

## § 6.2 Cutkosky rule

### \* Unitarity of the S-matrix

$$S_{\beta\alpha} = (\mathbb{1}_{\beta\alpha} + iM_{\beta\alpha}), \quad \boxed{S^\dagger S = \mathbb{1}}$$

$$\Leftrightarrow (M^\dagger)_{\beta\gamma} (M)_{\gamma\alpha} + iM_{\beta\alpha} - i(M^\dagger)_{\beta\alpha} = 0$$

$$\Leftrightarrow \boxed{[M_{\beta\alpha} - (M_{\beta\alpha})^*] / i = \sum_{\beta'} (M_{\beta\beta'})^* (M_{\beta'\alpha})}$$

$$M \sim e^{-iN(k)\alpha} \quad \begin{matrix} \times N(k) \\ \frac{1}{T} = 2\text{Im}(N(k)) \end{matrix}$$

$$\left( \text{for } |\beta\rangle = |\alpha\rangle : 2\text{Im}(M_{\alpha\alpha}) = \sum_{\beta'} |M_{\beta\alpha}|^2 \text{ optical thm} \right)$$

### \* Analyticity of scattering amplitudes (graph by graph)

$$[iM_{\beta\alpha}^{\beta'}(s+i\epsilon) - iM_{\beta\alpha}^{\beta'}(s-i\epsilon)] \stackrel{\text{a}}{=} \sum_{\text{any cut in the } s\text{-channel}} (iM_{\beta\beta'}^{\beta'}) (iM_{\beta'\alpha}^{\beta'})$$

[discontinuity in the s-channel]

is from propagators.

$$\stackrel{\text{b}}{=} \sum_{\text{any cut}} - (iM_{\beta\beta'}^{\beta'})^* (iM_{\beta'\alpha}^{\beta'})$$

①: See Stermann's lecture note appendix B (time-ordered perturbation theory is used)

Contributions on the RHS of ① come only from cuts where all the cut propagators can be on-shell simultaneously

⇒ in a cut graph, replace all the cut propagators by

$$\frac{i}{p^2 - m^2 + i\epsilon} \rightarrow \left\{ \frac{i}{p^2 - m^2 + i\epsilon} - \frac{i}{p^2 - m^2 - i\epsilon} \right\} \approx (2\pi) \delta^+(p^2 - m^2)$$

for use in the RHS of ① (2\pi) \delta(E - E\_p)

②:  $\mathcal{L}_{\text{int}} \supset c_j \phi_j$  is use for  $(iM_{\beta\beta'}^{\beta'}) \Leftrightarrow \mathcal{L}_{\text{int}} \supset c_j^* \phi_j^\dagger$  is used for  $(iM_{\beta'\alpha}^{\beta'})$

$$\bullet \left[ \# \text{ of extra } (\pm i)'s \text{ in } (iM) \right] = (\#I) + (\#V) + (\#L) \equiv \#(\text{connected components}) \pmod{2} = 1$$

(R or not)

$$\text{So, } \left[ M_{\alpha\alpha}^P(s+i\epsilon) - M_{\alpha\alpha}^P(s-i\epsilon) \right] / i = \sum_{\text{any cut in } s\text{-ch}} \left( M_{\beta\gamma}^{(P_R)} \right)^* \left( M_{\beta\alpha}^{P_L} \right)$$

computed by the Cutkosky rule

(sum over all possible cuts and replacement of all the cut propagators)

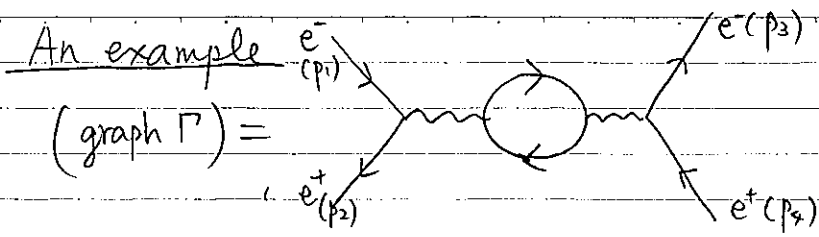
sum over graphs

$$\left[ M_{\alpha\alpha}(s+i\epsilon) - M_{\alpha\alpha}(s-i\epsilon) \right] / i = \sum_{\beta} \left( M_{\beta\gamma} \right)^* M_{\beta\alpha}$$

