Introductory blurb.

QCD … SU(3) with 2 (or 3) light quark.

confines.

chiral sym breaking → SU(2) π-on.

How do we understand ?

SU(3) A_μ : octet — $N_f \sim 10$ stop breaking chiral sym. — 16.5 : free.

ψ_i : triplet $i=1,2,(3)$

consider A_μ : octet

ψ : octet — } fluctuation largely cancels. } "Q" SUSY.

known there are three vacua.

or A_μ : octet

$Q \circlearrowleft$ ψ : octet — $Q \circlearrowleft$ ψ_i : triplet $i=1,2,\ldots$

ϕ : triplet — known at $\underline{i=4}$ stops breaking chiral sym.

or A_μ octet $\xleftrightarrow{Q'}$

$Q \circlearrowleft$ ψ — ψ' octet — $\boxed{\mathcal{N}=2 \text{ SUSY}}$

$\xleftrightarrow[Q']{}$ ϕ' octet } Q — almost completely solved.

can decouple ψ' Q'

monopole condensation. picture

. phenomenological application → ask Kitano san.

plan.

superfields. 7 pages

perturb. non renormaliz. } 2 pages — 4 pages per 1 コマ

one-loop + R-sym

pure $\mathcal{N}=1$ — 1 pages —] first day — 3 コマ

super QCD and Seib. dual. — 4 page —] second day — 2 コマ

pure SU(2) $\mathcal{N}=2$ — 4 pages

seminar. w/ Yonekura —] 2nd day seminar. — 1 コマ

more $\mathcal{N}=2$ + $\mathcal{N}=1$ duality. 7 pages. —] 3rd day — 2 コマ

SF① superfield

$$\boxed{\pm i = \pi = 2 = 1}$$ except for dimension-independent objects.

- Dirac spinor … 4 components, on which $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ act.

Let $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$. $(\gamma^5)^2 = 1$. There's a basis s.t.

$$\gamma^5 = \left(\begin{array}{cc|cc} +1 & & & \\ & +1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right).$$

Call $\gamma^5\psi = \psi$ left handed sp. 2 components each.

$\gamma^5\psi = -\psi$ right "

~~$SO(3,1)$: Lorentz group~~
~~$\cong SL(2,\mathbb{C})$~~

denote them by ψ_α $\bar\psi_{\dot\alpha}$ instead. $\alpha = 1,2$ $\dot\alpha = 1,2$

note that complex conjugates send $\alpha \leftrightarrow \dot\alpha$

$SO(3,1) \simeq SL(2,\mathbb{C})/\{\pm 1\}$

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $g \curvearrowright \mathbb{C}^2$ [illegible]

$ad - bc = 1$ $\bar g \curvearrowright \mathbb{C}^2$

hermite

$X = \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix}$ det $X = t^2 - x^2 - y^2 - z^2$.

$X \to g X g^*$ preserves the det, i.e. the Minkowski metric.

vector $X_\mu \Leftrightarrow$ hermitean $X_{\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\beta} X_\mu$.

Poincaré gp : rotations $M_{\mu\nu}$ and translations P_μ

super-Poincaré gp: add Q_α, $\bar Q_{\dot\alpha}$: fermionic.

$a_\alpha\ b_\beta$

$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is inv.

$\epsilon^{\alpha\beta} a_\alpha b_\beta$

反対称 $\epsilon^{\dot\alpha\dot\beta}$

$$\begin{cases} \{Q_\alpha, \bar Q_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu . \\ \{Q_\alpha, Q_\alpha\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\alpha}\} = 0 \\ [Q_\alpha, P] = 0 . \end{cases}$$

How can we write down a theory which has this symmetry?

Recall how we write a theory with Poincaré sym!

SF ②

$$S[\phi,\dots] = \int d^4x \, (\partial_\mu\phi\partial_\mu\phi) + \cdots$$

Let P_μ acts via $x_\mu \to x_\mu + \delta c_\mu$.

then $\delta \phi = \delta c_\mu \cdot \partial_\mu\phi$

$$\delta(\partial_\nu\phi\cdot\partial_\nu\phi) = \delta c_\mu \partial_\mu(\partial_\nu\phi\partial_\nu\phi).$$

$\Rightarrow$ $$\delta S[\phi,\dots] = \delta c_\mu \int \underbrace{d^4x \, \partial_\mu}\left[\partial_\nu\phi\partial_\nu\phi + \cdots\right]$$

use the fact $\int d^4x \, \partial_\mu[\cdots] = 0$.

$= 0$

$\Rightarrow$ Noether's theorem gives back P_μ as conserved charges.

Let's mimick this. P_μ still acts via $x_\mu \to x_\mu + \delta c_\mu$.

We need a place in which $Q_\alpha, \overline{Q}_{\dot\alpha}$ acts $\theta_\alpha, \theta_{\dot\alpha}$

$\theta_\alpha \to \theta_\alpha + \epsilon_\alpha$, $\overline{\theta}_{\dot\alpha} \to \overline{\theta}_{\dot\alpha} + \overline{\epsilon}_{\dot\alpha}$. θ needs to be fermionic.

But not quite: this makes $\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 0$.

instead.

$$\theta_\alpha \to \theta_\alpha + \epsilon_\alpha$$
$$\overline{\theta}_{\dot\alpha} \to \overline{\theta}_{\dot\alpha} + \overline{\epsilon}_{\dot\alpha}$$
$$x_\mu \to x_\mu - i\epsilon_\alpha \overline{\theta}_{\dot\alpha} \sigma^{\alpha\dot\alpha}_m + i\theta_\alpha\epsilon_{\dot\alpha}\sigma^{\alpha\dot\alpha}_m.$$

(to lowest order in ϵ.)

i.e.
$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\mu$$
$$\overline{Q}_{\dot\alpha} = \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} \quad i\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\partial_\mu$$

$$\boxed{\{Q_\alpha, Q_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} i\partial_\mu}.$$

Then, $\boxed{\text{namely}}$, you consider a field $\Phi(x,\theta,\overline{\theta})$ in the superspace

$$\int d^4x \, d^2\theta \, d^2\overline{\theta} \; \mathcal{L}[\Phi_1, \Phi_2, \dots, Q_\alpha\Phi, \partial_\mu\Phi, \dots]$$

~~first, what~~ This is not quite right. what do we mean by $\Phi(\dots)$ before correcting, or $\int d^2\theta$?

SF ③

$\theta_\alpha \theta_\beta = - \theta_\alpha \theta_\beta$ etc. $\theta_1 \theta_1 = 0$ in particular.

then $\Phi(x, \theta, \bar\theta) = \phi(x) + \phi_\alpha(x)\theta_\alpha + \phi_{\dot\alpha}(x)\bar\theta_{\dot\alpha}$

$\phi_\mu(x)\sigma^\mu_{\alpha\dot\beta}\theta_\alpha\theta_{\dot\beta} >$ $+ \underline{\phi}(x)\,\theta_\alpha\theta_\beta\varepsilon^{\alpha\beta} + \underline{\phi}(x)\,\theta_{\dot\alpha}\theta_{\dot\beta}\,\varepsilon^{\dot\alpha\dot\beta}$

$+ \underline{\phi}_{\dot\alpha}(x)\,\theta_\alpha\theta_\beta\varepsilon^{\alpha\beta}\bar\theta_{\dot\alpha} + \underline{\Phi}_\alpha(x)\,\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}\,\varepsilon^{\dot\alpha\dot\beta}\,\theta_\alpha$

$+ \underline{\underline{\phi}}(x)\,\theta_\alpha\theta_\beta\varepsilon^{\alpha\beta}\,\theta_{\dot\alpha}\theta_{\dot\beta}\,\varepsilon^{\dot\alpha\dot\beta}$.

$Q^\alpha_\alpha = \frac{\partial}{\partial\theta_\alpha} - i\,\theta^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ naturally gives SUSY to among coeffs

e.g. $\phi(x) + \phi_\alpha(x)\theta_\alpha + \phi_{\dot\alpha}(x)\theta_{\dot\alpha}$

$\cancel{\frac{\partial}{\partial\theta_\alpha}}$ $\theta^{\dot\alpha}\sigma_{\alpha\dot\alpha}{}^\mu$

$\delta_{\varepsilon_\alpha}\phi(x) = \varepsilon^\alpha\phi_\alpha(x)$

$\delta_{\varepsilon_\alpha}\phi_{\dot\alpha}(x) = \theta^{\dot\alpha}\varepsilon^\alpha\sigma^m_{\alpha\dot\alpha}\partial_\mu\phi(x) + \quad \cdots$ etc.

NOTE THAT $\underline{\phi}\ldots$ $\underline{\underline{\phi}}(x)\,\theta_\alpha\theta_\beta\varepsilon^{\alpha\beta}\theta_{\dot\alpha}\theta_{\dot\beta}\,\varepsilon^{\dot\alpha\dot\beta}$ $\cancel{\frac{\partial}{\partial\theta}}$ no term!

θ_α $\cancel{\theta\frac{\partial}{\partial\theta}}$

The Q_α top term of the is a total der.

$\delta_{\varepsilon_\alpha}\underline{\underline{\phi}}(x) = \varepsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu[\cdots\quad]_{\dot\alpha}$

We define $\int d\theta_\alpha = \frac{\partial}{\partial\theta_\alpha}$

Then $\int d^4x \int \underbrace{d^2\theta_\alpha d^2\bar\theta_\alpha}_{d^4\theta} \varepsilon^\alpha Q_\alpha [\qquad] = \int d^4x\, \varepsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu[\cdots]_{\dot\alpha}$

$= 0$.

