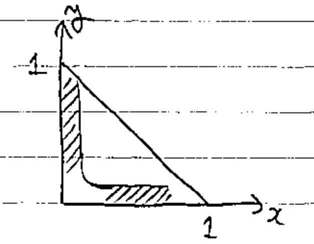
★ integral over the small-x small-y region

$$(1\text{-loop } \gamma^\mu) \simeq -2ieQ_f \frac{(Q_f e)^2}{16\pi^2}\int dx\,dy\,\frac{q^2-2m_f^2}{(x+y)^2 m_f^2 - xy\,q^2}$$



even for $m_f \neq 0$. $\int_0 d\lambda\, \lambda \frac{1}{\lambda^2}$ log. divergence.

Even after removing the UV divergence by renormalization,

$\boxed{\text{divergence remains in the amplitudes.}}$

Origin of those divergences

$$\Gamma^\mu_{(1)} \sim \int \frac{d^4k}{(2\pi)^4}\int dx\,dy\,dz\, \frac{2\delta(x+y+z-1)\ \overbrace{[p^\nu, k^\nu, \bar{p}^\nu, m_f \text{ etc.}]}^{(\text{no } x, y \text{ here})}}{\{x[(p-k)^2-m_f^2]+y[(\bar{p}+k)^2-m_f^2]+z\,k^2\}^3} \qquad \{\ldots\}^3 =: D(k)^3$$

If there is any divergence from integral over a $|k| <$ finite region,

there must be (x_*, y_*, z_*), k^μ_* where

- $D((x,y,z)_*, k^\mu_*) = 0$
- $\left[\frac{\partial}{\partial k^\mu} D\right]((x,y,z)_*, k^\mu_*) = 0$

} double root of $D(k^\mu)=0$
otherwise the integration contour can be deformed to avoid $D(k)=0$.

- $(p-k_*)^2 - m_f^2 = 0$ or $x_*=0$ or $x_*=1$
- $(\bar{p}+k_*)^2-m_f^2=0$ or $y_*=0$ or $y_*=1$
- $k^2=0$ or $z_*=0$ or $z_*=1$

} so that $D(k)=0$ cannot be avoided by deforming the integration contour of $x, y, (z)$

OK to drop this option ($x_*=1 \Rightarrow y_*=z_*=0 \Downarrow D(k)=0$, $(p-k_*)^2-m_f^2=0$ automatic)

(x_*, y_*, z_*, k^ν_*) satisfying all the conditions above

$\Longrightarrow$ $\boxed{\text{pinch (hyper) surface}}$

☆ $\boxed{x_* = y_* = 0 \ \& \ k_*^2 = 0}$ $\quad x_*(k_*-p)^\mu + y_*(k_*+\bar p)^\mu + z_* k_*^\mu = 0 \Longrightarrow \boxed{k_*^\mu = 0}$

Introduce a scaling parameter $\lambda \ll 1$ in the (x, y, k^ν) space around $(x_*, y_*, k_*^\nu) = (0, 0, 0^\nu)$.

$(x, y, k^\nu) \sim (\lambda x_0, \lambda y_0, \lambda k_0^\nu)$

- $dx\,dy\,d^4k \sim d^5\Omega\, d\lambda\, \lambda^5$
- $$[D(x,y,k)]^3 \cong \Big(\lambda x_0 \cdot \underbrace{[(p-k)^2-m_f^2]}_{\underbrace{(p^2-m_f^2)}_{=0} - \underbrace{2p\cdot k}_{\sim\lambda} + \underbrace{k^2}_{\sim\lambda^2}} + \lambda y_0 \cdot \underbrace{[(\bar p+k)^2-m_f^2]}_{\underbrace{(\bar p^2-m_f^2)}_{=0} + \underbrace{2\bar p\cdot k}_{\sim\lambda} + \underbrace{k^2}_{\sim\lambda^2}} + 1\cdot \underbrace{k^2}_{\sim\lambda^2 k_0^2}\Big)^3 \sim \lambda^6$$

so $\displaystyle\int d^5\Omega \left(\int_0 \frac{d\lambda\,\lambda^5}{\lambda^6} \sim \text{log divergence}\right)$

associated with $k^\nu \sim (k_*^\nu = 0^\nu)$ soft $\left\{\begin{matrix}\text{photon}\\ \text{gluon}\end{matrix}\right\}$.

