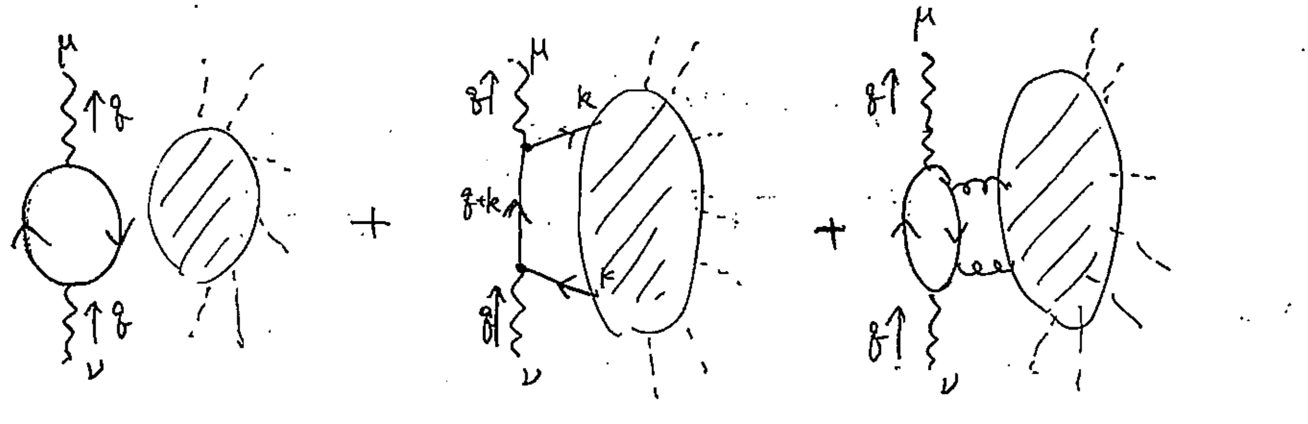
$$-e^2\iint \langle\Omega|T\{\cdots\; J^\mu(x)\, J^\nu(y)\;\cdots\}|\Omega\rangle\; e^{iq\cdot x}\, e^{-iq\cdot y}\; d^4x\, d^4y .$$

$\Updownarrow$



large momentum q^μ. ($q \simeq q'$)
necessarily flow from $J^\nu(y)$ to $J^\mu(x)$.

set μ. so that (ext. mom.) $\ll \mu \ll q^2$.
ok. to integrate out $\mu \lesssim k$'s.



$$\boxed{\int [ie J^\mu(x)]\,[ie J^\nu(y)]\, e^{iq'\cdot x}\, e^{-iq\cdot y}\, d^4y \quad \text{in } T\{\cdots\}.}$$

$$= i\,(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\Pi^{(1)}_{\text{ren}}(q^2)\cdot e^{i(q'-q)\cdot x}\ \mathbb{1}.$$

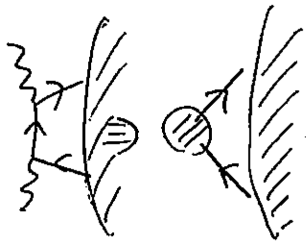
$$+ \int d^4y\ e^{iq'\cdot x}\,[ie\,\bar\psi(x)\gamma^\mu]\ \underbrace{\int \frac{d^4k}{(2\pi)^4}\,\frac{i[(\not q+\not k)+m]\, e^{-i(q+k)\cdot(x-y)}}{(q+k)^2-m^2+i\varepsilon}}_{\langle 0|T\{\psi_I(x)\bar\psi_I(y)\}|0\rangle.\ \text{propagator.}}\ (ie)\,[\gamma^\nu\psi(y)]\, e^{-iq\cdot y}$$

approximation. (or expansion).

$$\frac{i[(\not q+\not k)+m]}{(q+k)^2-m^2+i\varepsilon} \to \frac{i\not q}{q^2}$$

$$\int d^4y\,(-ie^2)\,\frac{q_\lambda}{q^2}\int\frac{d^4k}{(2\pi)^4}\, e^{i(q'-q-k)\cdot x}\, e^{i(q+k-q)\cdot y}\,[\bar\psi(x)\gamma^\mu\gamma^\lambda\gamma^\nu\psi(y)]$$

$$\left(\int\frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)} = \delta^4(x-y)\right)$$



$$\simeq\ -ie^2\,\frac{q_\lambda}{q^2}\, e^{i(q'-q)\cdot x}\,[\bar\psi(x)\gamma^\mu\gamma^\lambda\gamma^\nu\psi(y)]$$