How about $\int d^4x\, d^4\theta \left(Q_\alpha\Phi\, Q_\beta\Phi\ \varepsilon^{\alpha\beta}\right)$? $\underset{\downarrow}{\text{is it}}$ SUSY ? No !

$\Phi \to \Phi + \varepsilon^\gamma Q_\gamma \Phi$.

$Q_\alpha\Phi \to Q_\alpha\Phi + Q_\alpha(\varepsilon^\gamma Q_\gamma\Phi) \neq Q_\alpha\Phi + (\varepsilon^\gamma Q_\gamma)Q_\alpha\Phi$

because $\{Q_\alpha, Q_\gamma\} \neq 0$.

We need some op. D_α so that

$D_\alpha\Phi \to D_\alpha\Phi + D_\alpha(\varepsilon^\gamma Q_\gamma\Phi) = D_\alpha\Phi + (\varepsilon^\gamma Q_\gamma)D_\alpha\Phi$

with this, $\delta(D_\alpha\Phi D_\beta\Phi\,\varepsilon^{\alpha\beta}) = \varepsilon^\gamma Q_\gamma(D_\alpha\Phi D_\beta\Phi\,\varepsilon^{\alpha\beta})$

i.e. $\delta\int d^4x\,d^4\theta\,(D_\alpha\Phi D_\beta\Phi\,\varepsilon^{\alpha\beta}) = \int d^4x\,d^4\theta\,\varepsilon^\gamma Q_\gamma(\qquad) = 0$

$\Rightarrow$ SUSY

SF ④

is given by

SUCH $D_\alpha = \frac{\partial}{\partial\theta^\alpha} \boxed{+} i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$ $\qquad Q_\alpha = \frac{\partial}{\partial\theta^\alpha}\boxed{-} i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$

$\bar D_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} \boxed{+} i\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\partial_\mu \quad \longleftrightarrow \quad \bar Q_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\boxed{-} i\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\partial_\mu$

So,

$$\int d^4x\, d^4\theta \left[\text{poly. of } \begin{array}{l} \Phi(x,\theta,\bar\theta) \\ D_\alpha\Phi(x,\theta,\bar\theta) \\ D_\alpha\bar D_{\dot\beta} \;--- \end{array} \right]$$

is a nice SUSY lagrangian. But still not quite!

Expansion of Φ just have too many fields.

Impose $\bar D_{\dot\alpha}\Phi(x,\theta,\bar\theta) = 0$ this doesn't involve differential eq. of ϕ's.

In fact $\begin{cases} y^\mu = x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} \\ \theta^\alpha \end{cases}$ ~~are inv. under $\bar D_{\dot\alpha}$~~.

$$\Phi(y^\mu,\theta^\alpha) = \phi(y) + \psi^\alpha(y)\theta_\alpha + F(y)\theta_\alpha\theta_\beta\epsilon^{\alpha\beta}.$$

~~$\int d\theta^\alpha d\theta^\beta\, \Phi(y^\mu,\theta^\alpha)$~~

$$\int d^4x \int \epsilon_{\alpha\beta} d\theta^\alpha d\theta^\beta \underset{\bar Q_{\dot\alpha}}{\wedge} \boxed{\text{chiral}} \qquad \int \epsilon_{\alpha\beta} d\theta^\alpha d\theta^\beta\, \bar D_{\dot\alpha}\boxed{} + \partial_\mu = 0$$

$$\int d^4x \int \epsilon_{\alpha\beta}\, d\theta^\alpha d\theta^\beta\, Q_\alpha \boxed{\text{chiral}} = \int d^4x\, \partial_\mu[\,---\,] = 0$$

due to similar reasoning

$$\underset{Q_\alpha}{} \quad \phi \quad \psi^\alpha \quad F(y) \quad \bar\theta\partial_\mu$$

面倒臭いので x と y は区別しない. 状況に $\boxed{}$ よって判断です.

But note this is not real

$\bar D_{\dot\alpha}\Phi = 0 \quad \longleftrightarrow \quad D_\alpha\bar\Phi = 0.$ $\quad (\bar y = x^\mu - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha})$

note also $\int d^2\theta\, d^2\bar\theta \boxed{\text{chiral}} = 0$ up to $\theta\theta\bar\theta\bar\theta$..

So, a nice SUSY Lagrangian is

$$\int d^4x \int d^4\theta \left(\text{poly of } \Phi, \bar\Phi, D_\alpha\Phi, D_\alpha\bar\Phi, \ldots\right) + \int d^4x \int d^2\theta \text{ poly of } \Phi + \text{c.c.}$$

SF⑤

Consider something simple. just take one Φ : chiral

$$S = \int d^4x \int d^4\theta\ \bar{\Phi}\Phi + \int d^4x\, d^2\theta\ W(\Phi) + c.c.$$

without any derivatives. ↑

$$\cancel{\phi(x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}) + \psi \cdots + F(x^\mu + \cdots)\theta\theta}.$$

$$= \phi(x^\mu) + \underline{(\partial_\mu\phi)}\ \underbrace{\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}} + F(x^\mu)\underbrace{\theta\theta}$$

expanding, we got

$$= \int d^4x \left(\partial_\mu\phi\,\partial_\mu\bar\phi + F\bar{F}\right) + \frac{\partial W}{\partial\Phi}F + c.c. + \text{ferm.}$$

eom of F ($\bar{F} = \frac{\partial W}{\partial\Phi}$.

$$= \int d^4x\ \partial_\mu\phi\,\partial_\mu\bar\Phi + \left|\frac{\partial W}{\partial\Phi}\right|^2 + \text{ferm.}$$

$V(\phi) = \left|\frac{\partial W}{\partial\phi}\right|^2$.   note that $\delta\psi_\alpha = \cancel{\epsilon\,\partial_\mu\phi} = \epsilon_\alpha F \cdots$

$$\sim \bar{F} = \frac{\partial W}{\partial\phi} = 0 \leftrightarrow \delta\psi_\alpha = 0 \leftrightarrow \text{SUSY preserved!}$$

BUT WE WANT GAUGE THEORY !

again, let's recall the non-SUSY case.

$$\partial_\mu\phi\,\partial_\mu\bar\phi + \partial_\mu\tilde\phi\,\partial_\mu\bar{\tilde\phi} + \cancel{\phi\bar\phi\phi} + \tilde\phi\bar{\tilde\phi}$$ ~~because of the chinetic term.~~

$\tilde\phi \to \tilde\phi\cdot e^{-i\theta}$

invariant under $\phi \to \phi\cdot e^{i\theta}$, θ : constant.

not inv. under $\phi \to \phi\cdot e^{i\theta(x)}$ $\theta(x)$: x^μ dependent.

$$\partial_\mu\tilde\phi \to \partial_\mu\tilde\phi - i\,\partial_\mu\theta\,\tilde\phi$$
$$\partial_\mu\phi \to \partial_\mu\phi + i(\partial_\mu\theta)\phi.$$

introduce $A_\mu \to A_\mu + \partial_\mu\theta$.   note $|e^{i\theta}| = 1$.

Then $(\partial_\mu + iA_\mu)\tilde\phi =: D_\mu\tilde\phi$

$(\partial_\mu - iA_\mu)\phi \to (\partial_\mu - iA_\mu)\phi =: D_\mu\phi$.

U(1)

unitary 1×1 matrix.

$$\rightsquigarrow D_\mu\phi D_\mu\bar\phi + D_\mu\tilde\phi D_\mu\bar{\tilde\phi} + \cancel{\phi\bar\phi\phi\bar\phi}\ .\ \phi\bar\phi + \tilde\phi\bar{\tilde\phi}$$

$F_{\mu\nu} \sim \partial_\mu A_\nu - \partial_\nu A_\mu$ $|F_{\mu\nu}|^2$.

Follow this in superspace:

$$\int d^2\theta\ \tilde\Phi\,\bar{\tilde\Phi} \longrightarrow V = \tilde\Phi\tilde\Phi^\dagger + \Phi\Phi^\dagger.$$

invariant under $\Phi \to e^{i\varphi}\Phi$ $\tilde\Phi \to e^{-i\varphi}\tilde\Phi$.

make ~~theta~~ φ x, θ "dependent". making it general superf too much.

$$\Phi \to \underbrace{e^{\Lambda}}\Phi$$

↑ want to keep it chiral ⇒ Λ : chiral.

$$\tilde\Phi \to e^{-\Lambda}\tilde\Phi.$$

it doesn't make sense to require $|e^{\Lambda}| = 1$.

But the kin. term $\int d^4\theta\ \Phi\bar\Phi + \tilde\Phi\bar{\tilde\Phi}$ is not invariant.

SF⑥

~~[illegible]~~

general superf.
$\int$ ~~[illegible]~~. real.