☆ $\boxed{x_* = 0 \ \& \ (\bar p + k)^2 - m_f^2 = 0 \ \& \ k^2 = 0}$ $\quad \overset{\nearrow 0}{x_*}(k_*-p)^\mu + y_*(k_*+\bar p)^\mu + z_* k^\mu = 0$

$$\Longrightarrow \frac{(-k_*^\mu)}{y_*} = \frac{\bar p^\mu - (-k_*)^\mu}{(1-y_*)}$$

$k_*^\mu \parallel [\bar p - (-k_*)]^\mu$ and both on-shell
$\longrightarrow$ possible only if $m_f = 0$.

(photon/gluon collinear to a massless $\bar f$.)

$\left\{\begin{matrix}(-k_*^\mu) = y_*(\bar p^\mu)\\ [\bar p - (-k_*)]^\mu = (1-y_*)(\bar p^\mu)\end{matrix}\right\}$

light cone components of a four vector

$$\left(\frac{l^0+l^3}{\sqrt2}, \frac{l^0-l^3}{\sqrt2}, \vec l_T\right) =: (l^+, l^-, \vec l_T)$$

Introduce a scaling parameter $\lambda \ll 1$.

$\bar p^\mu = (E, -E, \vec 0)$ $\quad (-k^\mu) = \left(y_0 E - \frac{k_0^- + \lambda^2 k_0^+}{\sqrt2}, -y_0 E - \frac{-k_0^- + \lambda^2 k_0^+}{\sqrt2}, -\lambda \vec k_{T,0}\right)$ $\quad x = \lambda^2 x$

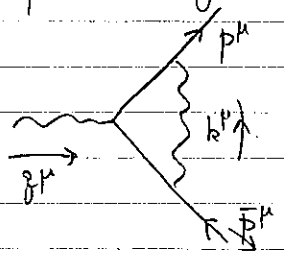
$\Longrightarrow (\bar p+k)^2 - m_f^2 \simeq O(\lambda^2)$ $\quad k^2 \simeq O(\lambda^2)$ $\quad$ so $D(x,y,k) \sim O(\lambda^2)$

- $dx\,dy\,d^4k \sim [dk_0^-\, dy_0]\,[dk^+ d^2\vec k_T\, dx \sim d^3\Omega\, d\lambda\, \lambda^5]$

so $\displaystyle dy_0\, dk_0^- \int d^3\Omega \left(\int \frac{d\lambda\,\lambda^5}{\lambda^6} \sim \text{log divergence}\right)$

$\boxed{\text{Recap}}$

soft divergence

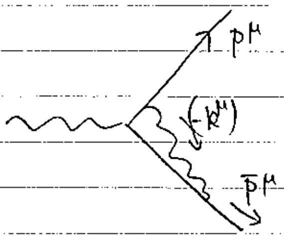


$k^\mu \approx 0$

(soft)

$$\left.\begin{array}{l} \dfrac{(\not{p}-\not{k})+m}{(p-k)^2-m_f^2} \\[2ex] \dfrac{-(\not{\bar{p}}+\not{k})+m}{(\bar{p}+k)^2-m_f^2} \end{array}\right\} \Rightarrow \boxed{\begin{array}{c}\text{nearly on-shell} \\ \text{(small virtuality)}\end{array}}$$

collinear divergence (for massless fermion)



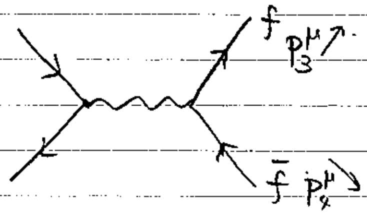
$$\left\{\begin{array}{ll} (-k^\mu) & \text{almost parallel} \\ {[\bar{p}-(-k^\mu)]^\mu} & \text{almost on-shell} \end{array}\right. \Leftarrow \boxed{\begin{array}{c}\text{nearly on-shell} \\ \text{intermediate} \\ \text{states}\end{array}}$$

