<u>OPE</u> $i\int d^4y\; e^{-iq\cdot y}\, T\{J_f^\mu(-y/2)\, J_f^\nu(+y/2)\}$

$$= (q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\Pi_{\rm ren}(q^2)\,\mathbb{1}$$

local operator

$$+\sum_{j=1}^{\infty} C^{\mu\nu}_{\lambda_1\cdots\lambda_j}(q)\left[\bar\psi_f\gamma^{\lambda_1}\left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_2}\cdots\left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_j}\psi_f\right](0) \longrightarrow \text{twist} = \boxed{(2+j)-j = 2}$$

$$+\;(\mu\leftrightarrow\nu \text{ anti symmetric part})$$

$$+\;(\text{coeff.})\times \text{operator such as } (\bar\psi_f\gamma^\kappa F^{\rho\sigma}\psi_f)(0)\;\cdots\; \underline{\text{twis} = 5-1 = 4}$$

$$(\text{twist}) := (\text{naive operator dim}) - \text{spin}\left(\text{repr. of } SO(3,1)\right)$$

Insert those local operators in $\langle h(\vec p)|\quad |h(\vec p)\rangle$.

$$\langle h(\vec p)|\underbrace{\left[\bar\psi\gamma^{\lambda_1}\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_2}\cdots\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_j}\psi\right]}_{\text{sym. traceless}}|h(\vec p)\rangle =: p^{\lambda_1}p^{\lambda_2}\cdots p^{\lambda_j}\,\underline{A_j}$$

non-perturbative information

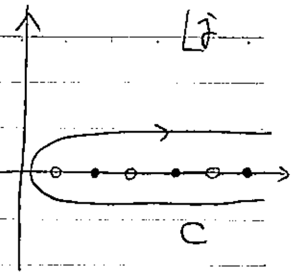
Use the explicit expressions of $C^{\mu\nu}_{\lambda_1\cdots\lambda_j}(q)$

to obtain (homework IX-4)

$$\left(T^{\mu\nu}_{\mu\nu\text{-sym, twist-2}}\right) = \left\{\left(-\eta^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)+\frac{1}{(p\cdot q)}\left[p^\mu - q^\mu\frac{p\cdot q}{q^2}\right]\left[p^\nu - q^\nu\frac{(p\cdot q)}{q^2}\right]2x\right\}$$
$$\times\sum_{j=1}^{\infty}\left[1+(-)^j\right]\left(\frac{1}{x}\right)^j\left(+\frac{A_j}{2}\right)(Q_f^2)$$

$\Rightarrow$ consistent with the Ward-Takahashi identity

$\Rightarrow$ Callan-Gross relation. $(F_2 = 2xF_1)$




$$T_1 = \frac{+1}{8\pi} \sum_{j=1}^{\infty} [1+(-)^j] \frac{1}{x^j} \sum_f A_{j,f} Q_f^2 \quad \Leftarrow \text{ expansion @ } 1 \ll x$$

$$= \frac{1}{8\pi} \int_C \frac{dj}{2i} \left[ -\frac{1+e^{-\pi i j}}{\sin(\pi j)} \right] \frac{1}{x^j} A_f^+(j) Q_f^2 \quad \text{continuation to } x<1.$$

- $A^+(j)$ : holomorphic fcn on $j$. $A^+(j \in 2\mathbb{N}) = A_{j \in 2\mathbb{N}}$.
- $\dfrac{1+e^{-\pi i j}}{\sin(\pi j)}$ : pole @ $j \in 2\mathbb{Z}$ residue $= \dfrac{2}{\pi}$.

$$\boxed{2\,\mathrm{Im}(T_1) = \frac{1}{4\pi} \int_{-i\infty}^{+i\infty} \frac{dj}{2i} \frac{1}{x^j} A_f^+(j)\, Q_f^2}$$

$$\boxed{\int_0^{+\infty} dx \left[2\,\mathrm{Im}(T_1)\right](x)\, x^{j-1} = \frac{1}{4} A_f^+(j)\, Q_f^2.}$$

Mellin transformation $\displaystyle\int_0^{+\infty} dx\, f(x)\, x^{j-1} =: \hat{f}(j)$

inverse Mellin transformation $\displaystyle f(x) = \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i} \left(\frac{1}{x}\right)^j \hat{f}(j)$

(Fourier transformation $\ln(x) \leftrightarrow j/\sqrt{-1}$.)

(Laplace transformation. $\ln(x) \leftrightarrow j$)

Structure functions are given by the inverse Mellin transform of the proton matrix elements of twist-2 quark operators.

