Causality in Discrete Models of Quantum Spacetime

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1 Introduction

The quantization of gravity is the most fundamental and long-standing problem in theoretical physics. Although advances in string theory may eventually provide a solution, other more radical approaches are under investigation. In particular, we can consider replacing a smooth spacetime with some kind of discrete model. This raises the possibility of quantizing spacetime at the Planck scale. In the same way that general relativity makes spacetime a dynamical object in the classical sense, so quantum gravity may make it a dynamical object in the quantum sense.

Much of our understanding of string theory is perturbative. We identify ab initio a background spacetime and consider fluctuations and excitations on it. Discrete theories are fundamentally non-perturbative. Instead of picking out a privileged manifold that models the universe on the large-scale, we attempt to derive that manifold’s features from other assumptions. This represents an adoption of an analogue of Mach’s principle: in the same way that in relativity we dispose of a privileged frame of reference, so we aim to dispose of the concept of a background in quantum gravity.

If we are not allowed to assume a background spacetime, what can we assume about the universe? A natural answer is ‘causality’. But how can we implement this assumption? Where should we start from? We begin by outlining the idea of a ‘causal set’, and we proceed to describe a variety of models which can be viewed as elaborations of simple causal sets. Our models are related to each other, in terms of complexity, in the following way:

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Causal sets

Regge calculus ↓ Penrose spin networks

Dynamical triangulations ↓ Quantum spin foams
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2 Causal sets

2.1 Causality is (almost) as important as geometry

A spacetime manifold $M$ is endowed with a metric tensor $g$, which in turn yields a local causality structure. To be specific, the space of tangent vectors $X$ at any point on the manifold (which we call $T_p$) can be partitioned into timelike, spacelike and null subsets, by considering the sign of $g(X,X)$. This partitioning is what is meant by a local causality structure.

If we strip the manifold of its metric structure, leaving the local causality structure, then we might suppose that almost all geometric information is lost. In fact, almost all geometric information is retained, as is shown in the following theorem:

**Theorem.** The metric is determined up to a local conformal factor by the local causality structure.

**Proof.** For any point $p$, take a timelike vector $T \in T_p$ and a spacelike vector $S \in T_p$. Consider the equation

$$g(T + \lambda S, T + \lambda S) = g(T, T) + 2\lambda g(T, S) + \lambda^2 g(S, S) = 0.$$  

Because $g(T, T) < 0 < g(S, S)$, the equation has two real roots $\lambda_1$ and $\lambda_2$. These roots can be found if we know the local causality structure, because $\lambda$ is a root precisely when $T + \lambda S$ is null. So we can determine the value of

$$\lambda_1\lambda_2 = g(T, T)/g(S, S),$$

and so we can find the ratio between the magnitudes of a given timelike and a given spacelike vector, simply by knowledge of local causality.

We now apply this technique to give us the value of $g(U, V)$ (up to a conformal factor) for arbitrary $U$ and $V$. Observe that

$$2g(U, V) = g(U + V, U + V) - g(U, U) - g(V, V)$$

Consider the terms on the right-hand side, each of the form $g(X, X)$. $X$ is either null, timelike or spacelike. If it is null, then the term in question vanishes. If it is timelike, we can obtain the ratio between $g(X, X)$ and $g(S, S)$ as before. If it is spacelike, then we can obtain the ratio between $g(X, X)$ and $g(T, T)$, also as before. Whatever the case, we can deduce the ratio between the right-hand side and $g(T, T)$.

In short, therefore, we have found the metric tensor up to a conformal factor. (This factor is determined by our initial choice of $T$, and is given by the corresponding value of $g(T, T)$.) We say we have found the ‘conformal’ metric tensor, $g/\|\det g\|^{1/d}$.  

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1This is an amended version of the proof given in [HE73]
The metric tensor yields a local causality structure. The local causality structure gives a global causality structure, which takes the form of a poset structure on the manifold $M$:

**Definition.** $(M, \leq)$: We say that $p \leq q$ if there exists a curve joining $p$ to $q$ whose tangent is everywhere non-spacelike.

What happens if we now strip the manifold of both the metric tensor and the local causality structure, leaving only the poset structure? Can we still recover the conformal metric tensor?

First note that we still have a notion of (continuous, or smooth) curves, which is given by the natural topology on the manifold. Non-spacelike curves are curves which are chains. Spacelike curves are curves which are antichains. Using these concepts, we can reconstruct the local causality structure on the tangent spaces, and thence reconstruct the conformal metric tensor.

Finally, we are led to ask what happens if we strip the manifold of its topological structure but retain the poset structure? This question motivates the definition of a causal set, given below.

### 2.2 Definition and motivation

[BLMS87]

**Definition.** Causal set $(C, \leq)$: A locally finite partially-ordered set.

If we want to found our treatment on causality, then modelling our spacetime as a set partially-ordered by a causality relation is natural. In particular, the antisymmetry of the relation eliminates closed timelike loops. The previous section shows that, modulo suitable smoothness condition, we can reconstruct a manifold with a conformal metric. But this still leaves us with no concept of length. This is where local finiteness\(^2\) comes in. It allows us to define the size of a region simply by counting the number of points in it.

The rationale for the local finiteness condition is two-fold. Firstly, it provides a concept of scale as described. Secondly, it provides an extremely natural regularization for field theories defined on the causal set: in some sense there is a lower limit on distance, and an upper limit on momentum.

In speaking about causal sets, we often replace poset terminology with more physically-motivated terms. For instance an antichain is referred to as an ‘acausal set’. We will introduce and use similar terms without comment.

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\(^2\)Local finiteness requires that every interval in the causal set has finitely many members. An interval is a set of the form $A(x, y) := \{z \mid x \leq z \leq y\}$, sometimes called an Alexandroff set.
2.3 Reconstructing smooth geometry

The sense in which we can reconstruct ‘smooth’ geometry from a causal set is mathematically sensitive. In a sense we do not want to perform the reconstruction, because theories defined on causal sets should describe physics in an intrinsically discrete manner. But they should also match continuum theories in some limit. To this end, we define:

**Definition. Embedding of a causal set** \((C, \leq)\) **into a pseudo-Riemannian manifold** \(M\). The pseudo-Riemannian metric defines a causal structure on \(M\), making it a poset \((M, \leq)\). An embedding is simply a poset homomorphism \(C \rightarrow M\).

Some light is thrown on the issue by the following theorem, from [HKM76].

**Theorem.** Define the path topology, \(P\), to be the finest topology on a spacetime manifold which induces the natural topology of the manifold on timelike curves. Then the group of \(P\)-homeomorphisms coincides with the group of conformal diffeomorphisms.

In other words, the \(P\)-structure on the manifold is just the conformal and differential structure in another guise. But if we have a causal set (which gives us a notion of timelikeness) embedded in a manifold (which gives us a topology and thus the notion of a continuous curve), then we can define a \(P\)-structure and thence a conformal and differential structure.

Now if we identify a class of manifolds which provide suitable models of spacetime (in particular, we specify the dimension), it seems unlikely that a general causal set can be embedded into a manifold of this class. When we equip causal sets with a metric, we will be able to assign Hausdorff dimensions to them. We will then see that we sometimes get causal sets of undesirable dimension. We hope that the dynamics of our theory will select those causal sets which are embeddable in suitable manifolds.

2.4 Quantum systems on causal sets

[Mar00a, Mar00b, Mar00c]

2.4.1 First approaches

In standard quantum mechanics, the quantum state at a particular time is given by an element of a Hilbert space of wave functions defined on flat space. If we replace our spacetime with a causal set, then it might seem natural to define a quantum state as follows:

**Definition. Quantum state space for a point:** A (finite-dimensional) Hilbert space \(H_p\) associated to a point \(p \in C\).