(collinear)

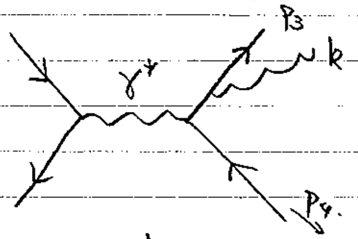
§5.2 Divergence in Real Emissions

$e^+ + e^- \longrightarrow f + \bar{f} + \gamma$ (or $q + \bar{q} + g$) in QED / QCD

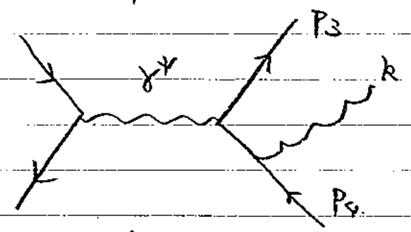
LO : 2-body final state.



NLO : 3-body final states



+



interfere in $\mathcal{M}$

$$(-ieQ_f)\,\bar{u}(\vec{p}_3)\left[-ieQ_f\gamma^\lambda\right]\frac{i\left[(\not{p}_3+\not{k})+m\right]}{(p_3+k)^2-m^2+i\varepsilon}\gamma^\mu v(\vec{p}_4)\varepsilon_\lambda(k)$$

$\boxed{\text{eikonal approximation}}$

numerator : $\cancel{(\not{p}_3+\not{k})}+m \Rightarrow \not{p}_3$

denominator : $(p_3^2-m^2)+2p_3\cdot k+k^2 = 2p_3\cdot k$.

$$\bar{u}(\vec{p}_3)\cdot\gamma^\lambda\not{p}_3 = \bar{u}\{\gamma^\lambda,\not{p}_3\} - \underbrace{\bar{u}(\vec{p}_3)\not{p}_3}_{0 \text{ (Dirac eq)}}\gamma^\lambda$$

$$\simeq 2\bar{u}(\vec{p}_3)\,p_3^\lambda$$

$$\Rightarrow -ieQ_f\left[\bar{u}(\vec{p}_3)\gamma^\mu v(\vec{p}_4)\right]\times eQ_f\frac{p_3^\lambda\varepsilon_\lambda}{(p_3\cdot k+i\varepsilon)}$$

$$\Rightarrow -ieQ_f\left[\bar{u}(\vec{p}_3)\gamma^\mu v(\vec{p}_4)\right]\times(-eQ_f)\frac{p_4^\lambda\varepsilon_\lambda}{(p_4\cdot k+i\varepsilon)}$$

So,

$$\sum_{s(\gamma)}\left|\mathcal{M}(\to f\bar{f}\gamma)^\lambda\varepsilon_\lambda(k)\right|^2 = \left[\sum_{s(\gamma)}(eQ_f)^2\times\left(\frac{p_3^\lambda}{p_3\cdot k}-\frac{p_4^\lambda}{p_4\cdot k}\right)\varepsilon_\lambda^s(k)\,\varepsilon_\kappa^s(k)^*\left(\frac{p_3^\kappa}{p_3\cdot k}-\frac{p_4^\kappa}{p_4\cdot k}\right)\right]\times\left|\mathcal{M}(\to f\bar{f})\right|^2$$

$$= \left[(eQ_f)^2\left(\frac{p_3^\lambda}{(p_3\cdot k)}-\frac{p_4^\lambda}{(p_4\cdot k)}\right)(-\eta_{\kappa\lambda})\left(\frac{p_3^\kappa}{p_3\cdot k}-\frac{p_4^\kappa}{p_4\cdot k}\right)\right]\times\left|\mathcal{M}(\to f\bar{f})\right|^2$$

$$\simeq \boxed{2(eQ_f)^2\frac{(p_3\cdot p_4)}{(p_3\cdot k)(p_4\cdot k)}\left|\mathcal{M}(\to f\bar{f})\right|^2}$$

