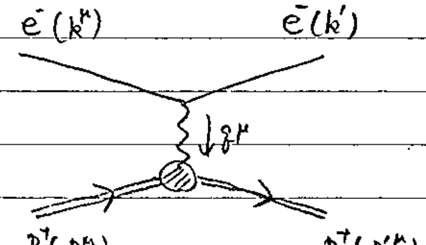
§ 7 Parton Distribution and Collinear Factorization

§ 7.1 Deep Inelastic Scattering

☆ experimental foundation for QCD (around 1970)

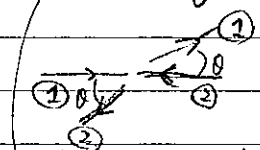
☆ another example of IR safe observable in QCD.

• $e^- + p^+ \longrightarrow e^- + p^+$ elastic scattering ⇒ ~~the same set of particles in the in-state and the out-state.~~



In 2 particle → 2 particle scatterings

(elastic processes in the center of mass frame) ⟺



just angle changes. no change in $|\vec{p}|$, E

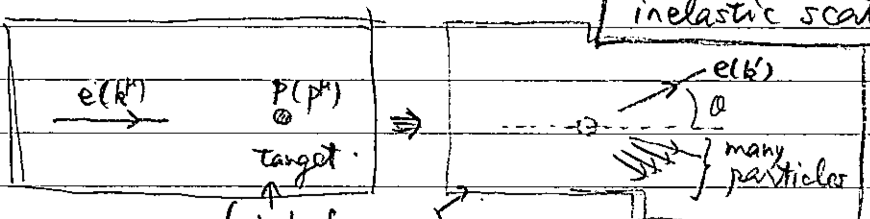
$$(p+q)^2 = (p')^2 = p^2$$

$$\Rightarrow \cancel{2(p\cdot q) + q^2 = 0} \longrightarrow \left(\frac{-q^2}{2p\cdot q} = 1\right)$$

$$\frac{d\sigma}{d\Omega} =: \left(\frac{d\sigma}{d\Omega}\right)_{\text{point particle elastic scatt.}} \times \underbrace{F(-q^2)}_{\text{form factor}}$$

form factor. describing how much $\left(\frac{d\sigma}{d\Omega}\right)$ differs from $\left(\frac{d\sigma}{d\Omega}\right)_{\text{pt. particle}}$.

• $e(k^\mu) + p(p^\mu) \longrightarrow e(k'^\mu) + (\text{anything})$ — inelastic scattering

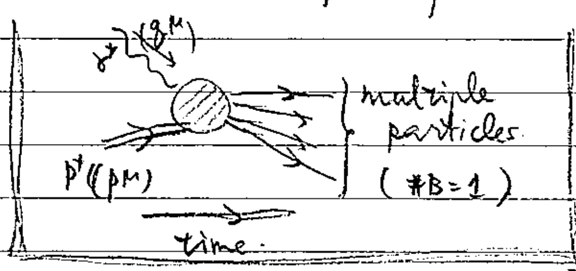


(Lab frame)

~~Three kinematical variables~~

$$\boxed{\begin{aligned} &\cdot\ s = (k+p)^2 = 2k\cdot p + m_p^2 \simeq 2k\cdot p \\ &\cdot\ (q+p)^2 = q^2 + 2q\cdot p + m_p^2 =: W^2 \\ &\cdot\ q^2 \end{aligned}}$$

The non-trivial part of this scattering



✓ $x := \frac{-q^2}{2p\cdot q}$ is not necessarily $= 1$ in inelastic scatterings.