Combine both the unitarity and the analyticity to obtain

$$\left[ M_{\alpha\alpha}(s+i\epsilon) - M_{\alpha\alpha}(s-i\epsilon) \right] / i = \left[ M_{\alpha\alpha} - \left( M_{\alpha\alpha} \right)^* \right] / i$$

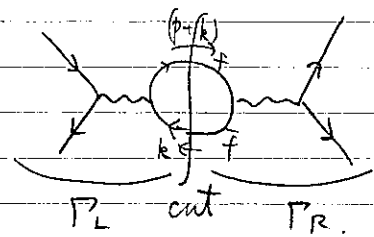
↑  
they are not necessarily  $\mathbb{R}$ -valued if  $|\alpha\rangle \neq |\alpha\rangle$ .



$$iM_{\Gamma} = \left. \begin{aligned} & (+ie [\bar{u}(\vec{p}_3) \gamma^\mu u(\vec{p}_1)]) \\ & (-ie [\bar{v}(\vec{p}_2) \gamma^\nu v(\vec{p}_4)]) \end{aligned} \right\} \left( \frac{-i}{s+i\epsilon} \right)^2 iS \eta_{\mu\nu} \Pi_{ren}^{(1)}(s)$$

• single photon in the s-channel cannot be on-shell.

• The only possible cut of this graph is



$$f: (P_{cm} + k)^\mu = (\sqrt{s} + k^0, \vec{k})$$

$$\bar{f}: (-k)^\mu = (-k^0, -\vec{k})$$

both are on-shell  $\Leftrightarrow (-k^0) = \sqrt{|\vec{k}|^2 + m_f^2}$  and  
 $\sqrt{s} + k^0 = \sqrt{|\vec{k}|^2 + m_f^2}$

The discontinuity  $[iM(s+i\epsilon) - iM(s-i\epsilon)]$  can arise

only from the discontinuity  $[i\Pi_{ren}^{(1)}(s+i\epsilon) - i\Pi_{ren}^{(1)}(s-i\epsilon)]$ .

⊙ Cutkosky rule

$$[i\Pi_{ren}^{(1)}(s+i\epsilon) - i\Pi_{ren}^{(1)}(s-i\epsilon)] \cdot S \eta_{\mu\nu} \sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{2E_f} \frac{1}{2E_{\bar{f}}} (2\pi)^2 \delta(\sqrt{s} + k^0 - E_f) \delta(-k^0 - E_{\bar{f}})$$

$$(-) \cdot \text{Tr} [\gamma^\mu (\not{k} + m_f) \gamma^\nu (\not{P}_{cm} + \not{k} + m_f)] (-ieQ_f)^2$$

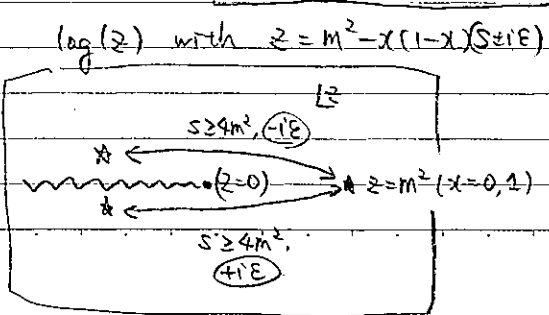
$$= \frac{-e^2 Q_f^2}{S (2\pi)^2} \int d^3k \frac{1}{|\vec{k}|} \int d^3k \delta(\sqrt{s} - 2E_k) \not{k} \not{k} \not{P}_{cm} \not{k}$$

$$\sim \frac{-e^2 Q_f^2}{S (2\pi)^2} \frac{|k| E_k}{2} \not{k} \not{P}_{cm} \not{k} \sim \left[ -(4\pi) \frac{S + 2m_f^2}{3} \eta^{\mu\nu} \right]$$

$$= \boxed{+ \frac{e^2 Q_f^2}{\pi} \frac{S + 2m_f^2}{6S} \sqrt{1 - \frac{4m_f^2}{S}}} \times S \eta_{\mu\nu}$$