~~In the case of $(\to q\bar{q}g.)$ final state.~~

~~replace~~ $\boxed{(eQ_f)^2}$ ~~by~~ $g^2\times\mathrm{Tr}_{\text{color}}\left[t^a t^b\right]\delta_{ab} = g^2\times\left(\overset{C_2\cdot N_c =}{\frac{4}{3}\times 3} = \frac{1}{2}\times 8\right) = \boxed{4g^2}$

No.
Date

$$\sigma(e^+e^- \to f\bar{f}\gamma) \propto \int \frac{d^3\vec{p}_3}{(2\pi)^3}\frac{1}{2E_{p_3}} \int \frac{d^3\vec{p}_4}{(2\pi)^4}\frac{1}{2E_{p_4}} \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k}(2\pi)^4\delta^4(p_{in}-p_{out})|\mathcal{M}|^2$$

$$\simeq \sigma(\to f\bar{f}\gamma)\times \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k}\frac{2(eQ_f)^2(p_3\cdot p_4)}{(p_3\cdot k)(p_4\cdot k)}$$

In the center of mass frame.

$$p_3^\mu = E_3(1,\vec{v}) \qquad p_4^\mu = E_4(1,\vec{v}') \qquad k^\mu = k(1,\vec{n})$$

$\vec{v}' \simeq -\vec{v}$ (almost back to back) $\quad \dfrac{(p_3\cdot p_4)}{(p_3\cdot k)(p_4\cdot k)} \simeq \dfrac{1}{k^2}\dfrac{(1-\vec{v}\cdot\vec{v}')}{(1-\vec{n}\cdot\vec{v})(1-\vec{n}\cdot\vec{v}')}$

$$\boxed{\begin{aligned}\sigma(e^+e^-\to f\bar{f}\gamma) &\simeq \sigma(e^+e^-\to f\bar{f})\times\frac{eQ_f^2}{(2\pi)^3}\int_0\frac{dk}{k}\int d^2\Omega_{\vec{n}}\frac{(1-\vec{v}\cdot\vec{v}')}{(1-\vec{n}\cdot\vec{v})(1-\vec{n}\cdot\vec{v}')}.\\ \sigma(e^+e^-\to q\bar{q}g) &\simeq \sigma(e^+e^-\to q\bar{q})\times\frac{g^2C_2}{(2\pi)^3}\int_0\frac{dk}{k}\int d^2\Omega_{\vec{n}}\frac{(1-\vec{v}\cdot\vec{v}')}{(1-\vec{n}\cdot\vec{v})(1-\vec{n}\cdot\vec{v}')}.\end{aligned}}$$

For a regin $\vec{n}$ almost parallel to $\vec{v}$ $\quad \left(|\vec{v}| \sim 1-\dfrac{m^2}{2E^2}\right)$

$$\frac{(eQ_f)^2}{(2\pi)^3}\int\frac{dk}{k}\,2\pi\cdot\int^1 d(\cos\theta)\frac{1}{1-\cos\theta\cdot\left(1-\frac{m^2}{2E^2}\right)} \simeq \frac{(eQ_f)^2}{(2\pi)^2}\int\frac{dk}{k}\ln\left(\frac{2E^2}{m^2}\right)$$

$$\boxed{\underset{\substack{|k_\gamma|\ll\sqrt{s},\\ \vec{n}_\gamma \text{ almost} /\!/ \vec{p}_3}}{\sigma(\to f\bar{f}\gamma)} \simeq \sigma(\to f\bar{f})\times\frac{(eQ_f)^2}{4\pi^2}\int\frac{dk}{k}\times\ln\left(\frac{s}{2m^2}\right)}$$

log div $\qquad$ large log

collinear divergence $\quad$ γ emission $/\!/$ to f, $\quad$ " $/\!/$ to $\bar{f}$ $\Big\}$ if $m_f \ll \sqrt{s} \Leftarrow \left\{\begin{aligned}\frac{1}{p_3\cdot k} &\sim \frac{1}{0}\\ \frac{1}{p_4\cdot k} &\sim \frac{1}{0}\end{aligned}\right.$

soft divergence $\quad$ propagator is nearly onshell $\forall \vec{n}$.

§ ~~10~~ 6 Cancellation of IR divergence.

Observation

- soft divergence : massless particle (γ) arbitrarily low energy

  $\Rightarrow$ can we see it?