• for a fixed s.

$$\frac{d\sigma}{dx\,dQ^2} = \frac{2\pi\alpha_e^2}{xQ^4}\left[2(1-y)\,\sigma^{(L)} + \{1+(1-y)^2\}\,\tilde{\sigma}^{(T)}\right]$$

$$\left(Q^2 := -q^2,\quad x := \frac{-q^2}{2p\cdot q}\quad y := \frac{p\cdot q}{p\cdot k}\right)$$

$\Rightarrow$ $\boxed{\sigma^{(L)} \ll \tilde{\sigma}^{(T)}}$ and $\boxed{\tilde{\sigma}^{(T)} \approx \text{fn of } x\text{, not much on } Q^2}$ Bjorken scaling

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A

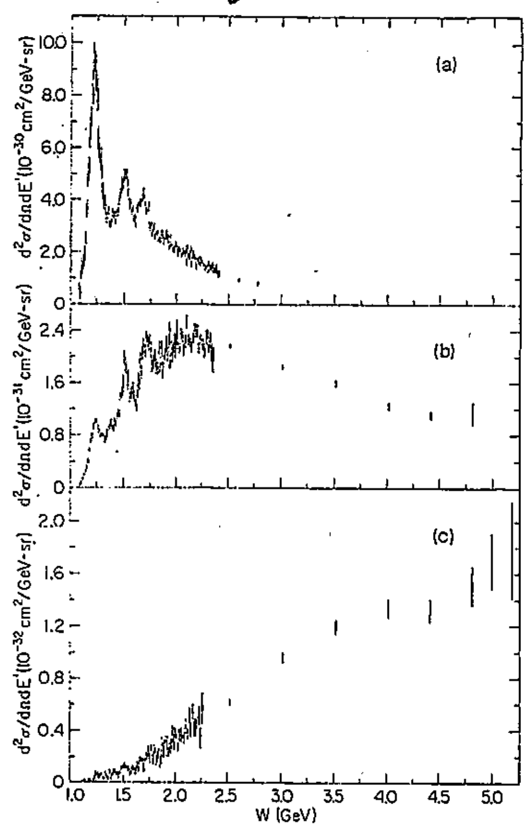

FIG. 2. Three representative radiatively corrected spectra at (a) $\theta = 6°$, $E = 7$ GeV; (b) $\theta = 6°$, $E = 16$ GeV, and (c) $\theta = 10°$, $E = 17.7$ GeV. The ranges of q^2 covered are (a) $0.2 \leq q^2 \leq 0.5$ (GeV/c)^2; (b) $0.7 \leq q^2 \leq 2.6$ (GeV/c)^2; and (c) $1.6 \leq q^2 \leq 7.3$ (GeV/c)^2. The elastic peaks are not shown.