⊙ We have computed directly.

$$\Pi_{ren}^{(1)}(s+i\epsilon) = \frac{e^2 Q_f^2}{2\pi^2} \int_0^1 dx \ x(1-x) \ln \left( \frac{m_f^2 - x(1-x)(s+i\epsilon)}{m_f^2 + x(1-x)s} \right)$$



$$[i\Pi_{ren}^{(1)}(s+i\epsilon) - i\Pi_{ren}^{(1)}(s-i\epsilon)] = \frac{e^2 Q_f^2}{2\pi^2} i \cdot (2\pi i) \int_{x_-}^{x_+} dx \ x(1-x)$$

$$= \frac{e^2 Q_f^2}{\pi} \left[ \frac{(x_+^2 - x_-^2)}{2} - \frac{(x_+^3 - x_-^3)}{3} \right] = \boxed{\frac{e^2 Q_f^2}{\pi} \frac{S + 2m_f^2}{6S} \sqrt{1 - \frac{4m_f^2}{S}}}$$

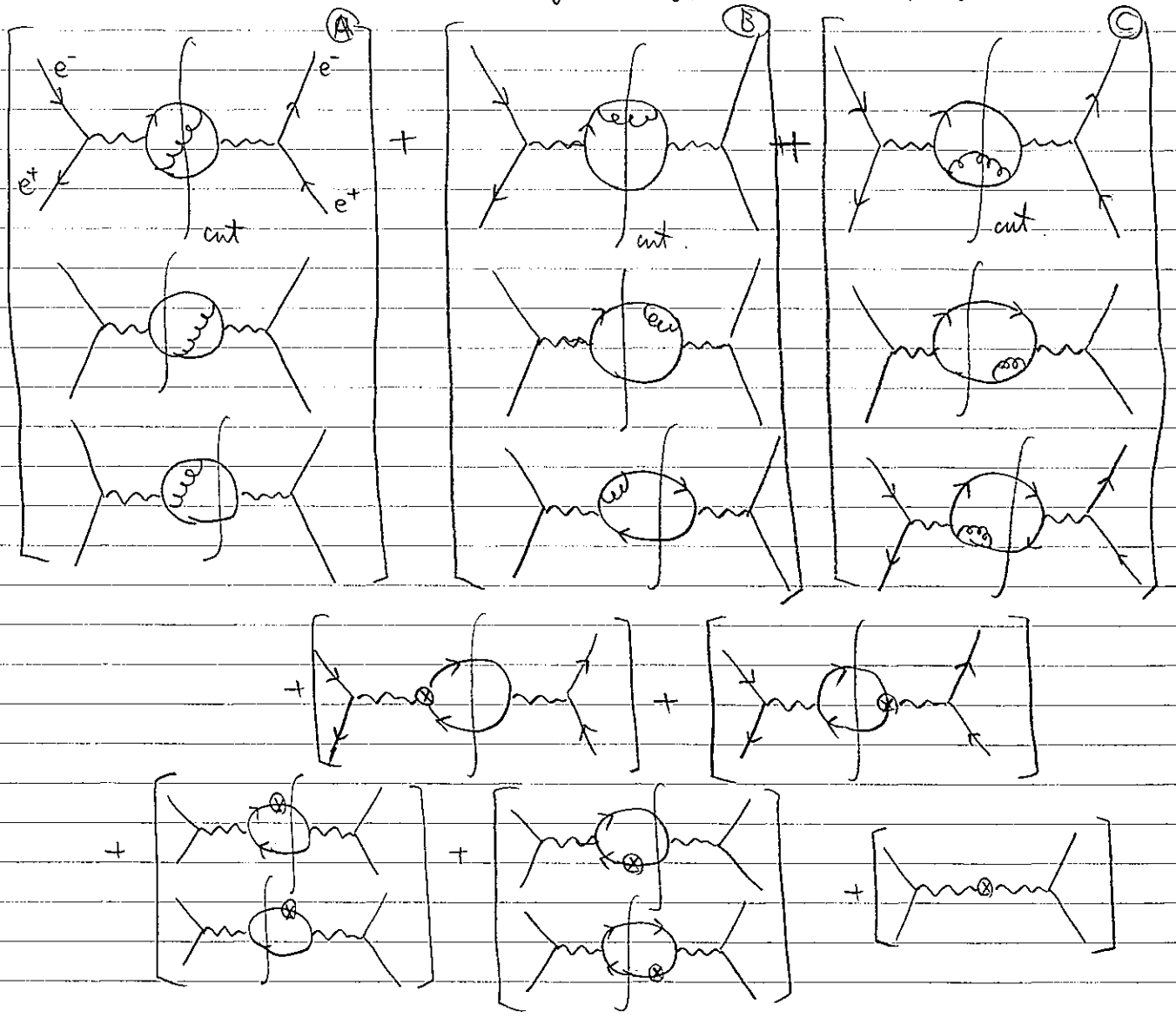
$$x_{\pm} = \frac{1}{2} \left( 1 \pm \sqrt{1 - \frac{4m_f^2}{S}} \right)$$