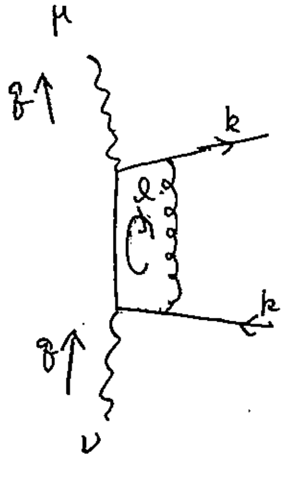
$$= \sum_I C_I(q^\kappa)\cdot \mathcal{O}_I(x).$$

k^μ's : momentum in $\psi(y)$ or $\bar\psi(x)$.

$\Rightarrow$ ∂ on ψ or ∂ on $\bar\psi$.

$\Rightarrow$ derivative expansion.. $\dfrac{q\cdot\overset{\rightarrow\text{or}\leftarrow}{\partial}}{q^2} \ll 1$.

loop correction.



$|q| \ll \underset{\text{UV}}{\overset{\ell}{\mu}}$: finite integral

$\underset{\ell}{\mu} \ll q$: log ~~divergence~~ correction.

propagator $\left\{ \dfrac{i(\not q+\not \ell+\not k)+m}{(q+\ell+k)^2-m^2} \simeq \dfrac{i\not q}{q^2} \right.$

$$C_I(q^2;\alpha_s;\mu)\ [\mathcal{O}_I(x)]_\mu.$$

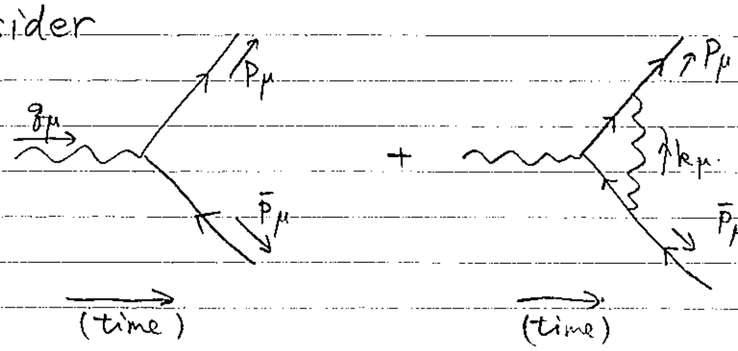
$\{\text{tree} + \text{1-loop } \mu<\ell\}$

only $k<\mu$.

§5 Soft and Collinear Divergence

§5.1 Divergence in Virtual Corrections

Consider



$\gamma^* \to \bar{f} + f$ in QED.

(or $\gamma^* \to \bar{q} + q$ in QCD)

$$\left(-ieQ_f \Gamma^\mu\right) = -ieQ_f\gamma^\mu - ieQ_f \frac{(Q_f e)^2}{16\pi^2}\int dx\,dy \left[2\gamma^\mu \left\{ \ln\left(\frac{(1-x-y)\Lambda^2 + (x+y)^2 m_f^2 - xy\,q^2}{(x+y)^2 m_f^2 - xy\,q^2}\right) + \frac{m_f^2(1-4z+z^2) + (1-x)(1-y)\,q^2}{(x+y)^2 m_f^2 - xy\,q^2} \right\} + [\gamma^\mu, \gamma^\nu] \text{ term} \right]$$

at 1-loop.

$z := (1-x-y)$

(UV divergence manifest in $\ln(\Lambda^2)$ is gone when $\ln(\Lambda^2) \to \ln(\mu^2)$)

We still need to carry out $dx\,dy$ integration.

☆ fixed y: integral over small x region.

$$(1\text{-loop } \gamma^\mu) \simeq -2ieQ_f \frac{(Q_f e)^2}{16\pi^2}\, dy \int_0^{x_*} dx\, \frac{(1-y)\,q^2 + m_f^2(1-4\cdots)}{y^2 m_f^2 + 2xy\,m_f^2 - xy\,q^2}$$

$$\simeq -2ieQ_f \frac{(Q_f e)^2}{16\pi^2}\, dy\, \frac{(1-y)\,q^2 + m_f^2(1-\cdots)}{(2y\,m_f^2 - y\,q^2)} \ln\left(\frac{(2y\,m_f^2 - y\,q^2)x_* + y^2 m_f^2}{y^2 m_f^2}\right)$$