$\Phi\bar\Phi \to e^{\Lambda+\bar\Lambda}\Phi\bar\Phi$. introduce a field $V \to V-\Lambda-\bar\Lambda$

then $\int d^4\theta\, \Phi e^V \bar\Phi$ is invariant. $\int d^4\theta\, \hat\Phi e^{-V}\bar{\hat\Phi}$

What's the corresponding thing to $F_{\mu\nu}$? $D_\alpha \bar D_{\dot\alpha} V \to D_\alpha \bar D_{\dot\alpha} V$.

$$W_\alpha = D_\alpha \bar D_{\dot\alpha} \bar D_{\dot\beta} V \varepsilon^{\dot\alpha\dot\beta}$$

This satisfies $\bar D_{\dot\alpha} W_\alpha = 0$. τ.

then one can consider $\int d^2\theta \left(\frac{1}{2e^2}\right) W_\alpha W_\beta \varepsilon^{\alpha\beta}$ + c.c.

now $V(x,\theta,\bar\theta) = \boxed{0+}\ A_\mu \sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\bar\theta^{\dot\alpha} + \lambda_\alpha(\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}\varepsilon_{\dot\alpha\dot\beta})\theta^\alpha$
$+\bar\lambda_{\dot\alpha}(\theta\theta)\bar\theta^{\dot\alpha}$
$+ D\,\theta\theta\bar\theta\bar\theta$

$\Lambda+\Lambda = 0+\ldots$
Λ: pure imaginary

then $W_\alpha = \lambda_\alpha + (\underset{\overset{\uparrow}{F_{\mu\nu}+i\tilde F_{\mu\nu}}}{F_{(\alpha\beta)}}\theta^\beta + D\theta^\alpha) + \partial_\mu\bar\lambda^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}(\theta\theta)$.

note that vector $= \mathbf{2}\otimes\dot{\mathbf{2}}$
asym $= \mathbf{3}\oplus\dot{\mathbf{3}}$

$\Rightarrow \int d^2\theta\,\tau W_\alpha W_\beta \varepsilon^{\alpha\beta} =$ ~~$\tau(F_{\mu\nu}+i\tilde F_{\mu\nu})(F_{\mu\nu}+iF$~~
$= \tau(iF_{\mu\nu}F_{\mu\nu} + F_{\mu\nu}\tilde F_{\mu\nu}) + i\tau D^2 + \ldots$

$\int d^2\theta - + \int d^2\bar\theta - = \underset{\overset{\|}{\frac{1}{e^2}}}{(\mathrm{Im}\,\tau)} F_{\mu\nu}F_{\mu\nu} + (\mathrm{Re}\,\tau)F_{\mu\nu}\tilde F_{\mu\nu} + \underset{\overset{\|}{\frac{1}{e^2}}}{(\mathrm{Im}\,\tau)}D^2$

$(\hat\Phi e^{-V}\bar{\hat\Phi})$

$\int d^4\theta\, \Phi e^V \bar\Phi \sim D(\phi\bar\phi - \hat\phi\bar{\hat\phi})$

then the bos. part of $\int$ is $D(\phi\bar\phi - \hat\phi\bar{\hat\phi}) + \frac{1}{e^2}|F|^2 + \theta F\tilde F + \frac{1}{e^2}D^2$.

$\sim$ ~~$\hat\phi(\phi\bar\phi$~~$)^2$ in the potential.
$e^2|\phi\bar\phi - \hat\phi\bar{\hat\phi}|^2$.

---

in the non abelian case, consider Q_{ia} $\tilde Q^{ia}$

$Q \to UQ$ $\qquad Q_{ia} \to U$~~[illegible]~~Q_{ib}

$\tilde Q \to \tilde Q U^{-1}$ or $\tilde Q \bar U^t$ $\qquad \tilde Q^{ia} \to$ ~~[illegible]~~ $\tilde Q^{ib}(U^{-1})^a{}_b$ $\qquad U$ unitary matrix.

not SUSY! $\qquad Q_{ia} \to U^b{}_a(y,\theta)\,Q_{ib}$

$\tilde Q^{ia} \to \tilde Q^{ia}(U^{-1})^b{}_a(y,\theta)$. $\qquad \int d^2\theta\, Q_{ia}\tilde Q^{ia}$

$\bar Q^{ia}$ $\bar Q^{ia} \to \bar Q^{ib}\bar U_b{}^a\, Q_{ia}$.

so we need $\bar Q^{ia}(e^V)^b{}_a Q_{ib}$

st $e^V \to (\bar U)(e^V U)$. $\qquad$ tr. law of V is complicated cause they don't commute!

SF ⑦

so $W_\alpha = D_\alpha \bar{D}\bar{D} V$ doesn't work either.

$V = \theta + \cdots$ unitary gp.

$e^V = 1 + \cdots$

$U e^V \bar{U} = 1 + \cdots \Rightarrow \boxed{U\bar{U} = 1}$

$W_\alpha = D_\alpha \bar{D}_{\dot\alpha} e^{-V} \bar{D}_{\dot\beta} e^{V} \varepsilon^{\dot\alpha\dot\beta}$ works. what's important is that

$$\int d^2\theta\, \tau\, \mathrm{tr}\, W_\alpha W_\alpha \to \tau(\mathrm{tr}\, FF + \mathrm{tr}\, F\tilde{F}) + (\tau) D^2 .$$

and $\int d^4\theta \left(\bar{Q}^{ia} e^V Q_{ia} + \bar{\tilde{Q}}_{ia} e^{-V} \tilde{Q}^{ia} \right)$ contains

$\bar{Q}^{ib} D^a{}_b Q_{ia} - \bar{\tilde{Q}}_{ia} D^a{}_b \tilde{Q}^{ib}$ in it.

$\rightsquigarrow \quad D^a{}_b = g^2 \left(Q^{ib} \bar{Q}_{ia} - \bar{\tilde{Q}}^{ib} \tilde{Q}_{ia} \right)$ − trace part

$\delta(\lambda_\alpha)^a{}_b = \varepsilon_\alpha D^a{}_b$

(note $D^a{}_a = 0$)

$\Rightarrow V_D = g^2 \left(Q^{ib} \bar{Q}_{ia} - \bar{\tilde{Q}}^{ib} \tilde{Q}_{ia} \right)^2$ sp. unitary.

SUSY vac. $\{ Q^{ib} \bar{Q}_{ia} - \bar{\tilde{Q}}^{ib} Q_{ia} = 0 \} / (Q, \tilde{Q}) \sim (UQ, \tilde{Q} U^{-1})$

−dim SU   −dim SU

If you tell it to mathematicians, they'll be surprised!

$\cong \{ \text{without condition} \} / (Q, \hat{Q}) \sim (UQ, \hat{Q} U^{-1})$

just det $U = 1$, $\bar{U} \neq U^{-1}$

−dim SL $= 2$ dim SU.

Kähler quotient

vs

Complex quotient.

apparent from the SUSY lagrangian. in any case,

$\int \bar{\tilde{Q}} e^{-V} Q + \cdots + \int W$ arb. chiral superf.

~~The system was invariant under~~ $Q \to UQ$, $\tilde{Q} \to \tilde{Q} U^{-1}$

$\Rightarrow$ supersym. inv. state $\simeq \{Q, \tilde{Q}\} /$ this action.

Pertubative non-renormalization. ①

usual global symmetry ~ flavor symmetry

$$\Phi(y,\theta) = \phi(y) + \psi_\alpha(y)\theta^\alpha + F(y)\theta\theta)$$

$$\Phi \to e^{i\varphi}\Phi \qquad \phi \to e^{i\varphi}\phi \qquad \psi \to e^{i\varphi}\psi \qquad F \to e^{i\varphi}F.$$

R-symmetry ... $\theta^\alpha \to e^{i\varphi}\theta^\alpha$.

$$\Phi \to e^{i g\varphi}\,\Phi(y, e^{i\varphi}\theta)$$

$$\phi \to e^{i g\varphi}\phi,\quad \psi \to e^{i(g-1)\varphi}\psi,\quad F \to e^{i(g-2)\varphi}F.$$

For vec superf, $W_\alpha = \underset{\substack{\uparrow \\ \text{R charge } 1}}{\lambda_\alpha} + \underset{\substack{\uparrow \\ \text{R-charge } 0}}{F_{\alpha\beta}}\,\theta^{(\alpha}\theta^{\beta)} + D(\theta\theta) + \cdots$

$$\int \underset{\substack{\uparrow \\ \text{note that this is } \frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^2}.}}{d^2\theta}\underset{\substack{\uparrow \\ \text{needs to be R-charge 2.}}}{\underbrace{\left(W(\Phi) + \tau\,\mathrm{tr}\,W^\alpha W_\alpha\right)}}$$

shift symmetry $\tau \to \tau +$ real number. adds $\theta F\tilde{F}$

WZ-model $\Phi_1 \cdots \Phi_n$ $\quad \mathcal{L} = \int d^4\theta\, \underset{\substack{\text{arb. complicated.} \\ \downarrow}}{K(\Phi,\bar\Phi)} + \int d^2\theta f(\Phi) + \text{c.c.}$

think of it as $= \int d^4\theta\, K(\Phi,\bar\Phi) + \int d^2\theta\, \underset{\text{R charge 2}}{\underbrace{Y(y,\theta)}}\,\underset{0}{\underbrace{f(\Phi)}}$

Y: external chiral bkgd supfield.

later we set it to 1.

$\downarrow$ perturbative calc.

$$= \int d^4\theta\ \boxed{▨} + \int d^2\theta\, Y(y,\theta)\, g(\Phi).$$

you can only use Y^n. R-charge.

Let Y very small. $f = g$. done.

