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1. Quantization of quadratic Hamiltonians

1. The quantiz. formalism.

H : vector space , $(\,,\,)$: non-deg. bilinear pairing

$$\mathcal{H} = H((z^{-1})) = \left\{ \sum_{n=N_0}^{\infty} f_n z^n \;\middle|\; f_n \in H \right\}$$

$$\Omega(f(z), g(z)) = \operatorname{res}_{z=0} (f(-z), g(z))\, dz \qquad \text{symplectic form}$$

$\{\phi_i\}_{i=1}^N$: basis of H and $\{\phi^j\}_{j=1}^N$: dual basis

$$(\phi_i, \phi^j) = \delta_i^j .$$

$$\mathcal{H} \ni f(z) = \sum_{k=0}^{\infty} \sum_{i=1}^{N} p_{k,i} (-z)^{-k-1} \phi^i + \sum_{k=0}^{\infty} \sum_{i=1}^{N} q_k^i \phi_i z^k$$

coordinates on $\mathcal{H}_+ = H[z]$ Lagrangian subspace

$\{p_{k,i}, q_k^i\}$ are Darboux coordinates on $\mathcal{H}$.

Remark: We can identify $\mathcal{H} \cong T^*\mathcal{H}_+$; then

$$\Omega = \sum_{k,i} dp_{k,i} \wedge dq_k^i \quad \text{is the standard sympl. form on } T^*\mathcal{H}_+ .$$

Symplectic transformations : $S(z) = 1 + S_1 z^{-1} + S_2 z^{-2} + \cdots$

(lower-triangular) $\qquad S_k : H \to H$ linear map

$$ {}^T S(-z)\, S(z) = 1$$

transposition w/ respect to $(\,,\,)$, i.e., $({}^T S f, g) = (f, Sg)$.

They form a group whose Lie algebra ~~is~~ $\mathfrak{g}$ consists of

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$$A(z) = A_1 z^{-1} + A_2 z^{-2} + \cdots \quad , \quad A_k : H \to H \text{ linear maps}$$

s.t. ${}^{T}A(-z) + A(z) = 0$

We define quadratic Hamiltonian:

$$(1) \qquad h_A(f) = \frac{1}{2}\,\Omega(Af, f) \quad , \quad f \in \mathcal{H}$$

$$= \frac{1}{2}\operatorname{res}_{z=0}\big(A(-z) f(-z) , f(z) \big)\, dz = \frac{1}{2}\operatorname{res}_{z=0}\Big(\sum_{k=1}^{\infty} A_k (-z)^k\, p_{\ell,i}\phi^i z^{-\ell-1} , p_n$$

$$= \frac{1}{2}\operatorname{res}_{z=0} \oint \sum_{\substack{k,\ell,n \\ i,j}} \Big(A_k (-z)^{-k}\big(p_{\ell,i}\phi^i z^{-\ell-1} + q_\ell^i \phi_i (-z)^{\ell}\big) , p_{n,j}\phi^j (-z)^{-n-1} + q_n^j \phi_j z^n \Big)$$

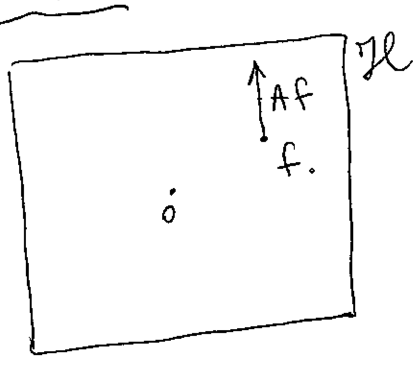
$$= \frac{1}{2}\sum_{m,\ell=0}^{\infty} (-1)^{m+1}\big(A_{m+\ell+1} q_\ell , q_m\big) + \sum_{k,\ell=0}^{\infty} (-1)^k \big(A_k p_\ell , q_{k+\ell}\big)$$

where $q_n = \sum_{i=1}^{N} q_n^i \phi_i$, $p_n = \sum_{i=1}^{N} p_{n,i}\phi^i$

Proposition 1: The map $A \mapsto h_A(f)$ is a Lie algebra homeomorphism, i.e.,

$$h_{[A,B]} = \{h_A , h_B\} \qquad \big(\{p_{k,i} , q_\ell^j\} := \delta_{k,\ell}\,\delta_i^j\big)$$