The optical theorem (+ analyticity)

$$\sigma(e^+e^- \rightarrow \text{anything}) = \frac{1}{2S} \left( \frac{M_{\alpha\alpha} - M_{\alpha\alpha}^*}{i} \right) = \frac{-1}{2S} (iM_{\alpha\alpha}(S+i\epsilon) - iM_{\alpha\alpha}(S-i\epsilon))$$

$M(e^+e^- \rightarrow \text{blob} \rightarrow e^+e^-)$  at  $\mathcal{O}(e^4 \sim \alpha_e^2)$   $\Rightarrow \sigma(e^+e^- \rightarrow f\bar{f})$  tree level

$M(e^+e^- \rightarrow \text{blob} \rightarrow e^+e^-)$  at  $\mathcal{O}(e^4 g^2 \sim \alpha_e^2 \alpha_g)$   $\Rightarrow \sigma(e^+e^- \rightarrow f\bar{f}g)$  NLO

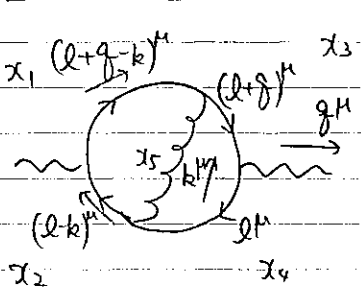


The discontinuity in

$$(iM_{\alpha\alpha})(S \pm i\epsilon) @ \mathcal{O}(e^4 g^2) \sim \alpha_e^2 \alpha_g \iff \left[ \begin{array}{l} \sigma(e^+e^- \rightarrow f\bar{f}g) \text{ incl. } \alpha_s \text{ correction} \\ + \\ \sigma(e^+e^- \rightarrow f\bar{f}g) \end{array} \right]$$

$$\left[ \begin{array}{l} \exists? \\ \text{well-defined limit} \\ \text{even if } m_g = m_q = 0 \end{array} \right] \iff \left[ \begin{array}{l} \exists? \\ \text{well-defined limit even if } m_g = 0 \\ m_q = 0 \end{array} \right] \text{ PLUS}$$

Does  $i\mathcal{M} = \text{[diagram of a loop with a bubble]} \text{ have a well-defined } m_f = m_f = 0 \text{ limit?}$



**Step 1** (A)

- for each  $i$ : either  $x_i = 0$  or  $(l_i^2 - m_i^2) = 0$
- for each loop momentum  $l^M$

$$\frac{\partial}{\partial l_a^M} \left( \sum_i x_i (l_i^2 - m_i^2) \right) \stackrel{D}{=} 0 \quad (*)$$