IT NEEDS TO BE EMPHASIZED THAT IT'S NOT THAT low energy is easy! It's just that there's a scheme, in holomorphic.

IR result is a func. of $f(\Phi)$ and Λ.

Similarly, with gauge fields. UV $\int d^2\theta\ ⓣ\ \underset{\text{R-charge 2.}}{\underbrace{\mathrm{tr}\,W^\alpha W^\alpha}}$

think of it as a superf.

τ

$i\frac{1}{g^2}(\Lambda_{UV}') + i\theta$

Λ_{UV}

Λ_{UV}'

$$= i\frac{1}{g^2}(\Lambda_{UV}) + i\theta + i b \log\frac{\Lambda_{UV}'}{\Lambda_{UV}}\quad \left(i\frac{1}{g^2} + i\theta\right) + \underset{\substack{\uparrow \\ \text{1-loop}}}{O(1)} + \underset{\substack{\uparrow \\ \text{2-loop}}}{g^2} + \underset{\substack{\uparrow \\ \text{3 loop}}}{g^4} + \cdots$$

$\Downarrow$

$$\Lambda^b \equiv (\Lambda_{UV}')^b\, e^{-\frac{1}{g^2}(\Lambda_{UV}') + i\theta} = (\Lambda_{UV})^b\, e^{-\frac{1}{g^2}(\Lambda_{UV}) + i\theta}$$

can't appear in $\int d^2\theta$ $\quad (\tau - \bar\tau)^{-1} \quad (\tau - \bar\tau)^{-2} \cdots$

$\Leftarrow$ complex! backgd superfield.

3 $\binom{\cdot}{\cdot}$ $\binom{\cdot}{\cdot}$ $-\frac{1}{2}$

1-loop renormalization & R-symmetry anomaly ①

<4/21>

$$\Lambda \frac{d}{d\Lambda}(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C(\text{adj}) - \frac{2}{3}\underset{\substack{\uparrow \\ \text{left handed}}}{C(\text{fermion})} \; \cancel{+} - \frac{1}{6}\underset{\substack{\uparrow \\ \text{real}}}{C(\text{scalars})}\right]$$

$$\boxed{\begin{aligned}&\operatorname{tr}\rho(T^a)\rho(T^b)\\ &= C(\rho)\,\delta^{ab}\end{aligned}}$$

$$\frac{11}{3} - \frac{8}{3} \cancel{+} - \frac{6}{6} = 0.$$

complex scalars.

$C(\text{adj}) = N$

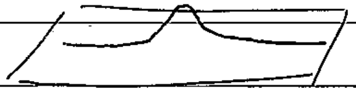
$C(\text{fund}) = \frac{1}{2}$

$$= -\frac{g^3}{(4\pi)^2}\Big[\underbrace{3C(\text{adj}) - C(\text{chiral mult.})}\Big]$$

$$\Lambda \frac{d}{d\Lambda}\frac{8\pi^2}{g^2} = 3C(\text{adj}) - C(\text{chiral mult.}) \; \boxed{b}$$

N.B. with three adjoint. $= 0$. $\mathcal{N}=4$.

<5/2>



$\mathbb{R}^4$

$$\int_{\mathbb{R}^4}\operatorname{tr} F_{\mu\nu}\tilde{F}_{\mu\nu} = ▨ \times \underset{\substack{\uparrow \\ \mathbb{Z}}}{k}$$

~~[illegible]~~ θ-angle. … $\propto \theta\int\operatorname{tr}F_{\mu\nu}\hat{F}_{\mu\nu}$ in the Lagrangian. normalize so that $\boxed{e^{i\theta k}}$

total derivative → doesn't affect perturbation.

$\theta \sim \theta + 2\pi$

$$e^{-\frac{1}{g^2}\int F_{\mu\nu}F_{\mu\nu}} \sim e^{-\frac{k}{g^2}} \text{ : derivatives at } g=0 \text{ are all zero}$$

$$e^{-\left(\frac{8\pi^2}{g^2}+i\theta\right)k} = e^{2\pi i \tau}$$

$\partial_\mu \operatorname{tr}\rho(A_\mu)$

$$\tau = i\frac{4\pi}{g^2} + \frac{\theta}{2\pi}$$

↑ the comb. which mult. $\int d^2\theta\, W_\alpha W^\alpha$

Consider charged fermion. $\bar{\psi}_{\dot{\alpha}} D_\mu \psi_\alpha \sigma^\mu_{\alpha\dot{\alpha}}$

there are $n = 2C(\rho)k$ fermion zero modes.

$\langle \ldots \rangle = 0$ unless you have $\langle \underbrace{\psi\psi\cdots\psi}_{2C(\rho)k} \rangle \neq 0$

path integral $\int[D\psi(x)] = \int dc_1\, dc_2 \cdots dc_n \cdots \quad \leftarrow = \frac{\partial}{\partial c_1}\frac{\partial}{\partial c_2}\cdots\frac{\partial}{\partial c_n}$

$$\psi(x) = \sum_{i=1} c_i \psi^i(x) + \sum \cdots$$

zero mode — nonzero modes

classically $\psi \to e^{i\varphi}\psi$ is a sym. QMlly, only $\psi \to e^{\frac{2\pi i}{2C(\rho)}}\psi$

or ~~by compensating~~ it shifts θ by $\theta \to \theta + 2C(\rho)\varphi$.

In SUSY. A_μ λ ψ ← R-charge $-\frac{1}{3}$.

renormaliz. massless R-charge 1. ϕ ← R-charge $2/3$ $\quad \int d^2\theta\, \Phi^3$

$\lambda \to e^{i\varphi}\lambda$

$\psi \to e^{-i\varphi/3}\lambda$

$\theta \to \theta + \frac{\varphi}{}\left(2C(\text{adj}) - \frac{2}{3}C(\text{chiral})\right)$

relation? significance?

$$\boxed{e^{-\frac{8\pi^2}{g^2}+i\theta}} \to \underline{e^{i\frac{2b}{3}\varphi}\, e^{-\frac{8\pi^2}{g^2}+i\theta}}$$

$$\Lambda^b \qquad e^{i\frac{2b}{3}\varphi}\,\Lambda^b$$

R-charge $= \frac{2b}{3}$.

Pure $\mathcal{N}=1$ SYM

Just A_μ λ_α both adjoint. UV input. $\int d^2\theta \, \tau_0 \, \mathrm{tr} W^\alpha W_\alpha + c.c.$

$G = SU(N)$ $C(adj) = N$ $\overset{\uparrow}{\Lambda^{3N}} = e^{-\frac{8\pi^2}{g^2}(\Lambda_0) + i\theta} \Lambda_0^{3N}$

R-charge $2N$

What's the IR superpotential? Should be R-charge 2. only candidate $= c\Lambda^3$.

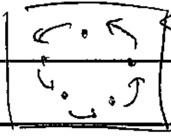
Assume ~~a~~ nonzero $c \neq 0$. $\langle \mathrm{tr} \underset{\sim}{W^\alpha W_\alpha} \rangle = \frac{\partial}{\partial \tau_0}$ (eff. superp.) $= \Lambda^3$

$\lambda^\alpha \lambda_\alpha$: chiral condensate of gaugino.

Note that there're N roots : $e^{\frac{2\pi i}{N} k} \Lambda^3 = (\Lambda^{3N})^{1/N}$. $\Leftarrow$ "$\frac{1}{N}$-instanton"

indeed. $\lambda_\alpha \to e^{i\varphi} \lambda_\alpha$ does $\theta \to \theta + (2N)\varphi$. $\therefore$ $\varphi = \frac{2\pi k}{2N}$ is unbroken.

$\langle \lambda_\alpha \lambda^\alpha \rangle \to e^{i\frac{2\pi k}{N}} \langle \lambda_\alpha \lambda^\alpha \rangle$: N vacua.



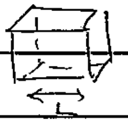
no such classical sol'n

cf. in non-SUSY QCD, [illegible] $\to \cancel{U(1)_A} \times SU(N_f)$

$\downarrow$ η'! $\quad\downarrow$ π-on

in general G. you expect $C(adj)$ vacua. $SO(n) \to$ $n-2$ vacua.

How do you know $c \neq 0$? "Witten index". put the system into a box. with periodic b.c.



$\times \uparrow \mathbb{R}_t$ perform KK reduction.

It's a QM. SUSY QM.

$P^0 = H$. $\underset{Q^2}{\{Q, Q\}} = H$. $H|E\rangle = E|E\rangle$.

$\left| Q|E\rangle \right|^2 = \langle E| QQ |E\rangle = \langle E|H|E\rangle = E\langle E|E\rangle$ assume $(-)^F$.