Remark2:



$f \mapsto A\cdot f$ gives a $\underbrace{\text{v.f.}}_{}$ linear X_A on $\mathcal{H}$

X_A is Hamiltonian $\iff$ A is inf. sympl. transf.

and the Hamiltonian: $dh_A = i_{X_A}\Omega$ is precisely given by (1)

Remark3: $\left(\frac{1}{z}\right)^{\wedge} = -\frac{1}{2}(q_0, q_0) - \sum_{\ell=0}^{\infty}\sum_{i=1}^{N} q_{\ell+1}^i\, p_{\ell,i}$

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Fock space: formal functions on $\mathcal{H}_+$

$$\mathbb{C}_\hbar[[q_0, q_1, q_2, \dots]], \quad \mathbb{C}_\hbar = \mathbb{C}((\sqrt{\hbar}))$$ formal Laurent series in $\hbar = 0$.

$$\mathcal{F}(\mathbf{q}) = \sum_{I=\{(k_1,i_1),\dots,(k_r,i_r)\}} a_I(\hbar)\cdot \mathbf{q}^I, \quad \mathbf{q}^I = q_{k_1}^{i_1}\cdots q_{k_r}^{i_r}$$

$\mathbf{q} := (q_0, q_1, q_2, \dots)$

$\mathbf{q}(z) := \sum_{k=0}^{\infty} q_k z^k$

We define:
$$\hat{p}_{k,i} = \sqrt{\hbar}\,\frac{\partial}{\partial q_k^i}$$
$$\hat{q}_{k,i} = q_k^i/\sqrt{\hbar}$$

we quantize h_A $\hat{A} :=$ by the above rules + normal ordering

(e.g. $(p_{k,i}\, q_\ell^j)^\wedge = q_\ell^j \frac{\partial}{\partial q_k^i}$)

This gives a projective representation of the symplectic Lie algebra:

$$[\hat{A}, \hat{B}] = ([A,B])^\wedge + C(A,B)\cdot 1$$

Lie algebra cocycle non-zero only in the following cases

where
$$C(p_a p_b, q_a q_b) = \begin{cases} 1 & \text{if } a \neq b \\ 2 & \text{if } a = b \end{cases}$$

Given a lower-triangular sympl. transformation

$$S = 1 + S_1 z^{-1} + S_2 z^{-2} + \dots = e^{A(z)}$$

we define $\hat{S} := e^{\widehat{A(z)}}$ – diff. oper. acting on the Fock space

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Given $S(z)$ define:

$$\sum_{k,l=0}^{\infty} W_{kl}\, z^{-k} w^{-l} = \frac{{}^{T}S(z)\,S(w) - 1}{z^{-1} + w^{-1}}$$

linear maps $H \to H$

Let $W(\mathbf{q},\mathbf{q}) = \sum_{k,l=0}^{\infty} (W_{kl}\, q_l, q_k)$.

<u>Thm 1.</u> If $\mathcal{F}(\hbar;\mathbf{q})$ is any function; then

$$\hat{S}^{-1} \cdot \mathcal{F} = e^{W(\mathbf{q},\mathbf{q})/2\hbar}\, \mathcal{F}([S\mathbf{q}]_+),$$

where $[\ ]_+$ truncation of terms w/ ~~negative~~ negative powers ~~degrees~~ of z, i.e.,

replace $q_0 \mapsto q_0 + S_1 q_1 + S_2 q_2 + \cdots$

$q_1 \mapsto q_1 + S_1 q_2 + S_2 q_3 + \cdots$

$\vdots$

<u>Pf</u>: Put $S = e^{A}$ and

$$G(t,\mathbf{q}) = e^{-t\hat{A}} \cdot \mathcal{F}(\mathbf{q});$$

then

$$\frac{dG}{dt} = -\hat{A} \cdot G = \frac{1}{2\hbar}\left(A(z)\mathbf{q}(z), \mathbf{q}(-z)\right)_0 G + \sum_{k,a} \left(A\phi^a(-z)^{-k-1}, \mathbf{q}(-z)\right)_0 \frac{\partial G}{\partial q_k^a}$$

take residue at $z=0$

where $\mathbf{q}(z) = \sum_{k,i} q_k^i \phi_i z^k$.