**Definition. Quantum state space for a spacelike region:** The product \(\prod_{p \in R} H_p\) of Hilbert spaces associated to the points of a space-like region \(R\) (i.e. an acausal set in \(C\)).
How is evolution to be defined? For a smooth manifold, we would expect to be able to evolve the state of a spacelike region to obtain the state of a spacelike subregion of its future Cauchy development. Recall

**Definition.** *Future Cauchy development* $F$ of spacelike region $R$: The set of all points $p$ with the property that every past-directed non-spacelike curve passing through $p$ intersects $R$.

The analogue of ‘$F$ is a spacelike subregion of future Cauchy development of $R$’ is given in the definition

**Definition.** $R$ is a complete past of an acausal set $F$: $R$ is an acausal set such that for every point $p$ which is to the past of some event in $F$, some event in $R$ is related to $p$.

We define a complete future similarly.

Given a region $F$ and a complete past $R$ of $F$, we might hope to construct an evolution operator which maps $R \to F$. We require this operator to be unitary and so, in particular, it must be invertible. We therefore focus on evolution operators between ‘complete pairs’ of acausal sets, defined as follows

**Definition.** $A$ and $B$ form a complete pair: $A$ and $B$ are both acausal. $A$ is a complete past of $B$ and $B$ is a complete future of $A$.

We note that this relation is a partial order on the acausal sets.

However, there is a problem here: evolution operators defined between complete pairs do not respect local causality. A specific example (from [Mar00a]) is the following causal set:

![Diagram of a causal set](image)

In this situation we have an evolution operator $E : H_a \otimes H_b \to H'_a \otimes H'_b$. We proceed as follows to show that it violates causality:

- Select an arbitrary state $|\psi_b\rangle \in H_b$.
- Extend this to a state $|\psi\rangle = |\psi_a\rangle \otimes |\psi_b\rangle \in H_a \otimes H_b$ (this extension is not unique).
• Evolve $|\psi\rangle$ to give $|\psi'\rangle = E|\psi\rangle \in H'_a \otimes H'_b$.

• Take a trace to give a density matrix $\rho'_a = \text{Tr}_{H'_b}|\psi'\rangle\langle\psi'|$ acting on $H'_a$.

The fact that we can construct $\rho'_a$ implies that the system $H'_a$ is influenced by the state of $H_b$. But this conflicts with the causal relations shown in the diagram.

One solution is to only construct evolution operators between 'full pairs' where

**Definition.** A and B form a full pair: A and B form a complete pair, and every element of A relates to every element of B.

However, say we consider the evolution of two disjoint regions $R_1$ and $R_2$ to give futures $F_1$ and $F_2$ respectively. If the two evolutions do not ‘interfere’ with each other, then we should be able to speak of an evolution from $R_1 \cup R_2$ to $F_1 \cup F_2$. Unions preserve the complete pair relation, but they cannot preserve the full pair relation. So defining evolution on full pairs is not satisfactory.

### 2.4.2 Associating Hilbert spaces to relations

The solution advocated by Markopoulou in [Mar00a] is to attach Hilbert spaces to the causal relations of the causal set. To implement this idea it is helpful to speak in terms of the causal set as a directed graph (a Hasse diagram), with vertices as elements of the set and directed edges as causal relations. Then we have the following:

**Definition.** Quantum causal history: A causal set $(C, \leq)$ with a Hilbert space $H_{pq}$ attached to every edge $e_{pq}$ of its graph. At each vertex $r$ of the graph there is an operator $O_r : \prod_{p \leq r} H_{pr} \to \prod_{r \leq q} H_{rq}$.

**Definition.** $p$ covers $q$ (denoted $p \rightarrow q$): For distinct $p, q \in C$, $p \leq q$ and there is no $x$ (distinct from both $p$ and $q$) for which $p \leq x \leq q$.

**Definition.** Edge-set $(E, \leq_e)$ of $(C, \leq)$: The set of all covers $p \rightarrow q$ with $p, q \in C$. We say $(p \rightarrow q) <_e (r \rightarrow s)$ if $q \leq r$. This relation is irreflexive, transitive and trichotomous, making it a strict partial order.

As before, antichains in $(E, \leq_e)$ are referred to as acausal sets. We can define evolution operators between complete pairs of such acausal sets by composing the $O_r$ operators. These evolution operators clearly respect causality.

When we come to discuss spin foams, it will be clear that they are an example of quantum causal histories.
2.5 Sum over histories

In the quantum gravity arena, causal sets suggest the possibility of regularizing a ‘sum over geometries’ to find transition amplitudes. Specifically, given two acausal sets $A$ and $B$, we define

**Definition.** Transitions $C_{AB}$ from $A$ to $B$. $C_{AB}$ is a causal set in which $A$ and $B$ for a complete pair. Additionally, $C_{AB}$ is entirely in the future of $A$, and entirely in the past of $B$.

Now a transition amplitude $Z$ can be obtained by using evolution operators $E$:

$$Z_{AB} = \sum_{C_{AB}} \langle \psi_A | E(C_{AB}) | \psi_B \rangle$$

We hope that an expression like this will take the place of the ill-defined Feynman path integral for general relativity, given by

$$Z_{AB} = \int_{\text{suitable geometries}} D[g_{\mu\nu}] e^{iS[g_{\mu\nu}]}$$
3 Regge Calculus

3.1 Introduction

[Reg61, RW]

In the same way that a sphere is approximated by a geodesic dome, so we can approximate a spacetime manifold by a piecewise-flat space. These, when considered as possible solutions to Einstein’s equations, are the subject of Regge Calculus. The curvature of a geodesic dome is not to be found in the edges between faces (because given two faces hinged together at an angle, we can bend the hinge to restore flatness). It is to be found instead at the vertices. The sum of the angles at a vertex is $2\pi - \epsilon$, and $\epsilon$ (which we call the deficit angle) is a measure of the curvature. Generalizing to $n$ dimensions, we find that curvature is concentrated in $(n-2)$-simplices. We call these hinges. Write $|\sigma_i|$ for the measure of the hinge, and $\epsilon_i$ for its associated deficit angle. We can then write down the Regge calculus equivalent of the Einstein action:

$$S = \frac{1}{2} \int R \sqrt{g} \, d^n x \quad \longrightarrow \quad S_{\text{Regge}} = \sum_i |\sigma_i| \epsilon_i$$

In the usual formulation, the dynamical variable is the $g$-field. What are the variables in the new formulation? They are the edge lengths (i.e. the measure of the 1-simplices). Specifying the edge lengths of a top-dimensional simplex suffices to specify the simplex, and so giving the edge lengths is equivalent to giving the geometry. To obtain the equations of motion, we need therefore to vary $I_R$ with respect to the edge-lengths. Happily, Schléfli’s identity (see [Reg61] for a proof) states that

$$\sum_i |\sigma_i| \delta \epsilon_i = 0$$

And so the variation gives us a single equation corresponding to each edge length $l_j$. To first order this equation involves only the geometry of those simplices containing $l_j$.

$$\sum_i \frac{\partial |\sigma_i|}{\partial l_j} \epsilon_i = 0$$

3.2 Causality

It should be noted in the present context of causality that the basic formulation of Regge calculus described here is strikingly non-causal and non-Lorentzian. In some sense, the

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3 To be precise, a piecewise-flat space is a combinatorial simplicial complex. It should be emphasised that each simplex is identified with a region of some model space (be it $M^n$ or $R^n$), and that this is the sense in which the space is piecewise-flat. But it is certainly not the case that the complex is (or can be) embedded in the model space.