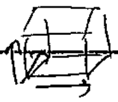
so two patterns: a state is zero energy. $|E=0, i\rangle_b$ $|E=0$. fermionic$\rangle$

non-zero energy $|E, i\rangle_b \underset{Q}{\overset{Q}{\rightleftarrows}} |E, i\rangle_f$

\# bosonic zero energy st $-$ \# ferm. zero energy can't be changed. $\mathrm{tr}(-1)^F$

Let the size of the box $L \ll (\Lambda)^{-1}$ perturbative. $e^{i\theta_{1k}}$ $k=1 \cdots N$

what's the zero-energy states? $F_{\mu\nu} = 0 \to$



$U_1 = \mathrm{diag}(\cdots)$ $\leftarrow e^{i\theta_{1k}}$

$U_2 = \mathrm{diag}(\cdots)$ $\leftarrow e^{i\theta_{2k}}$ $k=1 \cdots N$

$U_3 = \mathrm{diag}(\cdots)$ $\leftarrow e^{i\theta_{3k}}$ $k=1 \cdots N$

$\Sigma\theta_{3k} = \Sigma\theta_{1k} = \Sigma\theta_{2k} = 0$

$\underbrace{\lambda_{\alpha_1}^{i_1} \lambda_{\alpha_2}^{i_2} \cdots \lambda_{\alpha_k}^{i_k}}_{\text{needs to be gauge inv.}} \boxed{\psi(\theta_{1k}; \theta_{2k}; \theta_{3k})}$ also λ_1, λ_2 also diag.

needs to be const $|vac\rangle$

$|vac\rangle$, $(\lambda_\alpha^a \lambda_\beta^a \epsilon^{\alpha\beta})|vac\rangle$, $\cdots$, $(\lambda_\alpha^a \lambda_\beta^a \epsilon^{\alpha\beta})^{N-1} |vac\rangle$ | higher ones are zero!

We find N of them.

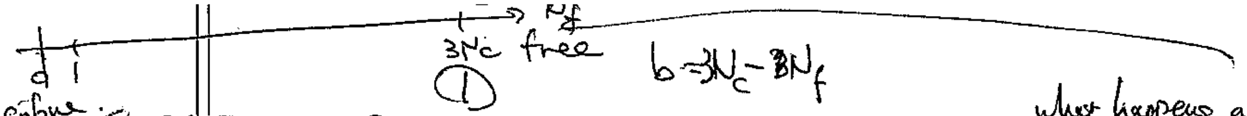
$SO(n) \Rightarrow SO(n)$ component $+$ $SO(n-7)$ component $\Leftarrow$ commutant of

$\lfloor \frac{n}{2} \rfloor + 1 + \lfloor \frac{n-7}{2} \rfloor + 1 = n-2$.

$U_1 = + \; + \; - \; - \; - \; + \; -$

$U_2 = + \; - \; + \; - \; + \; - \; -$

$U_3 = - \; + \; + \; + \; - \; - \; -$



$b = 3N_c - 3N_f$

draw a graph and fill them gradually

what happens at, We learned $N_f = 0$

when $N_f = 3N_c$, one loop $= 0$

two loop decreases g.

what happens in between?

crfine

SUPER QCD

$SO(N)$ N

A_μ			
λ_α	$\psi_\alpha^{\alpha i}\,\psi_{\alpha\, a i}$	$a = 1 \cdots N_c$	
	$Q^{ai}\ \tilde{Q}_{ai}$	$i = 1 \cdots N_f$.	

symmetry: $SU(N_f) \curvearrowright Q^i$ $\qquad U(1) \curvearrowright Q^i \to e^{i\varphi} Q^i$ $\qquad \Lambda^b \to e^{i\varphi \overset{N_f}{}} \Lambda^b$

$SU(N_f) \curvearrowright \tilde{Q}_i$ $\qquad U(1) \curvearrowright \tilde{Q}^q{}_i \to e^{i\varphi} Q_c$ $\qquad \Lambda^b \to e^{i\varphi N_f} \Lambda^b$

R-sym $\quad Q : 0 \quad \tilde{Q} : 0 \qquad A : 0 \qquad \lambda \to \lambda e^{i\varphi} \qquad \Lambda^b \to \Lambda^b e^{i\varphi(\overset{2}{}N_c - \overset{2}{}N_f)}$

$\psi \to \psi e^{-i\varphi}$

$\tilde{\psi} \to \tilde{\psi} e^{-i\varphi}$

gauge inv chiral op $M^i{}_j = Q^{ia} \tilde{Q}_{ja} \Rightarrow \det M$: inv under $SU(N_f) \times SU(N_f)$

$\Lambda^b / \det M$: inv under $U(1) \times U(1)$.

$C_{N_f} \int d^2\theta \left(\Lambda^b / \det M \right)^{\frac{1}{N_c - N_f}}$: R-charge 2.

sensible when $N_c > N_f$. $\quad N_c = N_f \Rightarrow \psi'' X \quad N_c < N_f$: $e^{+\frac{1}{g^2}}$ instead of $e^{-\frac{1}{g^2}}$. A''X, 消え

Affleck-Dine-Seiberg superpotential $\qquad C_{N_f}$ believed to be non-zero.

e.g. $N_f = 1 \quad W = \left(\frac{\Lambda^{3N_c - 1}}{Q\tilde{Q}} \right)^{\frac{1}{N_c - 1}}$. add the mass term $c \left(\frac{\Lambda^{3N_c-1}}{\boxed{Q\tilde{Q}}_M} \right)^{\frac{1}{N_c-1}} + m \boxed{Q\tilde{Q}}_M$

$c (\Lambda^{3N_c - 1})^{\frac{1}{N_c - 1}} (Q\tilde{Q})^{\frac{-N_c}{N_c - 1}} = m^{N_c - 1} \Longrightarrow (Q\tilde{Q}) = c(\quad)^{\frac{1}{N_c}}$

note that with $m = 0$ there's no vacuum. $\boxed{\text{"runaway"}}$ $\qquad \sim N_c$ vacuum. if $c \neq 0$.

$N_f = N_c - 1 \quad W = \left(\frac{\Lambda^{3N_c - N_c + 1}}{\det M} \right)^{1}$: 1-instanton factor.

consider n_f random vectors in $\mathbb{R}^{N_c}$ --> $SO(N_c - N_f)$ unbroken.

similarly, $SU(N_c - N_f)$ unbroken. $\Rightarrow$ if $N_f = N_c - 1$. no unbroken gauge

$\Lambda_A^{3N_c - N_f} = e^{-\frac{8\pi^2}{g^2}(\Lambda') + i\theta} \Lambda'^{\,3N_c - N_f}$ $\to$ weakly coupled. $\to$ calculable. (ADS)

A: $\quad SU(N_c) \quad N_f$

$\downarrow$ - - - - - - - at scale $\Lambda_{\text{transition}} \sim (\det M)^{1/2N_f}$

B: $\quad SU(N_c - N_f) \quad N_f = 0$ pure.

$\Lambda_B^{3N_c - 3N_f} = e^{-\frac{8\pi^2}{g^2}(\Lambda'') + i\theta} \Lambda''^{\,3N_c - 3N_f}$ $\qquad$ should be equal $\sim \det M^{1/2N_f}$.

$\therefore \Lambda_A^{3N_c - N_f} = \Lambda_B^{3N_c - 3N_f} (\det M)$.

B's superpotential $= \Lambda_B^3 = \left(\frac{\Lambda_A^{3N_c - N_f}}{\det M} \right)^{\frac{1}{N_c - N_f}}$.

what happens when $N_f = N_c$?

$B = \epsilon_{a_1 \cdots a_N} Q^{a_1 i_1} Q^{a_2 i_2} \cdots Q^{a_N i_N}$ $\quad \updownarrow \quad$ automatically antisym in i's. $SU(N_f)$-inv.

$\tilde{B} =$

classically, $\det M = B\tilde{B}$. $\qquad \epsilon^{a_1 \cdots a_N} \epsilon_{b_1 \cdots b_N} = \delta^{a_1}_{[b_1} \delta^{a_2}_{b_2} \cdots \delta^{a_N}_{b_N]}$.

SUSY QCD ②.

QM'lly, $\det M - B\tilde{B} = \Lambda^{3N_c - N_c}$: mass dim, R-sym, U(1)-charges all OK.

<u>1-instanton</u> effect.

SU(0) → calculable. (Beasly-Witten)

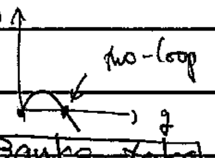
~~you might say how can a relation like~~

$\varepsilon^{\cdots}\varepsilon_{\cdots} = \delta_{\cdots}\cdots\delta$ be modified?

operators can't be at a same point. $Q^{ia}(x)\exp\left(\int_x^y A_\mu dx^\mu\right)\tilde{Q}_{jb}(y)$

$x \to y$.

What happens when $N_f \lessapprox 3N_c$?



at $g^2 \sim O(\varepsilon)$. $\varepsilon = 3 - \frac{N_f}{N_c}$.

very weakly coupled, SCFT. (Banks Zaks)

How is it compatible with "one-loop exactness" of the running of $\tau = i\frac{4\pi}{g^2} + \frac{\theta}{2\pi}$?

$$\int \begin{matrix} Q^\dagger e^V Q \\ + \tilde{Q}^\dagger e^{V}\tilde{Q} \end{matrix} + \int \left(\frac{4\pi}{g^2}(\Lambda') + \frac{\theta}{2\pi}\right) \mathrm{tr}\, W^\alpha W^\alpha$$

$$\hookrightarrow \int \underbrace{Z}_{}\begin{matrix}(Q^\dagger e^V Q) \\ + \tilde{Q} e^V \tilde{Q}\end{matrix} + \int \left(i\frac{4\pi}{g^2_{hol}}(\Lambda'') + \frac{\theta}{2\pi}\right) \mathrm{tr}\, W^\alpha W^\alpha \qquad \frac{4\pi}{g^2_{hol}}(\Lambda'') = \frac{4\pi}{g^2}(\Lambda') + \frac{b}{2\pi}\log\frac{\Lambda''}{\Lambda'}$$

$1 + \left(\log\frac{\Lambda''}{\Lambda'}\right)\gamma(g^2)$.