If $q^2 \gg m_f^2$ $\quad (e^+ e^- \to \gamma^* \to f + \bar{f}\ ;\ q^2 = s)$

$$\simeq 2ieQ_f \frac{(Q_f e)^2}{16\pi^2}\, dy\, \frac{1-y}{y} \ln\left(\frac{-q^2}{y\,m_f^2}\right)$$

$\left\{ \begin{array}{l} \text{large log if massive} \\ \text{log divergence if massless} \end{array} \right.$

☆ fixed x: integral over small y region

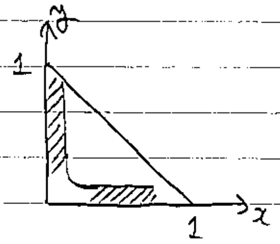
the same.

☆ QCD correction

$\boxed{(Q_f e)^2}$ is replaced by $\boxed{C(R)\, g^2 = \frac{4}{3} g^2}$ for $q\bar{q}$.

☆ integral over the small-x small-y region

$$\left(1\text{-loop } \gamma^\mu\right) \simeq -2ieQ_f \frac{(Q_f e)^2}{16\pi^2}\int dx\,dy\, \frac{q^2-2m_f^2}{(x+y)^2 m_f^2 - xy\, q^2}$$



even for $m_f \neq 0$. $\int_0 d\lambda\, \lambda \frac{1}{\lambda^2}$ Log. divergence.

Even after removing the UV divergence by renormalization,

$\boxed{\text{divergence remains in the amplitudes.}}$

Origin of those divergences

(no x, y here)

$$\Gamma^\mu_{(1)} \sim \int \frac{d^4k}{(2\pi)^4}\int dx\,dy\,dz\, \frac{2\delta(x+y+z-1)\quad [p^\nu,\ k^\nu,\ \bar{p}^\nu,\ m_f \text{ etc.}]}{\left\{x\left[(p-k)^2-m_f^2\right]+y\left[(\bar{p}+k)^2-m_f^2\right]+z k^2\right\}^3 =: D(k)^3}$$

If there is any divergence from integral over a $|k| <$ finite region,

there must be (x_*, y_*, z_*), k^μ_* where

- $D((x,y,z)_*, k^\mu_*) = 0$
- $\left[\frac{\partial}{\partial k^\mu} D\right]((x,y,z)_*, k^\mu_*) = 0$

} double root of $D(k^\mu) = 0$ — otherwise the integration contour can be deformed to avoid $D(k) = 0$.

- $(p-k_*)^2 - m_f^2 = 0$ or $x_* = 0$ or $x_* = 1$
- $(\bar{p}+k_*)^2 - m_f^2 = 0$ or $y_* = 0$ or $y_* = 1$
- ~~$k^2 = 0$ or $z_* = 0$ or $z_* = 1$~~

} so that $D(k) = 0$ cannot be avoided by deforming the integration contour of $x, y, (z)$

OK to drop this option $\left(x_* = 1 \Rightarrow y_* = z_* = 0 \Downarrow D(k_*) = 0 \Rightarrow (p-k_*)^2 - m_f^2 = 0 \text{ automatic} \right)$

(x_*, y_*, z_*, k^ν_*) satisfying all the conditions above.

$\Longrightarrow$ $\boxed{\text{pinch (hyper) surface}}$

☆ $\boxed{x_* = y_* = 0 \ \& \ k_*^2 = 0}$ $\quad x_*(k_*-p)^\mu + y_*(k_*+\bar{p})^\mu + z_* k_*^\mu = 0 \Longrightarrow \boxed{k_*^\mu = 0}$

Introduce a scaling parameter $\lambda \ll 1$ in the (x, y, k^ν) space around $(x_*, y_*, k_*^\nu) = (0, 0, 0^\nu)$

$$(x, y, k^\nu) \sim (\lambda x_0, \lambda y_0, \lambda k_0^\nu)$$

- $dx\,dy\,d^4k \sim d^5\Omega\, d\lambda\, \lambda^5$
- $[D(x,y,k)]^3 \cong \Big(\lambda x_0 \cdot \underbrace{[(p-k)^2 - m_f^2]}_{\underbrace{(p^2-m_f^2)}_{0} - \underbrace{2p\cdot k}_{\lambda} + \underbrace{k^2}_{\lambda^2}} + \lambda y_0 \cdot \underbrace{[(\bar{p}+k)^2 - m_f^2]}_{\underbrace{(\bar{p}^2-m_f^2)}_{0} + \underbrace{2\bar{p}\cdot k}_{\lambda} + \underbrace{k^2}_{\lambda^2}} + 1 \cdot \underbrace{k^2}_{\lambda^2 k_0^2}\Big)^3 \sim \lambda^6$