B

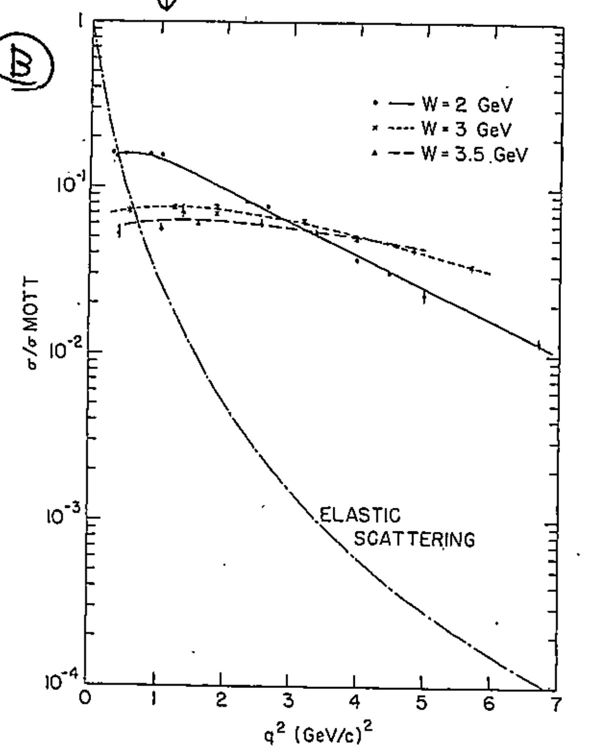

FIG. 1. $(d^2\sigma/d\Omega dE')/\sigma_{\rm Mott}$, in GeV^{-1}, vs q^2 for $W = 2$, 3, and 3.5 GeV. The lines drawn through the data are meant to guide the eye. Also shown is the cross section for elastic e-p scattering divided by $\sigma_{\rm Mott}$, $(d\sigma/d\Omega)/\sigma_{\rm Mott}$, calculated for $\theta = 10°$, using the dipole form factor. The relatively slow variation with q^2 of the inelastic cross section compared with the elastic cross section is clearly shown.

~~...ic interactions at high energies.~~ Since at only ~~cross-section measurements~~ at ~~angles are available, we are unable to~~

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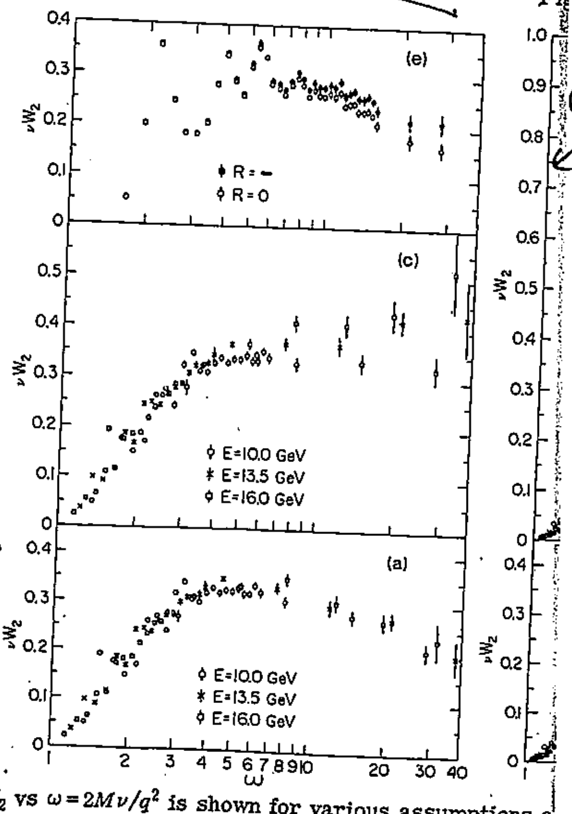

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(Fig 2).

νW_2 vs $\omega = 2M\nu/q^2$ is shown for various assumptions a $R = 0$. (b) 10° data for $R = 0$. (c) 6° data except for 7-Ge GeV spectrum for $R = 0$ and $R = \infty$.

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C

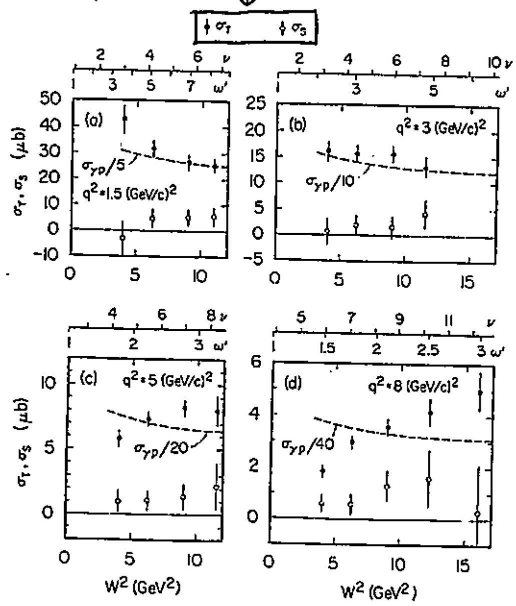
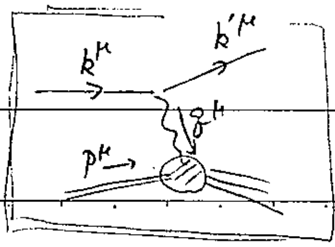


FIG. 8. The values of σ_T and σ_S given in Table III are shown at constant q^2 as a function of W^2 (or ν) for $q^2 = 1.5$, 3, 5, and 8 (GeV/c)^2. Also shown is the ν dependence of the total photoabsorption cross section.

ences ~~would point to a substantial nondiffractive~~ component of the ~~deep-inelastic~~ cross section for values ~~of ω less~~ than approximately six.