- collinear divergence : kinematically possible for massless particles.

$$p^\mu \to \lambda_1 p^\mu + (1-\lambda_1) p^\mu \to \lambda_1 p^\mu + \lambda_2 p^\mu + (1-\lambda_1-\lambda_2) p^\mu \to \cdots$$

$$\to \sum_i (\lambda_i p^\mu) \quad \text{so that } \left(\sum_i \lambda_i = 1\right)$$

can we distinguish them?

$|\vec{k}_\gamma| \ll \sqrt{s}$, $\vec{k}_\gamma \parallel \vec{p}_e$ or $\vec{p}_{\bar{e}}$ part of $\sigma(\to e\bar{e}\gamma)$ should be treated as a part of $\sigma(\to e\bar{e})$.

Look at the collinear part

$$\left\{\begin{aligned}
&\bullet\ \sigma(\to e\bar{e}) \simeq \sigma(\to e\bar{e})_{\text{tree}} \times \left|1 - \frac{e^2}{8\pi^2}\int dy \frac{1-y}{y} \ln\left(\frac{-s}{m_e^2}\right)\right|^2 \\
&\qquad \simeq \sigma(\to e\bar{e})_{\text{tree}} \times \left[1 - \frac{e^2}{4\pi^2}\int dy \frac{1}{y} \ln\left(\frac{s}{m_e^2}\right)\right] \quad \left(\overset{\text{approx.}}{1-y \simeq 1.}\right) \\
&\bullet\ \sigma(\to e\bar{e}\gamma) \simeq \sigma(\to e\bar{e})_{\text{tree}} \times \left[+\frac{e^2}{4\pi^2}\int \frac{dk}{k} \ln\left(\frac{s}{2m_e^2}\right)\right]
\end{aligned}\right.$$

$$\ln\left(\frac{1}{2m_e^2/s}\right) \Rightarrow \ln\left(\frac{1-\cos\theta_*}{2m_e^2/s}\right) + \ln\left(\frac{1}{1-\cos\theta_*}\right)$$

$$\boxed{\sigma(\to e\bar{e}\gamma)_{|k|>k_*,\, \theta>\theta_*} \simeq \sigma(\to e\bar{e}) \times \left[\frac{e^2}{4\pi^2}\int_{k_*} \frac{dk}{k} \ln\left(\frac{1}{1-\cos\theta_*}\right)\right]} \Leftarrow \text{positive}$$

$$\boxed{\begin{aligned}&\sigma(\to e\bar{e}) \\ &+ \sigma(\to e\bar{e}\gamma)_{|k|>k_*,\, \theta_*>\theta}\end{aligned} \simeq \sigma(\to e\bar{e}) \times \left[1 - \frac{e^2}{4\pi^2}\int_{2k_*/\sqrt{s}} \frac{dy}{y} \ln\left(\frac{2}{1-\cos\theta_*}\right)\right]} \quad \text{negative correcti}$$

§6.2 Cutkosky rule

☆ Unitarity of the S-matrix

$$S_{\beta\alpha} =: (\mathbb{1}_{\beta\alpha} + i\mathcal{M}_{\beta\alpha}), \qquad \boxed{S^\dagger S = \mathbb{1}}$$

$$\Longleftrightarrow (\mathcal{M}^\dagger)_{\gamma\beta}(\mathcal{M})_{\beta\alpha} + i\mathcal{M}_{\gamma\alpha} - i(\mathcal{M}^\dagger)_{\gamma\alpha} = 0$$

$$\Longleftrightarrow \boxed{\left[\mathcal{M}_{\gamma\alpha} - (\mathcal{M}_{\alpha\gamma})^*\right]/i = \sum_\beta (\mathcal{M}_{\beta\gamma})^*(\mathcal{M}_{\beta\alpha})}$$

$\boxed{\mathcal{M} \sim e^{in(k)kx} \quad \frac{1}{L} = 2\,\mathrm{Im}[n(k)]\,k}$ $\quad e^{-i\omega t}$ $\quad \boxed{\times n(k)}$ $\quad \boxed{\frac{1}{T} = 2\,\mathrm{Im}(n) \times \omega}$

$$\left(\text{for } |\gamma\rangle = |\alpha\rangle : \quad 2\,\mathrm{Im}(\mathcal{M}_{\alpha\alpha}) = \sum_\beta |\mathcal{M}_{\beta\alpha}|^2 \quad \text{optical thm}\right)$$