There's a relation $\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}(Q^\dagger e^V Q) = \mathrm{tr}\, W^\alpha W_\alpha$. "Konishi anomaly"

$\partial_\mu(\psi^\dagger \gamma^\mu \psi) = \mathrm{tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu}$ standard anomaly

also $\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}(\tilde{Q}^\dagger e^{-V}\tilde{Q}) = \mathrm{tr}\, W^\alpha W^\alpha$

so $\int \underbrace{d^4\theta}_{d^2\theta\, d^2\bar\theta} \log\frac{\Lambda''}{\Lambda'}\gamma(g^2)\, Q^\dagger e^V Q = \int d^2\theta \underbrace{}_{\left(\log\frac{\Lambda''}{\Lambda'}\right)\gamma(g^2)} \bar{D}^2(Q^\dagger e^V Q) = \int d^2\theta \left(\log\frac{\Lambda''}{\Lambda'}\right)\mathrm{tr}\, W^\alpha W$

Then $\hookrightarrow \int Q^\dagger e^V Q + \int\left(\left(i\frac{4\pi}{g^2_{phys}}(\Lambda'') + \frac{\theta}{2\pi}\right)\mathrm{tr}\, W^\alpha W_\alpha\right)$ can cancel.

where $\frac{4\pi}{g^2_{phys}}(\Lambda'') = \frac{4\pi}{g^2}(\Lambda') + \boxed{\frac{b}{2\pi}}\left(\log\frac{\Lambda''}{\Lambda'}\right) + \boxed{\gamma(g^2)}\left(\log\frac{\Lambda''}{\Lambda'}\right)$.

note that this is <u>not</u> holomorphic in τ. dep. on $\tau - \bar\tau$

γ is called the <u>anomalous dimension</u>, 'cause

$\underbrace{\int d^4x\, d^4\theta}_{\text{dim } -2.} \quad \underbrace{Q^\dagger e^V Q}_{\text{dim } 2.} \qquad x \to e^{\lambda} x \quad \theta \to e^{\lambda/2} x \qquad Q \to e^{\lambda} Q$ classically.

$\left(1 + \frac{\gamma}{2}\underbrace{\log\frac{\Lambda'}{\Lambda''}}_{\lambda}\right)$

$Q \to e^{\lambda + \frac{\gamma}{2}\lambda} Q$.

Q's dimension is $1 + \frac{\gamma}{2}$.

SQCD ③

$\gamma(g^*)$ at the conf. point can be exactly determined!

note $\gamma(g^2)$ is a perturbative series in g,
g^* itself is given by solving
$\beta(g)=0$, where $\beta(g)$ is a perturbative ...
both β & γ are scheme dependent. but $\gamma(g^*)$ is not

SCFT CFT.

$Q_\alpha \; Q^\dagger_{\dot\alpha}$ $\quad P_\mu \quad \boxed{K_\mu}$

$S_\alpha \; S^\dagger_{\dot\alpha}$ $\quad D$

$I : x^\mu \to \frac{x^\mu}{|x^\mu|^2}$ $\quad K_\mu = I \cdot P_\mu \cdot I$.

$S_\alpha = I \cdot Q_\alpha \cdot I$ superconformal $\leftrightarrow$ any R-symmetry

R "R-symmetry" $\quad [R, Q] = -Q, \quad [R, S] = +S$.

$$\{Q^\dagger_{\dot\alpha}, S^\dagger_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}(2iD - 3R) + M_{\dot\alpha\dot\beta}$$

now an operator is chiral $\leftrightarrow$ annihilated by $Q^\dagger_{\dot\alpha}$.

scalar $\leftrightarrow$ $M_{\dot\alpha\dot\beta}$ is zero. $\quad \theta \to e^{D(\theta)}\theta$

$\Rightarrow 2iD - 3R = 0$. $\quad \theta \to e^{iR(\theta)}\theta$

$\Rightarrow R(\theta) = \frac{2}{3}D(\theta)$.

e.g. a free chiral field. Φ: dimension 1. $\quad \int \underbrace{d^2\theta}_{\text{R-charge } 2} \Phi^3$ $\quad \Phi$: R-charge $\frac{2}{3}$.

~~superconformal R-sym is anomaly free.~~

A_μ

$\lambda_\alpha \quad \psi \quad \tilde\psi$

$\quad\quad Q \quad \tilde Q$

$\lambda_\alpha \to e^{i\varphi}\lambda_\alpha$

$\psi_\alpha \to e^{-i\varphi}\psi_\alpha$

$\tilde\psi_\alpha \to e^{-i\varphi}\tilde\psi_\alpha$

$\theta +$

$\theta \to 2N_c\varphi - 2N_f\varphi$.

$\therefore \; = \frac{N_c}{N_f}$.

$\therefore Q \to e^{(1-\frac{N_c}{N_f})i\varphi}Q \quad R(Q) = 1 - \frac{N_c}{N_f}$.

~~note that when $N_f \lesssim 3N_c$,~~ $\frac{N_c}{N_f} \gtrsim \frac{1}{3}$. $\leftarrow$ $D(Q) = \frac{3}{2} - \frac{3N_c}{2N_f}$

$\therefore D(Q) \lesssim 1$.

$D(M) = 3 - 3\frac{N_c}{N_f}$. becomes 1 at $N_f = \frac{3}{2}N_c$.

$\boxed{\text{gauge inv}}$ scalar op has $D(M) \geq 1$. $\leadsto$ $N_f < \frac{3}{2}N_c$ can't be SCFT.
use unitarity. note that gauge dep. Hilb. space is not unitary.

Seiberg duality when $N_f \geq N_c + 2$

$SU(N_c)$ with N_f $\quad \overline{\overline{\text{at IR}}} \quad$ $SU(N_f - N_c)$ with N_f

$Q^{ai}, \tilde Q_{ai}$ $\quad\quad q_{ai}\,\tilde q^{ai}$ plus $M^i{}_j$

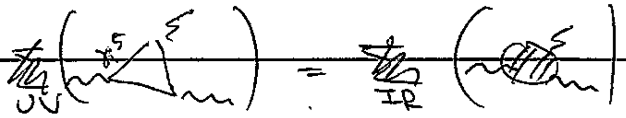
with $W = M^i{}_j\, q_i \tilde q^j$.

$M^i{}_j = Q^{ai}\tilde Q_{aj} \longleftrightarrow M^i{}_j$. $\quad$ (note $q_i \tilde q^j = 0$ due to $\frac{\partial W}{\partial M}$).

$\binom{N_f}{N_c} = \binom{N_f}{N_f - N_c}$ $\quad B^{i_1 \cdots i_N} = \epsilon_{a_1 \cdots a_N} Q^{a_1 i_1} \cdots Q^{a_N i_N} \longleftrightarrow B_{i_1 \cdots i_{N_f - N_c}} =$

SQCD ④.

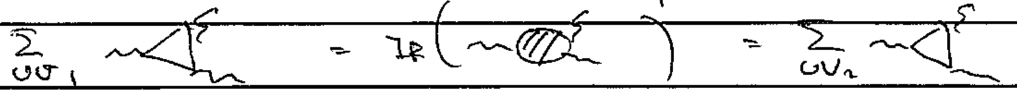
't Hooft anomaly matching.



't Hooft's argument : gauge it. you can't.

add a spectator fermion which cancels it.

Coleman - Grossman : more operator-ish.



because subscript. ①

eg. $SU(N_f)_L$ Q^{ac} : N_c mult. in $\triangleleft$ ⟵ $\tilde{q}_{a\bar{c}}$: $N_f - N_c$ mult. in $\triangleleft$

$M^{Ⓒ}_{j}$: $\times N_f$

you can check others.

at one extreme, $SU(N_c)$ with $N_f = N_c + 1$ = $SU(1)$ with $N_f = N_{c+1}$ q_i $\tilde{q}^i$

plus $M^{i}{}_{j}$.

$M^{i}{}_{j}$ ⟷

$B^{i_1 \cdots i_N}$ ⟷ q_i. $W = q_i M^{i}{}_{j} \tilde{q}^{j}$

$\boxed{+ \det M}$

why do we have it?

it just says you have superp

~~assuming this,~~ $W = B_i M^{i}{}_{j} \tilde{B}^{j} + \det M$

add $m Q_{N_c}^{N_c+1} Q_{N_c+1}$ ⟶ $m M^{N_{c+1}}{}_{N_{c+1}}$.

⟶ $B_{N_{c+1}} \tilde{B}^{N_{c+1}} + \det \hat{M} = m (\Lambda)^{\#}$. → reproduces mod.

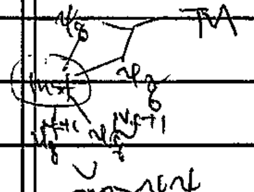
to understand the appearance of $\det M$, consider

$SU(\cancel{2}')$ with q_i $\tilde{q}^i$ $M^{i}{}_{j}$, $N_f' = N_f + 1$, $W = q_i \tilde{q}^j M^{i}{}_{j}$

add

~~give a vev to the last~~ $m M^{N_f+1}{}_{N_f+1}$. q_{N_f+1} $\tilde{q}^{N_f+1}$ ~~become massive.~~ have vevs → break SU(2).

$$\boxed{\underbrace{\left(\Lambda^{3\cdot 2 - (N_f+1)}\right)}_{\uparrow \text{ lowest factor.}} \left(\det M_{N_f \times N_f}\right)} \boxed{M^2}$$ ⇐ mass dim $\cancel{3}$?? R-charge 2



$6 - N_f - 1$

$+N_f$

$+2$

$U(1)_L$: $q \to e^{i\varphi} q$ $M \to e^{-i\varphi} q$.