APPENDIX: RADIATIVE

A brief note on DIS



Kinematics

- $\boxed{q^2<0}$ $\because$ $k^2 = \underset{(k-q)^2}{\underset{\parallel}{(k')^2}} = m_e^2$. $\Rightarrow$ $2k\cdot q = q^2$. $\Longrightarrow$ $q^2<0$.

  In the rest frame of $e^-(k^\mu)$, $k^\mu = (m_e, \vec{0})$, $k'^\mu = (E_e, \vec{k}')$, $q^\mu = (\underbrace{m_e - E_e}_{<0}, -\vec{k}')$ $\Big\}$ $k\cdot q = m_e(m_e - E_e) < 0$.

- $\boxed{x\le 1}$ note that $\sigma = 0$ if $W^2 = (q+p)^2 < m_p^2$.

  $\Rightarrow 0 \le \dfrac{W^2 - m_p^2}{-q^2} = \dfrac{q^2 + 2p\cdot q}{-q^2} = -1 + \dfrac{1}{x}$

  ($x=1 \Leftrightarrow W = m_p \Leftrightarrow$ elastic scattering)

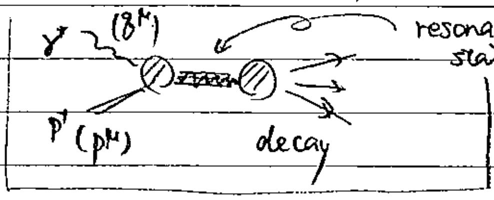
- At high energy (where m_p^2 can be ignored)

$$\boxed{s\, y\, x \simeq (2k\cdot p)\cdot\left(\frac{p\cdot q}{p\cdot k}\right)\left(\frac{Q^2}{2p\cdot q}\right) = Q^2}$$

How to read the data ?

Ⓐ instead of $\{s\sim(k+p)^2,\ W^2 = (q+p)^2,\ Q^2\}$ $\{(E=k^0),\ W,\ \theta\}$ is used. $\left(\begin{matrix}p^\mu = (m_p, \vec{0}) \\ \text{Lab. frame}\end{matrix}\right)$

In the pannel (a), many resonances are observed.



Those resonances (with a fixed mass W) dwindle as

(a) vs (b) : fixed θ. $s\nearrow$ $\approx$ $s\nearrow$ and $x \nearrow 1$. $\left(m_p^2 Q^2 = x(k\cdot p)(k'\cdot p) \times \dfrac{\tan^2(\theta/2)}{1+\tan^2(\theta/2)} \text{ is used} \right)$

(b) vs (c) : "fixed" s . $\theta\nearrow$ $\Rightarrow$ $x \nearrow 1$.

(varying Q^2 and one more parameter)

Ⓑ For a given W, data for $\left(\theta \in [6^\circ \sim 10^\circ] \quad \overset{\theta}{E_{=k^0}} \in [7 \sim 17\,\text{GeV}]\right)$
(fixed target frame)

show weak dependence on Q^2 (and also on the one more parameter).

This hints that the inelastic scattering @ $W \gg 1\,\text{GeV}$ is not due to a mechanism/object that is spatially spread.

(spatial spreading $\Rightarrow$ non-trivial form factor $\sim \text{fn}(Q^2 \underset{\text{spatial spread}}{R^2})$)

Ⓒ The non-trivial part of the DIS should depend only on $((q+p)^2 =: W^2)$ and $(q^2 = -Q^2)$, but in fact only the dimensionless ratio $\left(\frac{Q^2}{2p\cdot q} =: x =: \frac{1}{\omega}\right)$ " matters.