$$\text{so} \int d^5\Omega \left(\int_0 \frac{d\lambda\, \lambda^5}{\lambda^6} \sim \text{log divergence} \right)$$

associated with $k^\nu \sim (k_*^\nu = 0^\nu)$ $\underline{\text{soft}} \left\{ \begin{matrix} \text{photon} \\ \text{gluon} \end{matrix} \right\}$.

☆ $\boxed{x_* = 0 \ \& \ (\bar{p}+k)^2 - m_f^2 = 0 \ \& \ k^2 = 0}$ $\quad \overset{\nearrow 0}{x_*}(k_*-p)^\mu + y_*(k_*+\bar{p})^\mu + z_* k^\mu = 0$

$$\Longrightarrow \frac{(-k_*^\mu)}{y_*} = \frac{\bar{p}^\mu - (-k_*)^\mu}{(1-y_*)}$$

$k_*^\mu \,/\!/\, [\bar{p} - (-k_*)]^\mu$ and both on-shell
$\longrightarrow$ possible only if $m_f = 0$.

$\left(\begin{matrix} \text{photon/gluon} \\ \underline{\text{collinear to a massless } \bar{f}.} \end{matrix} \right)$

$\left\{ \begin{matrix} (-k_*^\mu) = y_*(\bar{p}^\mu) \\ [\bar{p} - (-k_*)]^\mu = (1-y_*)(\bar{p}^\mu) \end{matrix} \right\}$

light cone components of a four vector

$$\left(\frac{l^0 + l^3}{\sqrt{2}}, \frac{l^0 - l^3}{\sqrt{2}}, \vec{l}_T \right) =: (l^+, l^-, \vec{l}_T)$$

Introduce a scaling parameter $\lambda \ll 1$.

$$\bar{p}^\mu = (E, -E, \vec{0}) \qquad (-k^\mu) = \left(y_0 E - \frac{k_0^- + \lambda^2 k_0^+}{\sqrt{2}}, -y_0 E - \frac{-k_0^- + \lambda^2 k_0^+}{\sqrt{2}}, -\lambda \vec{k}_{T,0} \right) \quad x = \lambda^2 x_0$$

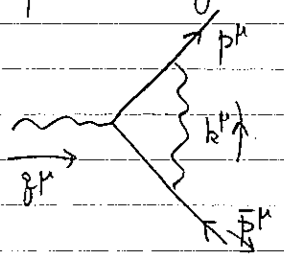
$$\Longrightarrow (\bar{p}+k)^2 - m_f^2 \simeq O(\lambda^2) \qquad k^2 \simeq O(\lambda^2) \qquad \text{so } \underline{D(x,y,k) \sim O(\lambda^2)}.$$

- $dx\,dy\,d^4k \sim [dk_0^-\, dy_0]\,[dk^+ d\vec{k}_T\, dx \sim d^3\Omega\, d\lambda\, \lambda^5]$.

$$\text{so} \quad dy_0\, dk_0^- \int d^3\Omega \left(\int \frac{d\lambda\, \lambda^5}{\lambda^6} \sim \text{log divergence} \right)$$

$\boxed{\text{Recap}}$

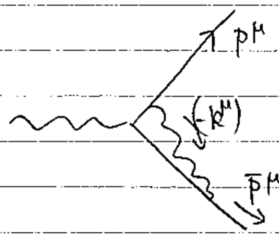
soft divergence



$k^\mu \approx 0$ (soft)

$$\left.\begin{array}{l} \dfrac{\cancel{(\not{p}-\not{k})}+m}{(p-k)^2-m_f^2} \\[2ex] \dfrac{-\cancel{(\not{\bar{p}}+\not{k})}+m}{(\bar{p}+k)^2-m_f^2} \end{array}\right\} \Rightarrow \boxed{\text{nearly on-shell (small virtuality)}}$$

collinear divergence (for massless fermion)



$$\left\{\begin{array}{l} (-k^\mu) \\ [\bar{p}-(-k^\mu)]^\mu \end{array}\right. \text{almost parallel, almost on-shell} \Leftarrow \boxed{\text{nearly on-shell intermediate states}}$$

(collinear)