4 The equivalence of these actions under certain conditions is addressed in [CMS84] and [FL84].

5 To justify this, count the degrees of freedom of a top-dimensional simplex in $R^n$. There are $n(n+1)$ degrees of freedom in the choice of co-ordinates for the vertices. Subtract from this $n$ translational freedoms and $\frac{1}{2} n(n+1)$ rotational freedoms (given by SO($n$)). This gives $nC_2$, which is the number of edges.
formalism maintains all the covariance of Einstein’s equations, and there is therefore no symmetry-breaking which specifies timelikeness, or any (discrete) symmetry-breaking which gives a time direction.

Local causality is however expected to be present in the dynamics of equations. For instance in [BGM’97] an evolution scheme for Regge calculus is presented. Given a vertex $v$ in a spacelike hypersurface, and the vertices immediately connected to it (its ‘star’), we can ‘erect a tent’ about $v$ to evolve it into the future. The apex of the tent is a point $v^+$, the canvas is traced out by edges $v^+\{\text{star}\}$ joining $v^+$ to the elements of the star, and the tent pole is the edge $v^+v$. Now the only equations involving the edge $v^+v$ and the edges $v\{\text{star}\}$ are those corresponding to those particular edges. As our unknowns are the edge lengths $v^+v$ and $v^+\{\text{star}\}$, we have the same number of equations as unknowns, so the geometry is locally determined. Because the equations are not all independent (they have relations analogous to the contracted Bianchi identities between them) we have precisely enough freedom to choose the lapse and shift of $v^+$. But if we choose $v^+$ to lie outside the Cauchy development of the star, then we can expect instability in the solution obtained, and this is the sense in which causality is encoded.

### 3.3 Incorporating a cosmological constant

In the presence of a cosmological constant, the Einstein action is altered as follows

$$S = \frac{1}{2} \int R \sqrt{g} \, d^n x \quad \rightarrow \quad S = \frac{1}{2} \int (R - 2\Lambda) \sqrt{g} \, d^n x$$

This corresponds to subtracting a factor proportional to the measure of the manifold. The analog in Regge calculus is just

$$S_{\text{Regge}} = \frac{1}{2} \int R \sqrt{g} \, d^n x \quad \rightarrow \quad S_{\text{Regge}} = \left( \sum_i |\sigma_i| \epsilon_i \right) - V$$

where $V$ is the measure of the piecewise-flat space.
4 Dynamical Triangulations

4.1 Definitions

A dynamical triangulation is a piecewise-flat manifold in which the edge-lengths of the simplices are restricted. In the Euclidean case, we demand that $l_i^2 = a^2$ for some fixed $a$. In the Lorentzian case, we require $l_i^2 = \pm a^2$. How do these structures model spacetime manifolds? The freedom to set arbitrary edge-lengths which we had in the full Regge calculus is gone, but we can still encode curvature by gluing together simplices in the right way. In a sense, connectivity of simplices is now the dynamical variable.

The 2-d Euclidean case provides a helpful picture. In flat space, we glue 6 equilateral triangles together at each vertex. To create positive curvature, we might glue 5 such triangles together (allowing us to make a dodecahedron; a coarse-grained model of $S^2$). Similarly, we create negative curvature by gluing more than 6 triangles. It is clear that curvature is now quantized. But despite this, we can still expect to model a smooth manifold if we are prepared to ignore features on scales smaller than $a$.

4.2 Implementing causality

In the Lorentzian case timelike and spacelike edges are distinguished by definition. But because we are working in the setting of combinatorial simplicial complexes, the words ‘timelike’ and ‘spacelike’ do not necessarily have the meaning we need. In particular, the causality structure may not be encoded properly by the definitions so far given. This turns out to be the case.

Given a Euclidean triangulation, we can turn it into a Lorentzian triangulation by defining which edges should be timelike and which spacelike. But an arbitrary assignment of this nature will not respect causality. For instance, consider a (1+1)-d triangulation in which one of the 2-simplices (a triangle) has 3 timelike edges. It can be shown graphically using a Minkowski diagram of flat space that such a situation is undesirable: it is impossible to draw an equilateral triangle on such a diagram with all of its edges timelike. To look at the situation another way, we have not yet broken the $t \mapsto -t$ symmetry in Einstein’s equations, and so a timelike edge is not equipped with a time arrow showing its direction. So a triangle of the type described might be a closed timelike curve.

To ensure that local causality holds, we impose the following condition on Lorentzian triangulations.

*Time-foliation condition:* There exists a time function $t$ which maps from the vertices of the triangulation to $\mathbb{Z}$. Spacelike edges connect vertices with equal $t$’s. Timelike edge connect vertices with consective $t$’s.
4.3 Dynamics

How do we put the dynamics into dynamical triangulations? The most natural solution is to use the Regge action in the natural discretized version of the Feynman integral over geometries, i.e.

$$Z_{AB} = \int_{\text{suitable geometries}} D[g_{\mu\nu}] e^{iS[g_{\mu\nu}]} \quad \longrightarrow \quad Z_{AB} = \sum_{\text{suitable triangulations}} e^{iS_{\text{Regge}}[T]}$$

One of the reasons why the Feynman integral on the left is ill-defined is because it performs an integration over the gauge group: we actually need to integrate over ‘suitable geometries’ modulo the diffeomorphism group to avoid over-counting. The implementation of this is far from obvious. However, in the integral on the right, we are working in a discrete system where the diffeomorphism group is replaced by the automorphism group of the triangulation. Let $C_T$ be the order of the group of automorphisms of $T$, the group of maps from $T$ to itself which preserve connectivity. Then we have the following analogy:

$$Z_{AB} = \int_{\text{suitable geometries}} D[g_{\mu\nu}] e^{iS[g_{\mu\nu}]} \quad \longrightarrow \quad Z_{AB} = \sum_{\text{suitable triangulations}} \frac{1}{C_T} e^{iS_{\text{Regge}}[T]}$$

The sum on the right gives the dynamics we use.

4.4 The $(1+1)$-d case

To illustrate some of the ideas in the field, we work through the $(1+1)$-d case in detail, indicating the generalisation to higher-dimensions as we go along. This will lead us to a discussion of the critical distinctions between the Lorentzian and Euclidean approach.

4.4.1 Regge action

Because all hinges have the same measure, we have that

$$S_{\text{Regge}} \propto \sum_{\text{hinges } h} \text{deficit angle at } h$$

In the 2-d case it is clear that if there are $k_v$ triangles meeting at a vertex $v$ (the vertices are hinges in this dimension), then the deficit angle is $2\pi - k_v \pi/3$. So

$$S_{\text{Regge}} \propto \sum_v 2\pi - k_v \pi/3$$

$$= 2\pi N_0 - \pi/3 \sum_v k_v$$

$$= 2\pi N_0 - \pi N_2$$
where \( N_i \) is the number of \( i \)-simplices. This generalizes to higher dimensions, and we write

\[
S_{\text{Regge}} = \kappa_{d-2}N_{d-2} - \kappa_dN_d
\]

The \( \kappa_i \) act as coupling constants. Physically, they are functions of the (bare) cosmological constant and Newton’s constant \( G \).

To generalise to the case where \( \Lambda \neq 0 \), we subtract \( \Lambda N_d \) from the action. This change is absorbed into the definition of \( \kappa_d \), so the above formula still applies.