Put $g(t,\mathbf{q}) = \ln G(t,\mathbf{q})$; then

(2) $$\frac{dg}{dt} = \frac{1}{2\hbar}\left(A\mathbf{q}, \mathbf{q}(-z)\right)_0 + \sum_{k,a}\left(A\phi^a(-z)^{-k-1}, \mathbf{q}(-z)\right)_0 \frac{\partial g}{\partial q_k^a}$$

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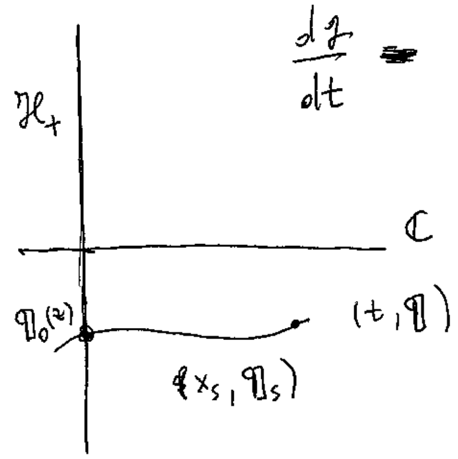
1) Solve first the homogeneous equation:

$$\frac{dg}{dt} = \sum_{k,a} \left(A\phi^a(-z)^{-k-1}, \mathbf{q}(-z)\right)_0 \frac{\partial g}{\partial q_k^a} = 0$$



characteristics:

$$\frac{dx}{ds} = 1$$

$$\frac{dq_k^a}{ds} = -\left(A\phi^a(-z)^{-k-1}, \mathbf{q}(-z)\right)_0$$

$$\frac{d\mathbf{q}}{ds} = -\left(A\phi^a(-z)^{-k-1}, \mathbf{q}(-z)\right)_0 \phi_a z^k \qquad {}^T\!A = -A(-z)$$

$$\left(\phi^a(-z)^{-k-1}, A(-z)\mathbf{q}(-z)\right)$$

$$\frac{d\mathbf{q}}{ds} = -\left[A(z)\cdot\mathbf{q}(z)\right]_+$$

$$-(-1)^\ell \left(A_\ell\phi^a, \phi_b\right) q^b_{k+\ell}\phi_a z^k$$

$$-(-1)^\ell\, {}^T\!A_\ell\, q_{k+\ell}\, z^k = -A_\ell q_{k+\ell} z^k = -\left[A(z)\cdot\mathbf{q}(z)\right]_+$$

$$\left[A(z)\cdot\left[A(z)\cdot\mathbf{q}\right]_+\right]_+ = \left[A^2\mathbf{q}\right]_+$$

$\Rightarrow$ the solutions are $x_s = s$

$$\mathbf{q}_s(z) = \left[e^{-s\cdot A(z)}\cdot\mathbf{q}_0(z)\right]_+$$

$$\mathbf{q}(z) = \left[e^{+tA(z)}\mathbf{q}_0(z)\right]_+ \quad \text{i.e.} \quad \mathbf{q}_0(z) = \left[e^{+tA(z)}\cdot\mathbf{q}(z)\right]_+$$

$$\Rightarrow g(t,\mathbf{q}) = f\left(\left[e^{+tA(z)}\cdot\mathbf{q}(z)\right]_+\right)$$

2) Check that

$$Q(t,\mathbf{q}) := \frac{1}{2\hbar}W_t(\mathbf{q},\mathbf{q}) = \frac{1}{2\hbar}\sum_{k,\ell\geq 0}^{\infty}\left(W_{k\ell}(t)q_\ell, q_k\right)$$

is a solution to (2), where

$$\sum_{k,\ell=0}^{\infty} W_{k\ell}(t) z^{-k} w^{-\ell} = \frac{e^{{}^T\!A(z)t}\, e^{A(w)t} - 1}{z^{-1}+w^{-1}}$$