$\Leftrightarrow$  pinch singularity surface

**Step 2** power counting around a pinch surface

$$D = x_1 (l+q-k)^2 + x_2 (l-k)^2 + x_3 (l+q)^2 + x_4 l^2 + x_5 k^2$$

eg.  $| x_1 \sim 0 \quad x_2 \sim 0 \quad x_3 \sim 0 \quad x_4 \sim 0 \quad (x_5 \sim 1) \& k^2 \sim 0 |$

(\* for  $l^M$ )  $\Rightarrow$  trivial      (\* for  $k^M$ )  $\Rightarrow x_5 k^M = 0$

power counting  $k^M \sim \mathcal{O}(\lambda)$ ,  $x_{1,2,3,4} \sim \mathcal{O}(\lambda^2)$

measure  $\Rightarrow d^4k d^4l \sim \mathcal{O}(\lambda^4 \cdot \lambda^4 = \lambda^8)$

denominator  $\Rightarrow D^5 \sim \mathcal{O}(\lambda^{2 \cdot 5} = \lambda^{10})$

$\int_0^\infty \frac{d\lambda^R}{\lambda^{10}}$  is finite

eg.  $| x_1 \sim 0 \quad x_2 \sim 0, (l+q)^2 \sim 0, l^2 \sim 0, k^2 \sim 0 |$

(\* for  $k^M$ )  $\Rightarrow x_5 k^M = 0$       (\* for  $l^M$ )  $\Rightarrow x_3 (l+q)^M + x_4 l^M = 0$

Impossible:  $(l+q)^M$  and  $l^M$  are parallel, light-like and  $(l+q)^M - l^M = q^M$  is time-like.

So, such a component of the pinch surface is absent.

Complete analysis:  $\mathcal{M}_{\text{box}}^{(A)}$  has a well-defined limit, and so do  $\mathcal{M}^{(B)}$ ,  $\mathcal{M}^{(C)}$ .

2-loop calculation  $\sigma_{\text{tot}} \approx \frac{4\pi\alpha_e^2}{3S} (N_c Q_f^2) \left[ 1 + C_2 \frac{3\mathcal{A}_S(M)}{4\pi} + \dots \right]$

$$\left[ \begin{aligned} \mathcal{O}(\rightarrow \overline{q}q) &\approx \frac{4\pi\alpha_e^2}{3S} (N_c Q_f^2) \left( 1 - C_2 \frac{\alpha_S}{\pi} \left[ \int_{\epsilon}^1 \frac{d\overline{y}}{\overline{y}} \ln\left(\frac{S}{m_f^2}\right) + \int_{\epsilon}^1 \frac{d\overline{x}}{\overline{x}} \ln\left(\frac{S}{m_f^2}\right) + \int_0^\epsilon \frac{d\overline{x}}{\overline{x}} \frac{d\overline{y}}{\overline{y}} \right] + (\text{finite}) \right) \\ \mathcal{O}(\rightarrow \overline{q}q) &\approx \frac{4\pi\alpha_e^2}{3S} (N_c Q_f^2) \left( C_2 \frac{\alpha_S}{\pi} \left[ \int_{m_f}^S \frac{dk}{k} \ln\left(\frac{S}{m_f^2}\right) + \int_{m_f}^S \frac{dk}{k} \ln\left(\frac{S}{m_f^2}\right) + \int_0^{m_f/S} \frac{d\overline{x}}{\overline{x}} \frac{d\overline{y}}{\overline{y}} \right] + (\text{finite}) \right) \end{aligned} \right] \text{ PLUS}$$

6.3  
~~§10.3~~ OPE and Non-perturbative corrections.

$e^-e^+ \rightarrow q\bar{q}g$  etc. = hadron production in fact.

In which sense is perturbative QCD approach justified for  $s \gg \Lambda_{QCD}^2$ ?

The optical thm: insertion of a complete system of the Hilbert space  
 either in  $q, \bar{q}$  or in hadron.  
 does not rely on perturbation.