### 4.4.2 Path integral

We have

\[
\begin{align*}
Z &= \sum_{\text{triangulations } T} \frac{1}{C_T} e^{iS_{\text{Regge}}[T]} \\
&= \sum_{N_d} e^{-i\kappa_dN_d} \sum_{N_{d-2}} e^{i\kappa_{d-2}N_{d-2}} \sum_T \frac{1}{C_T}.
\end{align*}
\]

In the 2-d case we can take advantage of the discrete analogue of the fact that 2-d GR is a topological field theory. We have that

\[
S = \frac{1}{2} \int \sqrt{-g} \, d^2x \left( R - 2\Lambda \right)
\]

\[
= \pi \chi - \Lambda \left( \int \sqrt{-g} \, d^2x \right)
\]

where we have simply used the definition of the Euler characteristic \( \chi \), which is a topological invariant. By analogy

\[
S_{\text{Regge}} = \pi \chi - \lambda N_2
\]

Putting this into the path integral, assuming for simplicity that \( C_T = 1 \), and dropping an overall multiplicative constant due to the \( \chi \) term, we get

\[
Z = \sum_T e^{-i\lambda N_2}
\]
4.4.3 Topology restriction

In addition to the time-foliation condition described earlier, we follow the treatment of
[ALNR99] in restricting the topology to $S_1 \times \mathbb{R}$, and identifying $\mathbb{R}$ as the time direction.
This gives a model of the following form:

![Diagram]

4.4.4 Wick rotation

The imaginary exponent in the above expression presents difficulties, both from the ana-
lytic point of view (in establishing, for instance, the convergence properties of $Z$) and in
numerical calculations. We can remove it by a ‘Wick rotation’, which is a procedure which
allows us to drop the causal structure of the (1 + 1)-d space, leaving us with a 2-d space.
It is of fundamental importance in more general contexts but we introduce it concretely
now.

We introduce for convenience a new complex parameter $\alpha$, and proceed as follows:

- We distinguish between spacelike edge-lengths $l_s$ and timelike edge-lengths $l_t$. We
take $l_s^2 = a^2$ as before, but we now set $l_t^2 = \alpha a^2$.

- We calculate our action $S$ in terms of the usual dynamical variables and the new
parameter $\alpha$, restricting $\alpha$ to a subset of the complex plane in which we can use a
geometric interpretation for the values of the $l_i$. In this example we would take $\alpha$
to be real and negative, giving the Lorentzian case).

- We analytically continue the action with respect to $\alpha$.

- We let $\alpha$ move along a curve joining $\alpha_0$ to $-\alpha_0$. The curve is confined to the lower
half-plane. ($\alpha_0$ is fixed, and in our current context $\alpha_0 = -1$, and so as $\alpha$ moves along
the curve we transfer from the Lorentzian to the Euclidean case.)
How is the Wick rotation implemented in the (1+1)-d case? We are restricted to a single type of simplex, with 2 timelike edges and 1 spacelike edge. Working initially in Lorentzian space, we calculate the measure of such a simplex to be $\frac{1}{2}\sqrt{-\alpha + \left(\frac{1}{2}\right)^2}$ by an application of Pythagoras. This is meaningful for $\alpha \in \mathbb{R}$ and $\alpha < -1/4$, and this is just the condition for triangle inequalities to hold. We have

$$S \propto N_2 \sqrt{-\alpha + \left(\frac{1}{2}\right)^2}$$

We analytically continue $S_{\text{Regge}}$ with a branch cut starting at the singularity at $\alpha = -\left(\frac{1}{2}\right)^2$ and going vertically upwards to infinity. Going from $\alpha = -1$ to $\alpha = +1$ then introduces a multiplicative factor of $-i\sqrt{\frac{5}{3}}$ to the action, giving

$$Z_{\text{Lorentzian}} = \sum \text{configurations} e^{-i\lambda N_2} \quad \longrightarrow \quad Z_{\text{Euclidean}} = \sum \text{configurations} e^{-\tilde{\lambda} N_2}$$

where $\tilde{\lambda} = \sqrt{\frac{5}{3}}\lambda$.

4.4.5 Calculating the one-step propagator

Following our topology restriction, our time-slices are just loops characterized by their length $l$. Consider a sum over possible one-step transitions from a state $l_1$ to a state $l_2$. Such a transition consists of $N_2 = l_1 + l_2$ triangles. To specify their configurations, we need only decide whether each one is ‘future-pointing’ or ‘past-pointing’. The future-pointing triangles have their bases on the $l_1$ loop, so there must be $l_1$ of these. Similarly there are $l_2$ past-pointing triangles. So the number of possible arrangements (modulo cyclic permutations of the strip of triangles) is

$$\frac{1}{l_1 + l_2} \binom{l_1 + l_2}{l_1}$$

This gives us

$$Z \equiv \langle l_1 | \hat{T} | l_2 \rangle = \sum_{\text{configurations}} e^{-\tilde{\lambda} N_2} = e^{-\tilde{\lambda}(l_1 + l_2)} \sum_{\text{configurations}} 1 = e^{-\tilde{\lambda}(l_1 + l_2)} \frac{1}{l_1 + l_2} \binom{l_1 + l_2}{l_1}$$

4.4.6 Renormalizing the propagator

How does the propagator behave when $N_2$ is large? In general, the number of configurations is exponentially bounded with respect to $N_2$. This can be seen in the expression above,
where the number of configurations is

\[
\frac{1}{l_1 + l_2} \left( \frac{l_1 + l_2}{l_1} \right) \sim \frac{1}{l_1 + l_2} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{l_1 + l_2}{l_1 l_2}} \left( \frac{l_1 + l_2}{l_1} \right)^{l_1 + l_2} \approx \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{l_1 l_2}} \left( \frac{l_1 + l_2}{l_1} \right)^{l_1 + l_2} e^{(l_1 + l_2)}
\]

Extrapolating this result, and choosing a suitable \( \tilde{\lambda}^{\text{crit}} \) we get

\[
Z = \sum_{N_2} e^{-\tilde{\lambda} - \tilde{\lambda}^{\text{crit}}} N_2 \times \text{subleading}(N_2)
\]

By comparison with our original expression for the propagator, we can see that \( \tilde{\lambda} - \tilde{\lambda}^{\text{crit}} \) acts as a cosmological constant. We perform the following renormalizations

\[
\tilde{\lambda} - \tilde{\lambda}^{\text{crit}} = a^2 \Lambda + O(a^3)
\]

\[
L = al
\]

\[
T = at
\]

\[
Z_\Lambda = \lim_{a \to 0} a^\nu Z
\]

were we have introduced continuum variables \( L \) and \( T \) and obtained a new continuum propagator \( Z_\Lambda \) in terms of a renormalized cosmological constant \( \Lambda \). Note that a positive cosmological constant is needed for convergence.

To obtain our final result, we perform an inverse Wick rotation to return to the Lorentzian setting. For details, see [ALNR99]. We obtain

\[
Z_\Lambda = e^{-\coth(i\sqrt{\Lambda}T)} e^{-\sqrt{\Lambda}(L_A + L_B)} \frac{\sqrt{\Lambda L_A L_B}}{\sinh(i\sqrt{\Lambda}T)} I_1 \left( 2 \frac{\sqrt{\Lambda L_A L_B}}{\sinh(i\sqrt{\Lambda}T)} \right)
\]

where \( L_A \) and \( L_B \) are the lengths of the initial and final states, and \( T \) is the time between them. \( I_1 \) is a Bessel function of the first kind.

### 4.5 Causality in the 2-d case

In numerous respects, a Lorentzian approach is better than a Euclidean one at providing a model of the continuum theory for \( d = 2 \). The continuum theory is not dynamical for \( d = 2 \) because the curvature part of the action is a topological invariant, as described. So it is particularly easy to identify what features the model should have. It turns out that the causal and topological restrictions that are imposed on the Lorentzian model dramatically improve its properties. We give some details in the following sections.
4.5.1 Dimensionality

[ALNR99]

The vertices of a dynamical triangulation form a metric space, where we take the metric to be the smallest number of links needed to travel between a pair of vertices - the geodesic distance. We can define the Hausdorff dimension $d_H$ at a point, for instance by counting the number of vertices $V(R)$ in a ball of radius $R$ centered at that point. Then we have, for a specific triangulation,

$$ V(R) \propto R^{d_H} $$

and more interestingly we have the dimension given by the ‘average over geometries’,

$$ \langle V(R) \rangle \propto R^{d_H} $$

This $d_H$ may depend on the position and the scale at which it is measured. In the model above, we get $d_H = 2$ at all scales.