$\boxed{\text{~~(a) and (c)~~}}$ ~~The non-trivial part depends primarily on~~ $\cancel{\omega = 1/x}$, not so much on Q^2.

$\theta = 6^\circ$, $E_{=k^0} \in [10 \sim 16]\,\text{GeV}$

~~(a) vs (c) : different ansatz ($R = 0$ or ∞) on $\sigma^{(L)}/\tilde{\sigma}^{(T)}$ ratio.~~

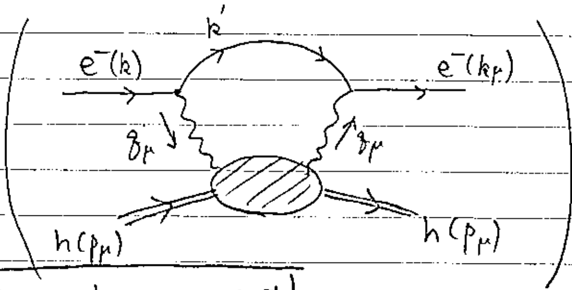
[(e) : $E_{=k^0} = 7\,\text{GeV}$. $\theta = 6^\circ$.]

Ⓒ "Fig 8" $\sigma_T \gg \sigma_L$ for $^\forall x, Q^2$.

§7.2 DIS Structure Functions

(Ignore m_e, m_p)

$$\sigma(e^- + p^+ \to e^- + \text{anything}) \simeq \frac{1}{2S}\, 2\,\mathrm{Im}\left(\mathcal{M}(e^-p^+ \to e^-p^+)\right) \qquad \text{optical thm}$$



$\boxed{\text{The unknown part}}$

$$\int d^4x \int d^4y\ (-ieQ_f)^2 \langle h(p)|\, T\{J_f^\mu(x)\, J_f^\nu(y)\}\, |h(p)\rangle\, e^{iq'\cdot x}\, e^{-iq\cdot y}$$

$\|$ translational invariance of the ME.

$$\int d^4x \int d^4y\ (-ieQ_f)^2 \langle h(p)|\, T\{J_f^\mu(0)\, J_f^\nu(y-x)\}\, |h(p)\rangle\, e^{iq'\cdot x}\, e^{-iq\cdot y}$$

$$= \underbrace{\int d^4x\, e^{i(q'-q)\cdot x}}_{(2\pi)^4\delta^4(q'-q)} \underbrace{\int d^4(y-x)\, (-ieQ_f)^2 \langle h(p)|\, T\{J_f^\mu(0)\, J_f^\nu(y-x)\}\, |h(p)\rangle\, e^{-iq\cdot(y-x)}}_{i e^2 T^{\mu\nu}(p,q)\ \text{Compton Tensor}}$$

If the e^- beam is not polarized, and the spin of $e^-(k')$ is not measured,

$$\sigma(e^-p^+ \to e^- + X) \simeq (\text{homework IX-5}) \simeq$$

$$\boxed{\begin{aligned} &\frac{1}{4p\cdot k}\int dQ^2 \int dy\ \frac{\alpha_e^2}{Q^4}\, 2\left[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')\right]\, 2\,\mathrm{Im}(T^{\mu\nu}) \\ &= \int dQ^2 \int dx\ \frac{y^2\alpha_e^2}{Q^6}\left[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')\right]\, 2\,\mathrm{Im}(T^{\mu\nu}) \end{aligned}}$$