But  $\sigma_{tot}(s) \propto e^2 [i\pi(s+i\epsilon) - i\pi(s-i\epsilon)]$  and we need to evaluate

$$(2\pi)^2 \delta^4(q-q') i(q^2 \eta^{\mu\nu} - q^\mu q^\nu) \pi(q^2+i\epsilon)$$

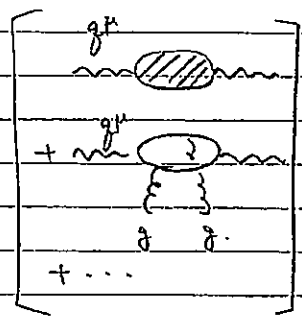
$$= (-ie)^2 \iint \langle 0 | T \{ J^\mu(x) J^\nu(y) \} | 0 \rangle e^{iq \cdot x} e^{-iq' \cdot y} d^4x d^4y$$

Remember that the operator product expansion (OPE) expands

$$(-ie)^2 \int d^4x T \{ J^\mu(x) J^\nu(0) \} e^{iq \cdot x} = i(q^2 \eta^{\mu\nu} - q^\mu q^\nu) \pi(q^2+i\epsilon) \mathbb{1}$$

$$+ i( \quad = \quad ) \times \left[ \mathcal{O} \left( \frac{1}{q^2 \Lambda^2} \right) \right] \text{tr}(F_{\mu\nu} F^{\mu\nu})(0)$$

$$+ \dots$$



\* Is OPE defined at non-perturbative level?  
 maybe if local operators are defined at non-perturbative level.

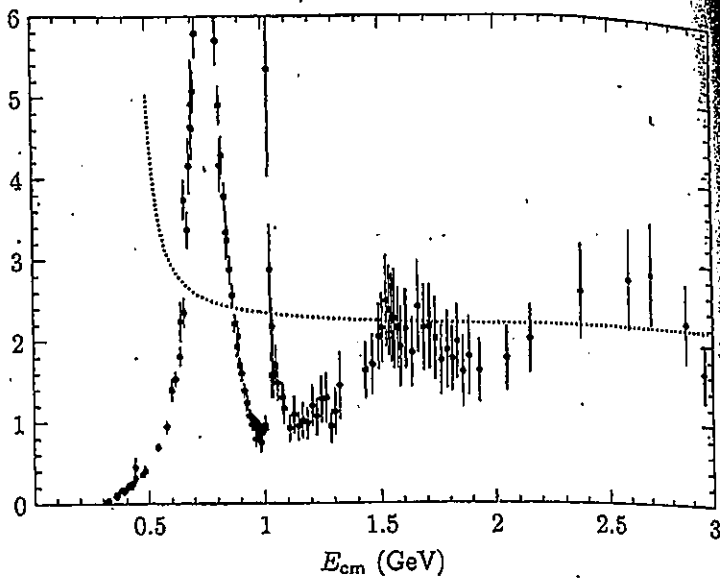
\* The perturbative computation  $\pi_{pert}(q^2)$  for  $\pi(q^2)$  may have zero convergence radius in  $\alpha_s$

still  $\left\{ \left( \text{Borel resummation of } \pi_{pert} \right) + \text{minimum non-P contributions to add} \right\}$   
 may become well-defined.  $=: [\pi_{pert}]_B$

\* also quite likely that  $\langle 0 | \text{tr}(F_{\mu\nu} F^{\mu\nu}) | 0 \rangle \sim (\Lambda_{QCD})^4 \neq 0$ .

This will give rise to (yet another) contribution supp. relatively by  $\left( \frac{\Lambda_{QCD}^4}{s^2} \right)_{\text{P.L.S}}$

from Peskin-Schroeder  $\frac{\sigma(e^+e^- \rightarrow \text{hadron})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} =: R(s)$  data vs theory  
 $\left[ \sigma(e^+e^- \rightarrow \mu^+\mu^-) = \frac{4\pi\alpha_e^2}{3s} \right]$



5.2  $e^+e^- \rightarrow \mu^+\mu^-$ : Helicity Structure  
 $R_0 = \sum_f N_c Q_f^2$   $K \approx 1 + \frac{\alpha_s(p)}{\pi} + \dots$