We can also correlate the dimensions of the dynamical variables by identifying dimensionless groups in the equations of our model. For instance, the final equation for $Z_\Lambda$ makes it clear that $T$ and $L$ have the same dimensions as $1/\sqrt{\Lambda}$. This is as expected: noting that the action is dimensionless, we see that $\Lambda$ has the dimensions [length]$^{-d}$, so $T$ and $L$ have dimensions [length].

This situation is a significant improvement on the analogous 2-d Euclidean model. To see this, consider the following expression for the Euclidean propagator, obtained by a similar calculation to the one outlined above:

$$ Z_\Lambda = \Lambda^{3/4} \frac{\cosh(\Lambda^{1/4} R)}{\sinh^3(\Lambda^{1/4} R)} $$

Here $R$ has dimensions [length]$^{1/2}$, again by inspecting the expression. This is an anomalous result. To investigate further, we can calculate the Hausdorff dimension:

$$ Z_\Lambda = \sum e^{-\Lambda N_2} \implies \langle V(R) \rangle = \langle N_2 \rangle \sim -\frac{\partial \log Z_\Lambda}{\partial \Lambda}_{\Lambda=0} \sim R^{d_H} $$

This gives $d_H = 2$ (Lorentzian) and $d_H = 4$ (Euclidean). So we see that the Euclidean model gives a fractal spacetime. Intuitively, given any point in the average Euclidean model, there are more points within a certain geodesic distance from it than is expected. The reason for this is that the models are full of what are (colourfully) referred to as ‘baby universes’. These are of various types. To construct one of the simplest, imagining gluing a ‘normal-looking’ triangulation and a triangulation of $S^2$ together at a point. Such structures are permitted in the Euclidean framework set up above - causality and topology constraints are what is required to eliminate them.

A renormalization procedure by which the baby universes are integrated out is outlined in [ACKL00]. This allows us to view the Lorentzian model as a renormalized Euclidean model.
4.5.2 Incorporation of matter

We briefly describe a model in which matter is incorporated. In effect, we place spin $\frac{1}{2}$ particles at each vertex of the triangulation, with spin $\sigma_i = \pm 1$. The following partition function models a situation in which neighbouring particles (at vertices $v_i$ and $v_j$ which share an edge $e_{ij}$) are predisposed to align:

$$Z_{\beta} = \sum_{\sigma_i = \pm 1} e^{\frac{1}{2} \beta \sum_{\text{edges } e_{ij}} \sigma_i \sigma_j}$$

This is the Ising model. We can use the analogy between partitions functions and propagators to simply combine the above expression with the propagator obtained above, giving:

$$Z_{\Lambda,\beta} = \sum_{\text{transition } \sigma_i = \pm 1} \sum_{e_{ij}} e^{-\Lambda N_2} e^{\frac{1}{2} \beta \sum_{\text{edges } e_{ij}} \sigma_i \sigma_j}$$

In flat space, the critical exponents (for the specific heat, magnetization and magnetic susceptibility - see statistical mechanics refs) are

$$\alpha = 0, \beta = 0.125, \gamma = 1.75$$

Numerical simulations ([AAL99]) have shown that Lorentzian models are compatible with these results. In the Euclidean case an exact solution ([BK87]) is available. The exponents are as follows:

$$\alpha = -1, \beta = 0.5, \gamma = 2$$

Again, the Euclidean model fails where the Lorentzian succeeds.

4.5.3 The preferred time

One might criticize the time-foliation constraint on the grounds that it goes against the spirit of GR to pick out a global time from the outset.

On one side of the argument, the analogy with the continuum case should not be overstated. We cannot consistently extend the domain of the time function from the set of vertices to the whole piecewise-flat manifold. Clearly within any given simplex this is possible, but because the curvature is concentrated at the vertices, extending over more than one time step is not possible in general.

However, given a series of embeddings of progressively finer dynamic triangulations into a spacetime manifold, we certainly do obtain a global time function (after renormalization). This is philosophically undesirable at least, even if there is no claim that the particular time function used is privileged – it is hoped that all quantum observables can be obtained using any time function.

It is at least provocative to note that the incorporation of causality requirements directly serves to renormalize the Euclidean model for $d = 2$. 
4.6 The 3-d and 4-d models

In the $d = 3$ case, analogous models to the one detailed above exist and their properties are under investigation. For $d = 4$, the case of physical interest, work is at a very early stage. Some details are given in [Lol].
5 Penrose spin networks

5.1 Definitions and motivation

Spin networks were first proposed by Penrose, and are defined as follows:

**Definition.** *Penrose spin network* $\Delta$: [Be71] A trivalent graph, whose edges are labelled by non-zero spins $j$. Open edges\(^6\) are allowed. The edges represent physical objects, and the vertices represent interactions. At each vertex, the rules for the addition of the angular momenta $j_1$, $j_2$, $j_3$ must be observed:

- $j_1 + j_2 + j_3$ is even.
- $j_1 \leq j_2 + j_3$, and the other two triangle inequalities also hold.

We now give an intimidatingly complex definition of a norm for spin networks. Note that although we use geometric terminology throughout, we are dealing with entirely combinatorial objects. But the (informal) geometric interpretation of spin networks is certainly very vivid and useful.

**Definition.** *The value of a Penrose spin network* $V(\Delta)$: Each edge is decomposed into $2j$ strands, and at each vertex these strands are joined up, ensuring that no two strands from the same edge are connected. (The rules for the addition of angular momenta are precisely the conditions for this trick to be possible.) This joining (which we call a routing) is non-unique, and we proceed to sum over all possible routings. We assign an orientation $\epsilon = \pm 1$ to each routing in such a way that if we change the sense of a crossing between two strands in an edge, then the sign of $\epsilon$ changes. We also count the number $N$ of closed loops formed by the routed strands. Then we have:

$$V(\Delta) = \prod_{edges} \frac{1}{j_1!} \sum_{routings} \epsilon (-2)^N$$

When Penrose originally proposed this definition, it was motivated (PWe71) by the theory of negative-dimensional tensors. We will not discuss these, but we will mention a link with 3-colourings.\(^7\) If we take a planar graph with $j = 1$ on all its edges, then it turns out that its value is simply the number of possible 3-colourings on it. This suggests a physical interpretation: the value of a $j = 1$ spin network is simply its entropy.

---

\(^6\)Open edge: an edge only connected to a vertex at one of its ends.

\(^7\)3-colouring: assigning one of 3 colours to each edge of a graph, so that edges with the same colour never meet at a vertex.
Theorem. \( V(\Delta) = \# \{3\text{-colourings of } \Delta \} \).

Proof. We have the following identities between the values of graphs:

\[
\begin{align*}
\circ & = 2 \\
\triangle & = 1 \\
\square & = 1 \\
\left( \begin{array}{c}
\end{array} \right) & = 0 \\
\circ & \\
\circ & \\
\end{align*}
\]

We can use these identities to remove faces of the graph one by one. We have to check that there always exists a face with at most 5 edges surrounding it, but this follows by an easy argument.\(^8\) Then we are in a position to perform an induction on the number of faces of the graph. The base case for our induction is a single closed loop: this can be coloured in 3 ways.

\[ \square \]

Definition. The norm of a Penrose spin network \( |\Delta|^2 \): Given a spin network \( \Delta \) with open ends, we construct another, \( \tilde{\Delta} \), which is a mirror-image of \( \Delta \). We connect corresponding open ends to obtain a new network, which we call \( \Delta \# \tilde{\Delta} \). Then we define \( |\Delta|^2 := V(\Delta \# \tilde{\Delta}) \).