☆ Analyticity of scattering amplitudes (graph by graph)

$$\left[i\mathcal{M}^\Gamma_{\gamma\alpha}(s+i\varepsilon) - i\mathcal{M}^\Gamma_{\gamma\alpha}(s-i\varepsilon)\right] \overset{ⓐ}{=} \sum_{\substack{\text{any}\\ \text{cut in the}\\ s\text{-channel}}} \left(i\mathcal{M}^{\Gamma_R}_{\gamma\beta}\right)\left(i\mathcal{M}^{\Gamma_L}_{\beta\alpha}\right)$$

$$\overset{ⓑ}{=} \sum_{\text{any cut}} -\left(i\mathcal{M}^{\overleftrightarrow{\Gamma_R}}_{\beta\gamma}\right)^*\left(i\mathcal{M}^{\Gamma_L}_{\beta\alpha}\right)$$

[discontinuity in the s-channel] is from propagators.

ⓐ: See Sterman's lecture note appendix B (time-ordered perturbation theory is used)

Contributions on the RHS of ⓐ come only from cuts where all the cut propagators can be on-shell simultaneously.

$\Longrightarrow$ in a cut graph, replace all the cut propagators by

$$\frac{i}{p^2 - m^2 + i\varepsilon} \longrightarrow \left\{\frac{i}{(p^2-m^2+i\varepsilon)} - \frac{i}{(p^2-m^2-i\varepsilon)}\right\} \simeq (2\pi)\delta^+(p^2-m^2) = \frac{(2\pi)}{(2E_{\vec p})}\delta(E - E_{\vec p})$$

for use in the RHS of ⓐ

ⓑ: • $\mathcal{L}_{int} \supset c_j \mathcal{O}_j$ is use for $(i\mathcal{M}^\Gamma_{\gamma\beta}) \Longleftrightarrow \mathcal{L}_{int} \supset c_j^* \mathcal{O}_j^\dagger$ is used for $(i\mathcal{M}^{\overleftrightarrow{\Gamma}}_{\beta\gamma})$

• $\underset{\text{in } (i\mathcal{M})}{[\#\text{ of extra}(\pm i)\text{'s}]} = (\# I) + (\# V) + (\# L) \underset{\substack{\boxed{\text{mod } 2}\\ (\mathbb{R}\text{ or not})}}{\equiv} \#(\text{connected components}) = 1$

So, $\boxed{\left[\mathcal{M}^{\Gamma}_{\gamma\alpha}(s+i\varepsilon) - \mathcal{M}^{\Gamma}_{\gamma\alpha}(s-i\varepsilon)\right]/i = \sum_{\substack{\text{any cut}\\ \text{in } s\text{-ch}}} \left(\mathcal{M}^{(\overleftrightarrow{\Gamma_R})}_{\beta\gamma}\right)^* \left(\mathcal{M}^{\Gamma_L}_{\beta\alpha}\right)}$

computed by the Cutkowsky rule

(sum over all possible cuts
and the replacement of all the cut propagators)

sum over graphs

$$\boxed{\left[\mathcal{M}_{\gamma\alpha}(s+i\varepsilon) - \mathcal{M}_{\gamma\alpha}(s-i\varepsilon)\right]/i = \sum_{\beta} (\mathcal{M}_{\beta\gamma})^* \mathcal{M}_{\beta\alpha}}$$

Combine both the unitarity and the analyticity to obtain

$$\boxed{\left[\mathcal{M}_{\gamma\alpha}(s+i\varepsilon) - \mathcal{M}_{\gamma\alpha}(s-i\varepsilon)\right]/i = \left[\mathcal{M}_{\gamma\alpha} - (\mathcal{M}_{\alpha\gamma})^*\right]/i}$$

they are not necessarily $\mathbb{R}$-valued if $|\gamma\rangle \neq |\alpha\rangle$.