## §7.4 Parton Distribution Function (PDF)

$$\boxed{f_q(x) := \frac{1}{4\pi}\int_{-\infty}^{+\infty} dk\ e^{ikx} \langle h(\vec{p})|\left[\bar{\psi}_q(-\tfrac{\bar{n}}{2}k)\,\bar{\not{n}}\,\psi_q(+\tfrac{\bar{n}}{2}k)\right]|h(\vec{p})\rangle}$$ quark PDF

$$f_{\bar{q}}(x) := \frac{1}{4\pi}\int_{-\infty}^{+\infty} dk\ e^{ikx} \langle h(\vec{p})|\left[\overline{(\psi_q^c)}(-\tfrac{\bar{n}}{2}k)\,\bar{\not{n}}\,(\psi_q^c)(+\tfrac{\bar{n}}{2}k)\right]|h(\vec{p})\rangle$$ anti-quark PDF

$$= \frac{-1}{4\pi}\int_{-\infty}^{+\infty} dk\ e^{ikx} \langle h(\vec{p})|\left[\bar{\psi}_q(+\tfrac{\bar{n}}{2}k)\,\bar{\not{n}}\,\psi_q(-\tfrac{\bar{n}}{2}k)\right]|h(\vec{p})\rangle$$

$$= \frac{-1}{4\pi}\int_{-\infty}^{+\infty} dk'\ e^{-ik'x} \langle h(\vec{p})|\left[\bar{\psi}_q(-\tfrac{\bar{n}}{2}k')\,\bar{\not{n}}\,\psi_q(+\tfrac{\bar{n}}{2}k')\right]|h(\vec{p})\rangle = -f_q(-x)$$

$$\bar{n}^\mu = \frac{\bar{P}^\mu}{(p\cdot q)} \qquad (\bar{P}^\mu : \text{light like vector with } P\cdot\bar{P} = p\cdot q)$$

$k$ : dimensionless parameter for integration

- (Wilson line, gauge-invariance etc. : we will come back later)
- For a quark with momentum $k^\mu$ collinear to $P^\mu$,

$$e^{\pm i\frac{\bar{n}}{2}\kappa\cdot k} \sim e^{\pm i\frac{\bar{n}'}{2}\kappa\cdot k} \quad \text{so long as } \bar{n}^- = (\bar{n}')^-.$$

So, it is OK to use $\bar{n}'^\mu := \frac{q^\mu}{(p\cdot q)}$ instead of $\bar{n}^\mu$.

For even $j$,

$$\int_0^{+\infty} dx\, x^{j-1}\{f_q(x)+f_{\bar{q}}(x)\} = \frac{1}{2}\int_{-\infty}^{+\infty} dx\, x^{j-1}\{f_q(x)+f_{\bar{q}}(x)\}$$

$$= \int_{-\infty}^{+\infty}\frac{dx}{2}\int_{-\infty}^{+\infty}\frac{dk}{4\pi}\left[(-i\partial_k)^{j-1}e^{ikx}\right]\begin{pmatrix}\langle\vec{p}|[\bar{\psi}(-\frac{\bar{n}}{2}k)\,\bar{\not{n}}\,\psi(+\frac{\bar{n}}{2}k)]|\vec{p}\rangle \\ -\langle\vec{p}|[\bar{\psi}(+\frac{\bar{n}}{2}k)\,\bar{\not{n}}\,\psi(-\frac{\bar{n}}{2}k)]|\vec{p}\rangle\end{pmatrix}$$

$$= \frac{1}{4}\left[(i\partial_k)^{j-1}\left(\langle\vec{p}|[-\cdots+]|\vec{p}\rangle - \langle\vec{p}|[+\cdots-]|\vec{p}\rangle\right)\right]\Bigg|_{k=0}$$

$$= \frac{1}{4}\langle\vec{p}|\left[\bar{\psi}\,\bar{\not{n}}\left(\tfrac{\bar{n}}{2}\cdot i\overleftrightarrow{\partial}\right)^{j-1}\psi\right]|\vec{p}\rangle\left[1-(-)^{j-1}\right]$$

$$= \frac{1}{2}A_j. \qquad (\bar{n}\cdot p = 1)$$

So, $$F_1(x) \stackrel{\text{tree}}{=} \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i} \frac{1}{x^j} \frac{1}{4}\left(\sum_f Q_f^2 A_f^+(j)\right)$$

$$\boxed{\left\{f_f(x) + f_{\bar{f}}(x)\right\} = \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i} \frac{1}{x^j} \frac{1}{2} A_f^+(j)}$$

$(f = u, d, s, \cdots)$

$$\Rightarrow \boxed{F_1(x) \stackrel{\text{tree}}{=} \frac{1}{2} \sum_f Q_f^2 \left\{f_f(x) + f_{\bar{f}}(x)\right\}}$$

At $O(\alpha_s)$ gluon PDF also contributes to the DIS structure functions

(The Callan–Gross relation $F_2 = 2x F_1$ no longer holds.)