Like causal sets, spin networks encode conformal structure. As before, the manner in which this is achieved is far from obvious. We give a heuristic description. [Be71, HW79]

- *Why are directions encoded?* Clearly the spin state of an object says something about direction: two neighbouring spin \(1/2\) systems in opposite spin states are ‘in different directions’, in an undeniable sense. We can only hope to obtain this kind of relative information between parts of the spin network system.\(^9\) The higher the spin, the

\[ \square \]

\(^8\)We take the numbers of vertices, edges and faces to be \( V, E \) and \( F \). We include the zone outside the graph in the count of faces. Then we have \( V - E + F = 2 \) (Euler’s formula) and \( 2E = 3V \) (by trivalence). This gives \( 2E = 6F - 12 \). But if all faces (not counting the zone outside the graph now) have at least 6 edges then \( 2E \geq 6F - 6 \). This gives a contradiction.

\(^9\)This is very much in accord with Mach’s principle, and in accord with our ambitions for background independent approaches to geometry.
more information we have: if the spin is zero, we have no information (the system is spherically symmetric), if the spin is higher, we have more information (there are more possible spin orientations). So we might expect to get a well-defined concept of direction in the high spin limit.

• Experiments to estimate direction. Take two open edges of high spin $M$ and $N$. What is the angle $\theta$ between them? If they are not both in the same connected component of the network, then Mach’s principle leads us to presume that the angle is undefined. So take $M$ and $N$ to be open edges of a connected network $X$, the body of which is unknown. Now imagine that $M$ and $N$ are systems such as atomic nuclei, and that an electron (spin $1/2$) passes between them, changing the spin of both $M$ and $N$. Depending on alignment, we get two different probable outcomes:

- $M$ and $N$ aligned parallel $\theta = 0$ $(M, N) \mapsto (M \pm 1, N \mp 1)$
- $M$ and $N$ aligned anti-parallel $\theta = \pi$ $(M, N) \mapsto (M \pm 1, N \pm 1)$

There is a continuum of states between these two, and these correspond to a mixing between the two possible types of outcome on the right. If the angle between $M$ and $N$ is $\theta$, we get probabilities

\[
(M, N) \mapsto (M \pm 1, N \mp 1) \quad P = \frac{1}{2} + \frac{1}{4} \cos \theta
\]

\[
(M, N) \mapsto (M \pm 1, N \pm 1) \quad P = \frac{1}{2} - \frac{1}{4} \cos \theta
\]

• Performing the experiments within the formalism. Penrose gives a detailed prescription in [Be71] allowing the probability of the outcomes of the experiment above to be measured. This allows us to recover the conformal structure of 3-dimensional space. As part of this program, we check that the ‘new angles’ behave like the old. There is a sense in which the algebraic information about $\mathbb{R}^3$ found in the spin representations of $SU(2)$ has been recovered.

5.2 Causality

Penrose spin networks use the non-relativistic theory of angular momentum, and as such are not naturally placed in a spacetime arena. Causality is not important, as is emphasised in [PWe71]. Interactions are represented by vertices, and the order in which the interactions occur is considered to be arbitrary.
6 Spin foam models

6.1 Motivation

6.1.1 Introduction

Remarkably, spin networks (of a slightly more general type than that first proposed by Penrose) provide a basis of states for Loop Quantum Gravity. Without giving a full introduction to the theory, we indicate the reasons for this fact.

Instead of making the metric the configuration variable, we use an $SU(2)$-connection instead, as originally proposed by Ashtekar in [Ash86, Ash87]. Putting the details aside, this means that we describe a given space-like hypersurface by giving it an $SU(2)$-connection $A_{a}^{j}$, and defining an $su(2)$-valued vector density of weight one $E_{a}^{j}$ on it.\footnote{To be specific, $A_{a}$ is an $SU(2)$-valued one-form, and $A_{a}^{j}$ are its components in some basis of $SU(2)$} $A_{a}^{j}$ and $E_{a}^{j}$ satisfy canonical commutation relations with respect to a suitable Poisson bracket. We therefore consider $A_{a}^{j}$ to be the configuration variable, and $E_{a}^{j}$ to be its canonically conjugate momentum. Together, they specify the geometry of a space-like hypersurface, and we can describe the geometry of a spacetime manifold by specifying them on all the slices of a foliation.

We wish to construct gauge-invariant quantum states on space-like hypersurface. By analogy with the wavefunction of a particle, which is a map from the configuration space of the particle to $\mathbb{C}$, we consider wavefunctionals $\Psi(A)$ of the connection $A$. To be more precise we take our Hilbert space of wavefunctionals to be $H = L^{2}[A]$, the space of square-integrable functions over the space of connections $A$.

The gauge freedoms of gravity in this formulation are the usual diffeomorphisms, plus a local $SU(2)$ freedom which transforms the connection. Because we have identified favoured space-like surfaces, the diffeomorphisms split into spatial and temporal reparametrizations. $H_{\text{kin}}$, the Hilbert space of kinetic states, consists of objects which are invariant under $SU(2)$ and spatial gauge transformations. Such objects would, for instance, suffice to specify an initial state on a space-like hypersurface. $H_{\text{phys}}$, the space of physical states, consists of objects which are invariant under all gauge freedoms. To construct these spaces, we need to construct suitable gauge-invariants; remarkably these are provided by spin networks.

Consider the holonomy $h_{c}(A) \in SU(2)$ along a path $c$ lying in the space-like hypersurface which joins points $c(0) = p$ and $c(1) = q$. This is defined by

\[
h_{c}(A) = P \exp \int_{t} A \equiv 1 + \sum_{n=1}^{\infty} \int_{0}^{t_{1}} dt_{1} \int_{t_{1}}^{t_{2}} dt_{2} \cdots \int_{t_{n-1}}^{1} dt_{n} A(t_{1}) \cdots A(t_{n})
\]

where $A(t) := \dot{c}_{a}(t)A_{a}(c(t))$, and $0 < t_{1} < \ldots < t_{n-1} < 1$. This is the result of parallely transporting $A$ along $c$. We want to see how this object changes under gauge
transformations. Under an $SU(2)$ transformation $g$ and a diffeomorphism $\phi$ we get:

\[
A \mapsto g A g^{-1} - d g g^{-1} \\
A \mapsto \phi^* A
\]

This gives

\[
h_c(A) \mapsto g(p) h_c(A) g(q)^{-1} \\
h_c(A) \mapsto \phi h_{\phi^{-1} c}(A)
\]

These transformation properties are very useful. Consider for instance the case when $c$ is a closed loop $l$. Then we have

\[
\text{Tr}(h_l(A)) \mapsto g \text{Tr}(h_l(A)) \\
\text{Tr}(h_l(A)) \mapsto \phi \text{Tr}(h_{\phi^{-1} l}(A))
\]

This expression, $\text{Tr}(h_l(A))$, is called a Wilson loop wavefunctional. It serves as our starting point for defining gauge-invariant quantities. It is already $SU(2)$ invariant. If we integrate it over a knot class (in other words, a diffeomorphism-equivalence class of loops) then we get a quantity that is diffeomorphism invariant too.

### 6.1.2 Trivalent spin networks

We now move on to the important connection with spin networks. We start with a specific case; trivalent spin networks, and define corresponding wavefunctionals as follows.