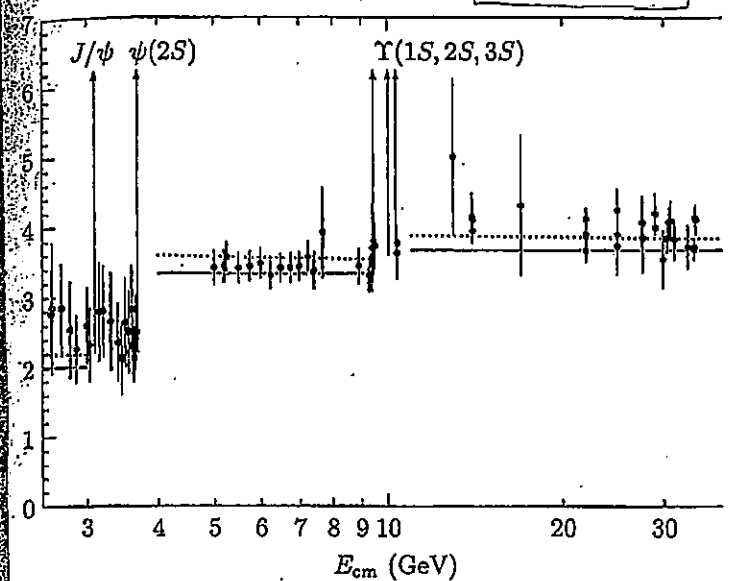


Figure 18.8. Experimental measurements of the total cross section for the reaction  $e^+e^- \rightarrow \text{hadrons}$  at energies below 3 GeV, compared to the prediction of perturbative QCD for 3 quark flavors. The data are taken from the compilation of M. Swartz, *Phys. Rev. D* 53, 5268 (1996). Complete references to the various experiments are given there.

Figure 5.3. Experimental measurements of the total cross section for the reaction  $e^+e^- \rightarrow \text{hadrons}$ , from the data compilation of M. Swartz, *Phys. Rev. D* 53, 5268 (1996). Complete references to the various experiments are given there. The measurements are compared to theoretical predictions from Quantum Chromodynamics, as explained in the text. The solid line is the complete prediction (5.16).

functions, which reflect the form of the proton wavefunction and are determined by soft QCD dynamics. However, we saw in Section 17.5 that effects of higher-order perturbation theory cause the parton distributions to change their form as a function of the momentum scale  $Q$ .

$e^+e^- \rightarrow \mu^+\mu^-$ : Helicity Structure

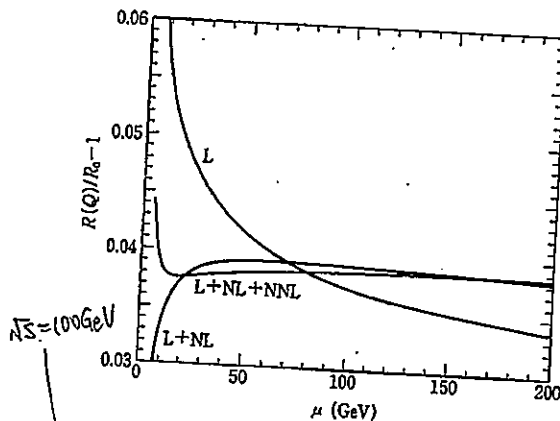


図 5.6  $R = \frac{\sigma(e^+e^- \rightarrow \text{ハドロン})}{\sigma(e^+e^- \rightarrow \mu^+\mu^- : \text{QED})}$  の QCD 効果のスケール依存性  
 図には  $R(Q=100\text{GeV}) - 1$ ,  $R_0 = \frac{11}{3}$  を書いてある。  
 L = 対数第 1 近似 (leading log approximation), NL = 第 2 近似 (next-to-LLA),  
 NNL = 第 3 近似 (next next-to-LLA).

a reaction is generally easy to c

By using  $\alpha_s(\mu)$  with  $\mu \sim \sqrt{s}$   
 $K^L$  can be a good approximation.

from Nagashima p. B9.