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$$(W_{k'',\ell}\, q_\ell,\ A_{k'} q_k)$$

$$k'+k''=k \quad A_{k'} q_k \underbrace{z^k \cdot z^{-k'}}_{z^{k''}}$$

$$(W_{k'',\ell}\, q_\ell,\ [A\cdot \mathbf{q}]_{k''}) \qquad -\Omega(A\mathbf{q},\ \phi^a(-z)^{-k-1}) \cdot \phi_a z^k$$

$$\frac{1}{2}\left(W_t(\mathbf{q}, [A\mathbf{q}]_+) + W_t([A\mathbf{q}]_+, \mathbf{q})\right) + \frac{1}{2}(A(z)\mathbf{q}(z),\ \mathbf{q}(-z))_0$$

$$\frac{\partial}{\partial t} W_t(\mathbf{q},\mathbf{q}) = -\Omega(A\mathbf{q},\ \phi^a(-z)^{-k-1})\ \frac{\partial}{\partial q^a_k} W_t(\mathbf{q},\mathbf{q}) + \frac{1}{2}(A(z)\mathbf{q}(z),\mathbf{q}(-z))_0$$

$$(-A(-z)\,\mathbf{q}(-z),\ \phi^a(-z)^{-k-1})_0$$

$${}^TA(z)$$

$$({}^TA(z)\,\phi^a(-z)^{-k-1},\ \mathbf{q}(-z))_0 \cdot \frac{\partial}{\partial q^a_k} W_t + \frac{1}{2\hbar}(A(z)\mathbf{q},\mathbf{q}(-z))_0$$

$$\sum_{k,\ell=0}^{\infty} \frac{\partial W_{k\ell}}{\partial t} z^{-k} w^{-\ell} = {}^TA(z)\left(W_{k\ell}\, z^{-k} w^{-\ell} + \frac{1}{z^{-1}+w^{-1}}\right) +$$

$$+ \left(W_{k\ell}\, z^{-k} w^{-\ell} + \frac{1}{z^{-1}+w^{-1}}\right) A(w) =$$

$$= \left({}^TA(z)\, W_{k\ell} + W_{k\ell} A(w)\right) z^{-k} w^{-\ell} + \frac{{}^TA(z) + A(w)}{z^{-1}+w^{-1}}$$

$$\frac{-A(-z)+A(w)}{z^{-1}+w^{-1}} = \sum_{n=1}^{\infty} A_n \frac{w^{-n} - (-z)^{-n}}{w^{-1} - (-z)^{-1}}$$

$$= \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} A_n\, w^{-(n-1-i)} (-z)^{-i}$$

$$\frac{\partial W_{k\ell}}{\partial t} = \sum_{k'+k''=k} {}^TA_{k'} W_{k''\ell} + \sum_{\ell'+\ell''=\ell} W_{k\ell'} A_{\ell''} + (-1)^k A_{k+\ell+1}$$

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⇒ the general solution is

$$g(t, \mathbf{q}) = \frac{1}{2\hbar} W_t(\mathbf{q}, \mathbf{q}) + f\left(\left[e^{+tA(z)}\mathbf{q}(z)\right]_+\right)$$

$$G(t, \mathbf{q}) = e^{\frac{1}{2\hbar}W_t(\mathbf{q},\mathbf{q})} \cdot \tilde{\mathcal{F}}\left(\left[e^{+tA(z)}\mathbf{q}(z)\right]_+\right)$$

when $t=0$, $W_t = 0$ and $G(0, \mathbf{q}) = \tilde{\mathcal{F}}(\mathbf{q}) = \mathcal{F}(\mathbf{q})$ ⇒

~~At~~ at $t=1$; then the theorem follows:

$$\hat{S}^{-1}\mathcal{F} = G(1, \mathbf{q}) = e^{\frac{1}{2\hbar}W(\mathbf{q},\mathbf{q})} \mathcal{F}([S\mathbf{q}]_+)$$