**Definition.** Trivalent spin network wavefunctional $\Psi_{\gamma, \{j_e\}, \{\iota_v\}} \in H$. $\gamma$ is a trivalent graph in a given space-like hypersurface. A spin $j_e$ is assigned to each edge and an intertwiner $\iota_v$ to each vertex. Spins are, of course, irreducible unitary representations of $SU(2)$. Intertwiners are $SU(2)$-invariant maps from the tensor product of the representation spaces $j_e$ on the edges meeting at a particular vertex to the representation space of the identity representation. The latter is just the base field $K$. So, at a given vertex

\[
\iota_v \left( \bigotimes_{e \ni v} [j_e[h_e(A)]] \right)
\]

is in $K$, and so we can construct a wavefunctional as follows:

\[
\Psi_{\gamma, \{j_e\}, \{\iota_v\}}(A) := \prod_{v \in \gamma} \iota_v \left( \bigotimes_{e \ni v} [j_e[h_e(A)]] \right)
\]

It can be shown quite easily that these wavefunctions are $SU(2)$-invariant. (An $SU(2)$ gauge transformation modifies the $h_e$ as described above, but the $SU(2)$-invariance of the intertwiners is just what is needed to ensure that this does not modify $\Psi$.)
6.1.3 Intertwiners

There are a number of equivalent ways in which to view the intertwiner. We have first the definition of an intertwining map \( f : V \mapsto V' \) between representations \( \rho : G \mapsto V \) and \( \rho' : G \mapsto V' \). This is simply an \( SU(2) \)-invariant map, ie. one for which

\[
f \circ \rho(g) = \rho'(g) \circ f \text{ for all } g \in G
\]

If we put arrows on our spin network, then we can distinguish between ‘in-edges’ and ‘out-edges’ incident at a vertex. Then it is common to consider an intertwiner as an intertwining map

\[
\iota^* : \bigotimes_{\text{in-edges } e} j_e \mapsto \bigotimes_{\text{out-edges } e} j_e
\]

It is now natural to consider \( \iota^* \) as an assignment of a spin state for the out-edges given a spin state for the in-edges. The \( SU(2) \)-invariance is just the condition that this assignment respect the \( SU(2) \) action, which is precisely what we need. Note that looking at intertwiners in this way identifies spin foams as an example of quantum causal histories.

In the trivalent case considered up to now, \( \iota \) or \( \iota^* \) depending on our viewpoint) is uniquely specified (up to a scalar multiple). This corresponds with what we know about Clebsch-Gordon coefficients, which are a simple type of intertwiner, combining two spins on ‘in-edges’ to give one on an ‘out-edge’. The conditions for the 3 spins to be allowable is precisely that the intertwiner should be non-vanishing, and this means that the wavefunctional corresponding to a disallowed spin network is zero, as we might expect.

We now consider the generalization to a vertex \( v \) of valency \( n > 3 \). In this case, the intertwiner is not uniquely specified by the spins. However, it is clear enough that if we replace \( v \) by a trivalent spin network \( \gamma_0 \) with \( n \) exterior edges, then we get an intertwiner for \( v \) by taking the product of the intertwiners of \( \gamma_0 \). Conversely, we find that we can always perform this replacement if the combination of spins at \( v \) is allowable. Working in the directed graph setting, we are giving a procedure to produce the out-edge spins from the in-edge spins, and we only need to use trivalent intertwiners at each stage. The new trivalent vertices are sometimes referred to as virtual nodes.
How does all this relate to the definition of an intertwiner given previously? We need to set up a correspondence between maps of the forms $\iota^*$ and $\iota$. This is constructed as follows:

$$\iota^* : \bigotimes_{\text{in-edges } e} \hat{e} \mapsto \bigotimes_{\text{out-edges } e} \hat{e}$$

$$\downarrow \text{Hom}(U, V) \cong U^* \otimes V$$

$$\sigma : \left( \bigotimes_{\text{in-edges } e} \hat{e}^* \right) \otimes \left( \bigotimes_{\text{out-edges } e} \hat{e} \right)$$

$$\downarrow \hat{e}^* \cong \hat{e}$$

$$\rho : \bigotimes_{\text{edges } e} \hat{e}$$

$$\downarrow V \cong V^*$$

$$\iota : \bigotimes_{\text{edges } e} \hat{e} \mapsto \text{Id}$$

Once this correspondence has been set up, the following are equivalent:

- $\iota^*$ is an intertwining map.
- $\sigma$ is a vector invariant under the action of $SU(2)$.
- $\rho$ is a vector invariant under the action of $SU(2)$.
- $\iota$ is an intertwining map.

It is often the case therefore that $\rho$ is taken to be the intertwiner.

### 6.1.4 More general spin networks

It should now be clear that the definition of a wavefunctional for trivalent spin networks given above generalizes immediately to networks based on any graph $\gamma$.

The graph $\gamma$ is embedded in a space-like hypersurface. In the same way that we took diffeomorphism-equivalence classes of loops, we can take a spatial diffeomorphism-equivalence class $[\gamma]$ of graphs. We averaging over the wavefunctionals $\Psi_\gamma : \hat{\gamma} \in [\gamma]$. This gives us a state which is not only $SU(2)$-invariant but also spatial diffeomorphism invariant, namely a kinematical state. This completes our construction of $H_{\text{kin}}$.11

### 6.1.5 Geometry

We described earlier the manner in which Penrose spin networks encode geometry. The new spin networks do too, and in a more transparent sense. Their edges are ‘flux lines’

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11The details of this procedure are given in [AL95]
of area: to put it another way, area is concentrated in the edges of the network. More specifically, we construct an area operator given by

$$\hat{A}_S(E) = \int_S dx^2 \sqrt{\text{Tr}[n_a n_b E^a E^b]}$$

where \( n \) is a co-normal to the surface \( S \) lying in a space-like hypersurface. A spin network is an eigenstate of \( \hat{A}_S(E) \) when \( S \) is punctured by an edge of the network. The eigenvalue is proportional to \( \sqrt{j(j+1)} \) where \( j \) is the spin on the edge. So area is quantized.

### 6.1.6 Spin foams

We represent space-like slices of a 4-geometry by spin networks. The 4-geometry itself is represented by an evolution of spin networks, a spin foam.

**Definition.** Spin foam \( \mathcal{F} : s \mapsto s' \), representing a transition between spin networks \( s = (\gamma, \{j_e\}, \{\iota_v\}) \) and \( s' = (\gamma', \{j'_e\}, \{\iota'_{v'}\}) \). The foam is a 2-complex \( \mathcal{J} \), bordered by \( \gamma \) and \( \gamma' \). Spins \( \{j_f\} \) are associated to faces and intertwiners \( \{\iota_e\} \) to edges. We require that the spins and intertwiners match with the spins and intertwiners on \( s \) and \( s' \), so that we have

$$j_e = j_f \quad \text{when} \quad e \in f$$

$$\iota_v = \iota_e \quad \text{when} \quad v \in e$$

We have an operation of composition, so that \( \mathcal{F} : s \mapsto s' \) and \( \mathcal{F}' : s' \mapsto s'' \) can be glued together by identifying the two copies of \( s' \). This gives \( \mathcal{F}\mathcal{F}' : s \mapsto s'' \). We get a natural concept of a

**Definition.** Spin foam model. An assignment of amplitudes \( A[\mathcal{F}] \) to spin foams in such a way that

$$A[\mathcal{F}\mathcal{F}'] = A[\mathcal{F}] A[\mathcal{F}']$$

The rest of this paper will consider spin foam models as implementation of the space of physical states, and particularly of the inner product which we hope to define by

$$\langle s, s' \rangle_{\text{phys}} = \sum_{\mathcal{F} : s \mapsto s'} A[\mathcal{F}]$$
6.2 Causal evolution of spin networks

6.2.1 Introduction

An approach to the construction of a spin foam model which is in line with our present emphasis on causality is given by [MS97].

Notice that spin foams are not causal sets by construction. Although their constituent spin networks give a time-foliation of some sort, there is no obvious way to define light-cones on them, and so no local causality structure. The proposal in [MS97] is that the dynamics (i.e. which spin foams are allowed, and what amplitudes they have) should be informed by causality, and ensure that we can define a causal structure.