~~mass dim~~ requires. $\Lambda^{b_0} \det M$.

R charge m : 0 $q, \tilde{q} = 0$ M : 2

$\Lambda^{3\cdot 2 - (N_f+1)}$: $4 - 2(N_f+1)$.

~~$\det M$~~ : $2N_f$.

2.

~~$N_f = \frac{3}{2} N_c$ ⟺ $SU(\frac{N_f}{2})$ $\frac{3}{2}N_f$: upper bound!~~

in addition to chiral ops & anomalies, we can now compare part. func. on $S^1 \times S^3$

also called "supercnf. index" Dolan-Osborn 0801.4947 Römelsberger 0707.3702

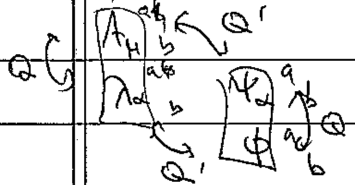
Spiridonov-Vartanov ... Rastelli 1011.5278

~~take whatever dual pair in the literature,~~ check it, ~~and write a paper!~~

hep-th / 9407087) extremely readable.
/ 9408099

SW ①

So far, we considered Q in the fundamental. what'll happen Φ : adj ?



automatically $\underline{\mathcal{N}=2}$ supersymmetric
↖ it's unfortunate, curly $\mathcal{N}$.

$$\mathcal{L} = \int d^4\theta \, \mathrm{tr}\, \Phi^\dagger e^{[V,\ ]}\Phi + \int \tau \,\mathrm{tr}\, W^\alpha W_\alpha + c.c.$$

change normalization of Φ

$$= \mathrm{Im}\,\tau \int \Phi^\dagger e^{[V,\ ]}\Phi + \int \tau \,\mathrm{tr}\, W^\alpha W_\alpha + c.c.$$

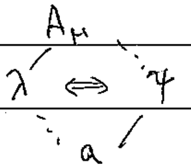
$$V = \frac{1}{\mathrm{Im}\,\tau} \mathrm{tr}[\Phi, \Phi^\dagger]^2$$

→ $\phi = \mathrm{diag}(a_1 \cdots a_N)$ $\quad \Sigma a_i = 0$ is a classical vac. $SU(N) \to U(1)^{N-1}$.

for simplicity, let's only consider $N=2$. $\quad \phi = \mathrm{diag}(a, -a)$

$u = \frac{1}{2}\mathrm{tr}\,\phi^2 = a^2$ classically. $\qquad |D_\mu \phi|^2 \cdots [A_\mu, \phi]^2 \Rightarrow A_\mu^2 |a|^2.$

$= a^2 + \cdots$ ← quantum corrections. $\qquad \Rightarrow$ W-boson mass $\simeq |a|$.



$$\int d^4\theta \; K(\bar{a}, a) + \int d^2\theta \; \tau(a) \, W^\alpha W_\alpha ,$$

$$\overset{\psi\psi \text{ kin}}{\nearrow} \frac{\partial^2 K}{\partial a \, \partial \bar{a}} = \underset{\lambda\lambda \text{ kin.}}{\mathrm{Im}\,\tau} = \tau - \bar{\tau}.$$

let's say $\left\{ \tau = \frac{\partial a_D}{\partial a} \quad \underset{\text{dual.}}{a_D} = \frac{\partial F}{\partial a} \right.$

Then $K = \underline{\bar{a}_D a - \bar{a} a_D}$ solves the eq. where

by determining F, you also determine K.

consider a monopole solution: $\displaystyle\int \frac{1}{g^2}(\phi) F_{\mu\nu}F_{\mu\nu} + \underline{\theta(a) F_{\mu\nu}\tilde{F}_{\mu\nu}}$
↖ not a total derivative!

Bianchi : $\partial_{[\mu} F_{\nu\rho]} = 0$

EOM : $\partial_\mu \big(\underbrace{\tfrac{1}{g^2} F_{\mu\nu} + \theta \tilde{F}_{\mu\nu}}_{\text{"electric field"}} \big) = 0.$

"electric field" $= 0$

→ $F_{0i} = g^2\theta \, F_{jk}$.

$$\frac{1}{2} a x^2 + \frac{1}{2}\frac{y^2}{a}$$

$$\swarrow \quad = \frac{1}{2} a \left(x + \frac{y}{a}\right)^2 + xy$$

$$H = \frac{1}{2}(\mathrm{Im}\,\tau)\, \partial_\mu \bar{a}\, \partial_\mu a + \frac{1}{2}\vec{B}\,\tau\,(\mathrm{Im}\,\tau)^{-1}\bar{\tau}\,\vec{B}.$$

$$= \frac{1}{2}\underline{\boxed{\mathrm{Im}(-\tau^{-1})}}\, \partial_\mu \bar{a}_D\, \partial_\mu a_D + \frac{1}{2}\vec{B}\,\underline{\left(\mathrm{Im}(-\tau^{-1})\right)^{-1}}\vec{B}.$$

equality if $\mathrm{Im}(-\tau^{-1})\partial_\mu a = B$.

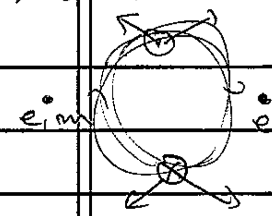
$$\geq |\vec{B} \cdot \vec{\nabla} a_D|. \quad \longleftarrow \vec{\nabla} \cdot \vec{B} = 0$$

$$\int d^3x \, H \geq \int d^3x \, |\vec{B} \cdot \vec{\nabla} a_D| \geq \left| \int d^3x \, \vec{B} \cdot \vec{\nabla} a_D \right| = \left| \int_{S^2} d\vec{n} \, (\vec{n} \cdot \vec{B}) \, a_D \right|$$

$$= |\underset{\substack{\uparrow \\ \text{magnetic charge.}}}{m}\, a_D|$$

in general, mass $= |ea + m a_D|$ $\quad$ # supercharge $\to 2^4 = 16$
'saturated' → # states.

SW ②



ang. mom. $l = em' - e'm = \frac{\hbar}{2} \times$ integer.

$\begin{pmatrix}e\\0\end{pmatrix}\begin{pmatrix}0\\m\end{pmatrix}$ $l = em$

$\begin{pmatrix}0\\e\end{pmatrix}\begin{pmatrix}-m\\0\end{pmatrix}$ $l = em$

$S = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$

$\frac{\partial a_D}{\partial a} = \tau$

$\frac{\partial -a}{\partial a_D} = -\frac{1}{\tau}$.

$\begin{pmatrix}e\\m\end{pmatrix} \to \begin{pmatrix}e+m\\m\end{pmatrix}$ $T = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$

$\tau \to \tau + 1$

$\left|(a, a_D)\begin{pmatrix}e\\m\end{pmatrix}\right|$.

in general, $\begin{pmatrix}p & q\\r & s\end{pmatrix}$: $\tau \mapsto \frac{p\tau + q}{r\tau + s}$

QM lly one-inst factor $\Lambda^4 = \Lambda_{scale}\, e^{-\frac{8\pi}{g^2}(\Lambda_{scale}) + i\theta_{UV}}$.

R-charge 0 A
1 ψ_1 ψ_2
2 ϕ

$\theta_{UV} \to \theta_{UV} + 2\pi$ $\Phi \to i\Phi$, $u \to -u$.

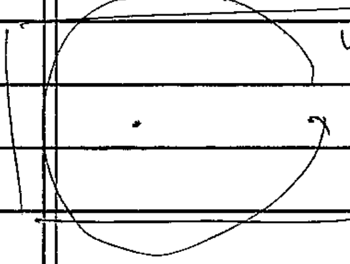
$\phi \to e^{i\varphi}\phi$
$\psi \to e^{i\varphi/2}\psi$
$\theta \to \theta + 4\varphi$

convention.

$F(\Phi) = \frac{1}{2}\tau_{UV}\Phi^2 \rightsquigarrow F(a) = \tau_{UV}a^2 - \frac{4a^2}{2\pi i}\log\frac{a}{\Lambda_{UV}} + \cdots$

$\rightsquigarrow \tau(a) = \boxed{2\tau_{UV}} - \frac{8}{2\pi i}\log\frac{a}{\boxed{\Lambda_{UV}}}$.

$= -\frac{8}{2\pi i}\log\frac{a}{\Lambda}$.



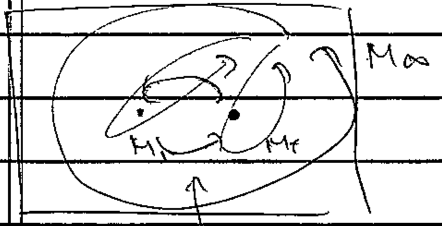
$u = \langle \frac{1}{2}\mathrm{tr}\,\phi^2 \rangle$

$a \to -a$
$\tau \to \tau - 4$
$a_D \to -a_D + 4a$

monodromy.