From the gauge invariance of QED,

$$q_\mu T^{\mu\nu} = 0 \qquad T^{\mu\nu} q_\nu = 0$$

$$\Rightarrow \boxed{T^{\mu\nu} = (4\pi)\left\{\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1 + \frac{1}{(p\cdot q)}\left[p^\mu - \frac{(p\cdot q)}{q^2} q^\mu\right]\left[p^\nu - \frac{(p\cdot q)}{q^2} q^\nu\right] T_2\right\}}$$

parametrized by 2 functions

$$F_1 := 2\,\mathrm{Im}(T_1) \qquad F_2 := 2\,\mathrm{Im}(T_2)$$

Then

$$\boxed{\frac{d\sigma_{DIS}}{dQ^2 dx}} = \frac{y^2 \alpha_e^2}{Q^6}(4\pi)\left[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')\right]\left\{\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1 + \frac{1}{p\cdot q}[\quad]^\mu[\quad]^\nu F_2\right\}$$

$$= \frac{y^2\alpha_e^2}{Q^6}(4\pi)\left(\left[2(k\cdot k') + \underbrace{\left\{\frac{2(q\cdot k)(q\cdot k')}{q^2} - (k\cdot k')\right\}}_{\to 0}\right] F_1 + \frac{F_2}{(p\cdot q)}\left[\begin{array}{l} 2(p\cdot k)(p\cdot k') \\ -2\frac{p\cdot q}{q^2}\left\{(p\cdot k)(q\cdot k') + (p\cdot k')(q\cdot k) - (p\cdot q)\underbrace{(k\cdot k')}_{\to 0}\right\} \\ + \frac{(p\cdot q)^2}{(q^2)^2}\underbrace{\left\{2(q\cdot k)(q\cdot k') - q^2(k\cdot k')\right\}}_{\to 0}\end{array}\right]\right)$$

use $0 \simeq (k')^2 = (k-q)^2 \simeq -2k\cdot q + q^2 \Rightarrow k\cdot q \simeq q^2/2$ (Ward-Takahashi id.)

$q^2 = (k-k')^2 \simeq -2k\cdot k' \Rightarrow k\cdot k' \simeq -q^2/2$

$$= \boxed{\frac{4\pi\alpha_e^2}{Q^4}\left(y^2 F_1 + \frac{(1-y)}{x} F_2\right)}$$

reorganized

$$= \boxed{\frac{4\pi\alpha_e^2}{Q^4}\left[\frac{(1-y)}{x}(F_2 - 2xF_1) + \{1+(1-y)^2\} F_1\right]}$$

$F_1(x, Q^2)$, $F_2(x, Q^2)$: structure functions

(experimental data) $\Rightarrow$

- Bjorken scaling: F_1, F_2 depend primarily on x not on Q^2
  $\longleftrightarrow$ point-like constituent.
- Callan-Gross relation: $(F_2 - 2xF_1) \ll F_2$
  $\Longleftrightarrow$ spin-1/2 constituent (quark)

$$\overline{|\mathcal{M}|^2} \propto \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} \propto \frac{(k\cdot p)^2 + (k'\cdot p)^2}{Q^4} = \frac{(k\cdot p)^2}{Q^4}\{1+(1-y)^2\}$$

$(e f_{1/2} \to e f_{1/2})$ (homework IX-3)

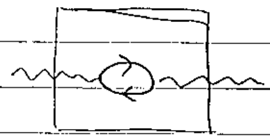
§ 7.3 Evaluation of $T^{\mu\nu}$ by OPE

$$T^{\mu\nu} := iQ_f^2 \int d^4y \; \langle h(\vec{p}) | T\{J_f^\mu(0)\, J_f^\nu(y)\} | h(\vec{p})\rangle \, e^{-iq\cdot y} \quad \text{w/ space-like } q^2<0.$$

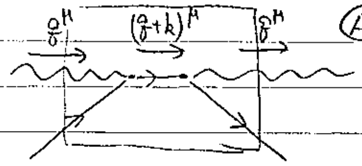
$$\left(\text{instead of } \Pi^{\mu\nu} := ie^2Q_f^2 \int d^4y \; \langle \Omega | T\{J_f^\mu(0)\, J_f^\nu(y)\} | \Omega\rangle \, e^{-iq\cdot y} \quad \text{w/ } q^2>0 \right)$$