We restrict the form of the 2-complexes interpolating between successive spin networks using a pair of evolution rules which we describe. We refer to these 2-complexes as ‘internets’, and we envisage a series of spin networks (‘spinnets’) interpolated by internets to give a spin foam. The edges of the internets will be interpreted as null lines, joining consecutive spinnets. This allows us to put a causal set structure on the vertices and edges of the spin foam. A point \( p \) is to the past of \( q \) if there is a path from \( p \) to \( q \) formed by a sequence of edges lying in the internets, with each internet in the sequence lying immediately to the future of the last. The causal past of a point is given by all points in its past, plus all edges joining pairs of such points. The causal past of an edge is given by the union of the causal pasts of its ends. We formulate the following heuristic principle:

- **Principle of discrete causality:** The connectivity of a vertex \( v \) and the intertwiner assigned to it are functions solely of information contained in the causal past of \( v \). The representation on an edge is similarly a function of the information in its causal past. (We make an exception for correlations induced between edges that share a common vertex.)

We make some further assumptions of micro-causality. For instance we assume that the connectivity of a node should only depend on information on the previous spinnet, and not on information from earlier spinnets. This gives some motivation for the following rules, which specify the allowable set of spin foams. We start our evolution from a trivalent spinnet.

6.2.2 Evolution rules for the 3-d case

The theory is easier to visualize in \( d = 3 \), so we give a full analysis of the \( d = 3 \) case, and indicate the generalization to \( d = 4 \) later. We subdivide each of the 2 evolution rules given by [MS97] into 3 sub-rules:

- **Restriction Rules** constrain the form of the network.
- **Spin Rules** restrict the spins assigned to edges.
- **Amplitude Rules** give transition amplitudes.
• **Restriction Rule 1:** Start with a trivalent spinnet $\Gamma_0$ with vertices $v_i$ and edges $e_{\{i,j\}}$ joining them. Define a successor spinnet $\Gamma_1$ with vertices $v'_{\{i,j\}}$, one for each edge $e_{\{i,j\}}$. Join pairs of these vertices with an edge $e'_{\{i',j'_1\};\{i_2,j'_2\}}$ if the intersection $\{i'_1,j'_1\}\cap\{i'_2,j'_2\}$ is non-empty. The edges of internet are then given by joining vertices $v_i$ to vertices $v'_{\{i',j'\}}$ whenever $i \in \{i',j'\}$.

The result of this is that we have a tetravalent spinnet $\Gamma_1$. So that we can repeat the evolution with rule 1, we reduce the valency of the spinnet:

• **Restriction Rule 2:** Given a spinnet $\Gamma_1$, we replace each tetravalent vertex with one of the three possible decompositions into a pair of trivalent vertices, by inserting a ‘virtual edge’. This gives a new spinnet $\Gamma_2$. The edges of the internet are of two types. Those of the first type join a trivalent vertex $v' \in \Gamma_1$ to its (unchanged) successor $v'' \in \Gamma_2$. Those of the second type join tetravalent vertices $v' \in \Gamma_1$ to the two trivalent vertices $v''_1, v''_2 \in \Gamma_2$ which replace them.

We can use causality to deduce some of the spins on edges of the spin foam $F : \Gamma_0 \mapsto \Gamma_2$, as follows:

• **Spin Rule 1:** The null edges in the internet $\Gamma_0 \mapsto \Gamma_1$ which meet at the vertex $v'_{\{i',j'\}}$ have the same spin as the edge $e'_{\{i',j'\}} \in \Gamma_0$ from which they are constructed.

• **Spin Rule 2:** Null edges of the first type in the internet $\Gamma_0 \mapsto \Gamma_1$ have spin zero. Those of the second type should be labelled with the spin assigned to the ‘virtual edge’ that they share a vertex with.

It will be observed that we have not yet uniquely specified the spin foam $F$, having not assigned spins to the spinnets $\Gamma_1$ and $\Gamma_2$. Finally we assign an amplitude to each possible choice.

• **Amplitude Rule 1:** Each vertex $v_i \in \Gamma_0$ is trivalent, with edges $e_{\{i,j_1\}}, e_{\{i,j_2\}}, e_{\{i,j_3\}} \in \Gamma_0$. To these three edges correspond three vertices $v_{\{i,j_1\}}, v_{\{i,j_2\}}, v_{\{i,j_3\}} \in \Gamma_1$. Now the triangle spanned by these vertices has three edges, each with an undetermined spin. Denote these by $\{j'_i\}$. The triangle forms the base of a past-pointing tetrahedron $T$, the vertex of which is $v_i$. Let the (previously determined) spins on the other 3 (null) edges of this tetrahedron be denoted by $\{j_i\}$. Then, for some suitable choice of $J$, to be discussed later, we get the amplitude

$$A[F : \Gamma_0 \mapsto \Gamma_1] := \prod_{\text{vertices } v_i} J(\{j'_i\}; \{j_i\})$$

• **Amplitude Rule 2:** The amplitudes for the spinnets $\Gamma_2$ with the 3 different possible orientations of an inserted virtual edge are taken to be equal.
6.2.3 The tetrahedron amplitude

The amplitude-assigning map $J$ must be consistent with the causality-preserving symmetries of $T$. Specifically, it must be invariant under those symmetries that map null edges to null edges and spacelike edges to spacelike edges. These are just rotations of $T$ around its spacelike base. This means that we require

$$J(j'_1, j'_2, j'_3; j_1, j_2, j_3) = J(j'_2, j'_3, j'_1; j_1, j_2, j_3) = J(j'_3, j'_1, j'_2; j_1, j_2, j_3)$$

The simplest choice of $J$ satisfying this constraint is referred to as the ‘tetrahedral’ or ‘6$j$’ symbol. (See [RW] for details.)

6.2.4 Generalization to the 4-d case

We make the following modifications:

- We start with a tetravalent spinnet, rather than a trivalent one.
- 15$j$ symbols are used. They generalize 6$j$ symbols.
- The splitting in Restriction Rule 2 is more complex, with a hexavalent node splitting into a pair of tetravalent nodes in 20 ways, rather than a tetravalent node splitting into a pair of trivalent nodes in 3 ways.

6.3 The Barrett-Crane model

The Barrett-Crane model is a much-studied example of a spin foam model. It can be derived from the SO(4) Plebanski formulation of GR, as described in [Per]. Without describing its structure in detail, we outline the argument given in [LO03], which shows how implementing causality within the model gives us reason to hope for a continuum limit.

We make an analogy with quantum field theory, where we have an acausal propagator given by $\langle 0 | \phi(x) \phi^\dagger(y) | 0 \rangle$, and a causal propagator given by $\langle 0 | T[\phi(x) \phi^\dagger(y)] | 0 \rangle$. Similarly, we can have acausal transition amplitudes in spin foam models. A spin-foam $F : s \mapsto s'$ can be time-reversed to give $F^{-1} : s' \mapsto s$. Then, if $A[F] = A[F^{-1}]$, we say that $A$ is acausal.

[LO03] gives a causal transition amplitude for the Barrett-Crane model. So the model is now intrinsically defined on a causal set, and not just on a triangulation of Euclidean space. This represents a step forward because, as we have shown previously, if the model has a continuum limit then we can determine the metric structure of that continuum limit from the causal set structure.
7 Conclusions

We have seen how causality can provide motivation, regularization and continuum limits for models of quantum gravity. Dynamical triangulations and spin foam models (many of which we have not had time to mention) continue to be fruitful areas of research. The field is interesting and eclectic, drawing on techniques from combinatorics, topology and numerical relativity. Its future development will certainly be interesting, and will hopefully shed some light on the most fundamental problems of physics.
References