**parton model.** (idea)

replace $\langle h(\vec p)| \;\cdots\; |h(\vec p)\rangle$

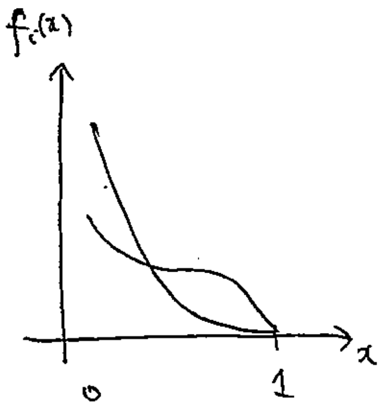
by $\sum_i \int_0^1 \frac{dz}{z} f_{q_i}(z) \langle q_i(z\vec p)| \cdots |q_i(z\vec p)\rangle$

$q_i = u, d, \cdots$
$\quad\; u^c, d^c \cdots$

$\Longrightarrow$ direct computation of $2\,\mathrm{Im}\,T_1 \equiv F_1$, $2\,\mathrm{Im}\,T_2 \equiv F_2$

$\rightarrow F_1 = \frac{1}{2} Q_q^2 \left[ f_q(x) + f_{\bar q}(x) \right]$

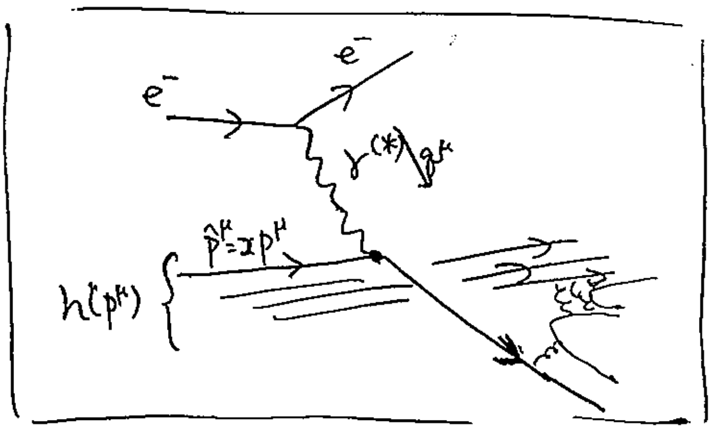
(homework).




hadron : parton w/ longitudinal momentum fraction. $x$. $f_{q_i}(x)$

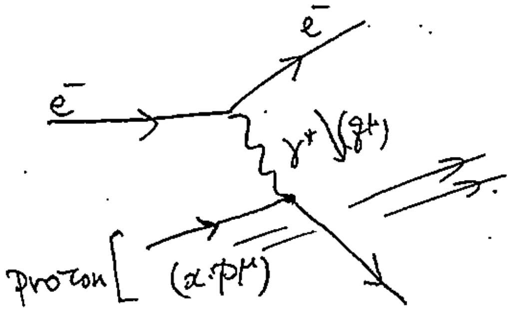
other partons : remain unimportant.

if $(-q^2) \gg \Lambda_{QCD}^2$

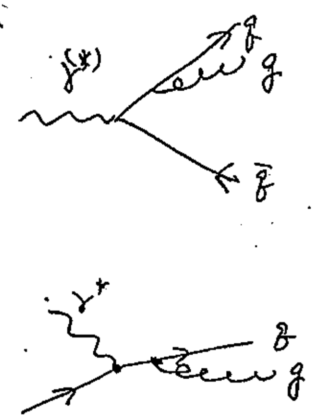


## § ~~11~~ 7.5 Initial State Radiation (ISR)

(cf. FSR )




DIS



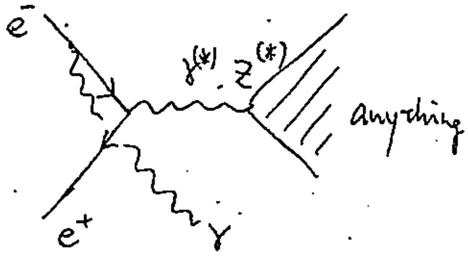

(s-channel)

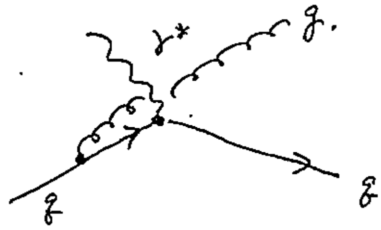
(t-channel)

FSR
↓
soft & collinear divergence.

ISR

(s-channel) (t-channel)




- ☆ divergence for the same reason as in FSR.
- ☆ un-observable (as in FSR) because the radiation goes down the beam pipe.

Does divergence cancel in observables? (in DIS)

- QED: see Peskin-Schroder. § 6.5
- QCD: ? confinement. hadron...