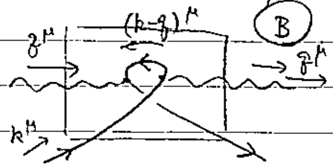
vacuum polarization

$$i\int d^4y \; e^{-iq\cdot y} \, T\{J^\mu(0) J^\nu(y)\} \qquad \text{OPE}$$

- begins with $(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\Pi_{ren}(q^2)\,\mathbb{1}_{operator}$ 

But $\mathrm{Im}(\Pi_{ren}(q^2)) = 0$ if $q^2<0$.

- 

$+$ 

$$(A):\; i\int d^4y \; e^{-iq\cdot y} \int \frac{d^4k}{(2\pi)^4}\, \bar\psi(-\tfrac{y}{2})\, \gamma^\mu \frac{i(\cancel{q}+\cancel{k})\, e^{i(k+q)\cdot y}}{(q+k)^2+i\varepsilon}\gamma^\nu \psi(+\tfrac{y}{2})$$

$$= (-)\int d^4y \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot y}\, e^{\frac{y}{2}\cdot(\vec\partial - \overleftarrow\partial)} \frac{\bar\psi(0)\,\gamma^\mu(\cancel{q}+\cancel{k})\gamma^\nu\psi(0)}{(q+k)^2+i\varepsilon}$$

$\overleftrightarrow{\partial}$ $\shortparallel$ $(2\pi)^4\delta^4(k-\frac{i}{2}\underbrace{(\vec\partial-\overleftarrow\partial)})$

$$= (-)\;\bar\psi(0)\frac{\gamma^\mu\, i(\cancel{q}+\frac{i}{2}\overleftrightarrow{\cancel\partial})\gamma^\nu}{(q+\frac{i}{2}\overleftrightarrow\partial)^2+i\varepsilon}\psi(0)$$

expand $[q^2 + i q\cdot\overleftrightarrow\partial - \frac{1}{4}(\overleftrightarrow\partial)^2]$

with respect to $(q\cdot\overleftrightarrow\partial / q^2)$

(equation of motion $\cancel{D}\psi = 0 \Rightarrow \cancel{D}\cancel{D}\psi = (D^2 + \frac{ig}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu})\psi = 0$

$$(B):\; i\int d^4y \; e^{-iq\cdot y} \int \frac{d^4k}{(2\pi)^4}\, \bar\psi(+\tfrac{y}{2})\, \gamma^\nu \frac{i(\cancel{k}-\cancel{q})\, e^{-i(k-q)\cdot y}}{(k-q)^2+i\varepsilon}\gamma^\mu \psi(-\tfrac{y}{2})$$

$$= \cdots = (+)\;\bar\psi(0)\frac{\gamma^\nu(\cancel{q}-\frac{i}{2}\overleftrightarrow{\cancel\partial})\gamma^\mu}{(q-\frac{i}{2}\overleftrightarrow\partial)^2+i\varepsilon}\psi(0)$$

OPE $i\int d^4y \; e^{-iq\cdot y} \, T\left\{ J_f^\mu(-y/2) \, J_f^\nu(+y/2) \right\}$

$$= \left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right) \Pi_{\mathrm{ren.}}(q^2) \, \mathbb{1}$$

local operator

$$+ \sum_{j=2}^{\infty} C^{\mu\nu}_{\lambda_1 \cdots \lambda_j}(q) \left[\bar{\psi}_f \gamma^{\lambda_1} \left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_2} \cdots \left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_j} \psi_f \right](0) \rightarrow \boxed{\text{twist} = (2+j) - j = 2}$$

$$+ \left(\mu \leftrightarrow \nu \text{ antisymmetric part.}\right)$$

$$+ (\text{coeff.}) \times \text{operator such as } \left(\bar{\psi}_f \gamma^\kappa F^{\rho\sigma} \psi_f\right)(0) \cdots \text{twis} = 5 - 1 = 4$$

$$(\text{twist}) := (\text{naive operator dim}) - \text{spin}\left(\text{repr. of } SO(3,1)\right)$$

Insert those local operators in $\langle h(\vec{p})| \quad |h(\vec{p})\rangle$.

$$\langle h(\vec{p})| \underbrace{\left[\bar{\psi} \gamma^{\lambda_1} \left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_2} \cdots \left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_j} \psi \right]}_{\text{sym. traceless}} |h(\vec{p})\rangle =: p^{\lambda_1} p^{\lambda_2} \cdots p^{\lambda_j} \underline{A_j}$$

non-perturbative information

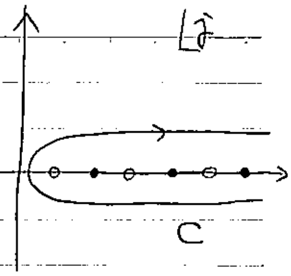
Use the explicit expressions of $C^{\mu\nu}_{\lambda_1 \cdots \lambda_j}(q)$

to obtain (homework IX-4)

$$\left(\underset{\substack{\mu\nu\text{-sym.,} \\ \text{twist-2}}}{T^{\mu\nu}} \right) = \left\{ \left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{1}{(p\cdot q)} \left[p^\mu - q^\mu \frac{p\cdot q}{q^2} \right] \left[p^\nu - q^\nu \frac{(p\cdot q)}{q^2} \right] 2x \right\}$$
$$\times \sum_{j=1}^{\infty} \left[1 + (-)^j \right] \left(\frac{1}{x} \right)^j \left(+ \frac{A_j}{2} \right) \left(Q_f^2 \right)$$

$\Rightarrow$ consistent with the Ward-Takahashi identity

$\Rightarrow$ Callan-Gross relation. $\left(F_2 = 2x F_1 \right)$



$$T_1 = \frac{+1}{8\pi} \sum_{j=1}^{\infty} [1+(-)^j] \frac{1}{x^j} A_j Q_f^2 \quad \Leftarrow \text{expansion @ } 1 \ll x$$

$$= \frac{1}{8\pi} \int_C \frac{dj}{2i} \left[-\frac{1+e^{-\pi i j}}{\sin(\pi j)} \right] \frac{1}{x^j} A_f^+(j) Q_f^2 \quad \text{continuation to } x<1.$$

- $A^+(j)$: holomorphic fcn on j. $\quad A^+(j\in 2\mathbb{N}) = A_{j\in 2\mathbb{N}}$.
- $\frac{1+e^{-\pi i j}}{\sin(\pi j)}$: pole @ $j \in 2\mathbb{Z}$ residue $= \frac{2}{\pi}$.

$$\boxed{2\,\mathrm{Im}(T_1) = \frac{1}{4\pi} \int_{-i\infty}^{+i\infty} \frac{dj}{2i} \frac{1}{x^j} A^+(j) Q_f^2}$$

$$\boxed{\int_0^{+\infty} dx \, [2\,\mathrm{Im}(T_1)](x) \, x^{j-1} = \frac{1}{4} A^+(j) Q_f^2 .}$$

Mellin transformation $\displaystyle\int_0^{+\infty} dx \, f(x) \, x^{j-1} =: \hat{f}(j)$

inverse Mellin transformation $\displaystyle f(x) = \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i} \left(\frac{1}{x}\right)^j \hat{f}(j)$

(Fourier transformation $\ln(x) \leftrightarrow j/i$.)

(Laplace transformation $\ln(x) \leftrightarrow j$)

Structure functions are given by the inverse Mellin transform of the proton matrix elements of twist-2 quark operators.