$(a, a_D) \to (a, a_D)\underbrace{\begin{pmatrix}-1 & 4\\0 & -1\end{pmatrix}}_{M_\infty}$.

$\mathrm{Im}\,\tau = \frac{1}{g^2}$ becomes negative when $a \ll \Lambda$. bad!



$M_\infty = M_+ M_-$

$\begin{cases} M_- = T^2 M_+ T^{-2} \\ M_+ \end{cases}$

solution $M_+ = STS^{-1} = \begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}$

$\therefore$ $M_- = \begin{pmatrix}-1 & 4\\-1 & 3\end{pmatrix}$.

$u \to -u$, $\theta_{UV} \to \theta_{UV} + 2\pi$
$\theta_{IR} \to \theta_{IR} + \underset{\sim}{4\pi}$. $\therefore T^2$.

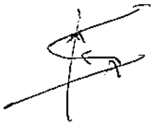
Why is it any better?

we show $\underline{\mathrm{Im}\,\tau \text{ is always positive}}$.

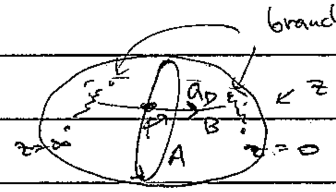
Consider a family of curves $\Sigma: \Lambda^2 z + \frac{\Lambda^2}{z} = x^2 - u$.

$\lambda_{SW} = x\frac{dz}{z}$.

SW ③



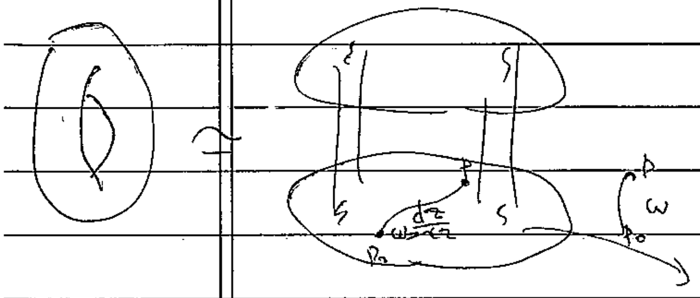
branch pt of x. as a fun of z : $x = \sqrt{\Lambda^2 z + \frac{\Lambda^2}{z} + u}$



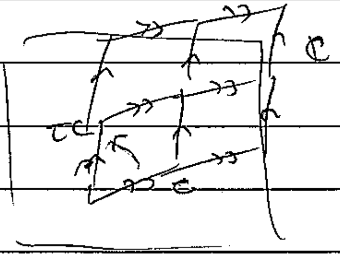
$a = \frac{1}{2\pi i}\int_A \lambda$

$a_D = \frac{1}{2\pi i}\int_B \lambda$.

satisfy the "correct" monodromy.



$$\tau = \frac{\partial a_D}{\partial a} = \frac{\partial a_D/\partial u}{\partial a_D/\partial u} = \frac{\int dz/xz}{\int dz/xz} =$$



$\mathrm{Im}\,\tau > 0$!

~~(if you choose a, a_D appropriately.)~~

Let's see the monodromy.

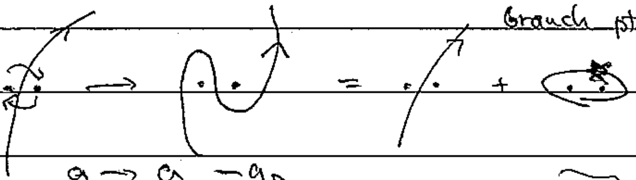
when $u \gg \Lambda^2$, $z + \frac{1}{z} = \frac{x}{\Lambda^2} - \frac{u}{\Lambda^2}$. ~~$x \sim \sqrt{u}$~~ branch pts at $z = -\frac{u}{\Lambda^2}, -\frac{\Lambda^2}{u}$.

$a_D = \frac{1}{2\pi i}\cdot 2\cdot 2\cdot\int_1^{u/\Lambda^2} a\frac{dz}{z}$ $a = \frac{1}{2\pi}\int x\frac{dz}{z} \sim \sqrt{u}$.

$\sim -\frac{8a}{2\pi i}\log\frac{a}{\Lambda}$. $\rightsquigarrow$ $\tau = -\frac{4}{\pi i}\log\frac{a}{\Lambda}$.

$z + \frac{1}{z} = c$ ← ± 2 is special. $\rightsquigarrow$ $u = \pm 2\Lambda^2$. $u = 2\Lambda^2 + \delta u$.

Branch pts : $z = 1 \pm \sqrt{\delta u}$



$a \to a - a_D$
$a_D \to a_D$. : M_+ .

what does it physically mean ? $a' = -a_D$, $a_D' = a$ $\rightsquigarrow$ $a_D' = \frac{a'}{2\pi i}\log(u-u_0)$

$a' = c(u-u_0)$.

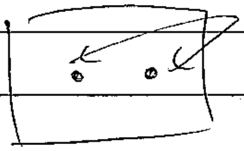
$\tau_D = \frac{\log u-u_0}{2\pi i}$ τ_D goes to infty.

g^2 goes to zero.

$U(1)$ with fixed UV coupling, with a quark of mass $(u-u_0)$

" monopole in the original description.

$m\underbrace{\mathrm{tr}\phi^2}_{u} + (u-u_0)q\tilde{q}$ $\longrightarrow$ $(u-u_0)q = 0$, $(u-u_0)\tilde{q} = 0$, $q\tilde{q} = m$: $\begin{cases} u = u_0 \\ q\tilde{q} = m \end{cases}$ monopole condensation!

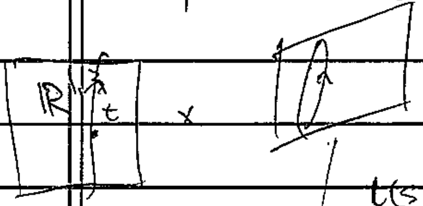


and two vacua!

SW ④.

6d? no!

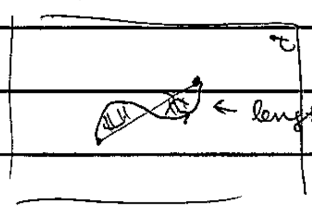
6d theory with string. not perturbative! tension ~ $|\lambda|$



tension = $\int|\lambda| \geq \left|\int\lambda\right|$

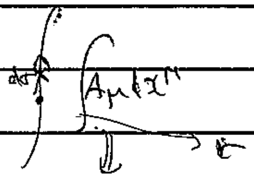
$= |ea + m a_D|$.

$t(s) = \int_0^s \lambda$

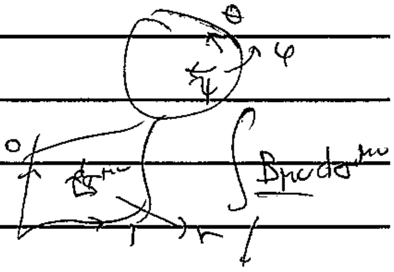


← length = $\int|\lambda|$.

self-dual tensor theory



$\int A_\mu dx^\mu$

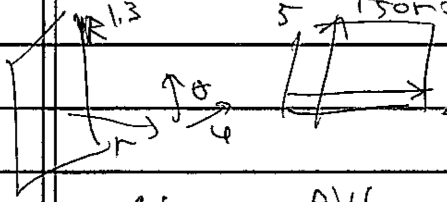


$\int B_{\mu\nu} d\sigma^{\mu\nu}$

$F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ creates F_{0r}.

$F_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$. creates F_{01r}. also $\overset{\|}{F_{\theta\varphi\psi}}$.

$F_{\mu\nu\rho} = \frac{1}{6}\varepsilon_{\mu\nu\rho\sigma\tau\upsilon}F^{\sigma\tau\upsilon}$.



$F_{50r} = F_{4\theta\varphi}$

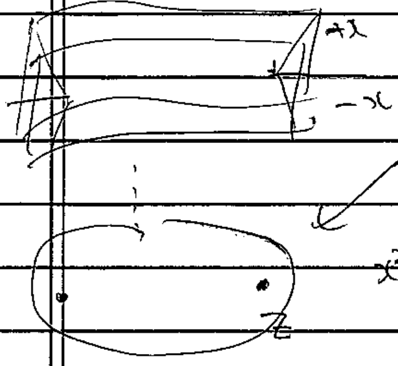
$F_{40r} = F_{5\theta\varphi}$

~~F_{40r}~~ ~~$F_{5\theta\varphi}$~~

$F_{5\theta\varphi}$	$F_{4\theta\varphi}$
$\|$	$\|$
F_{40r}	F_{50r}
↓	↓
E_r	B_r
B_r	E_r

4d gauge field can either be $F_{\mu\nu} = F_{4\mu\nu}$

or $\tilde{F}_{\mu\nu} = F_{5\mu\nu}$.



2 M5s on a sphere.

$x^2 = \Lambda^2(z + \frac{1}{z}) + u$

N M5s ⟶ SU(N) $\mathcal{N}=2$ SW.

$N=1$ curves for trifundamentals

May 26 2011

- Trifund. of $SU(2)$. Q_{ijk} is an intriguing object. would like to have some $\overset{\wedge}{\text{more}}$ fun.

- $N=1$ Abelian Coulomb phase.

moduli u_i
U(1) fields $W_\alpha{}^a$ $\rightarrow$ $\underbrace{\tau_{ab}}(u)\, W_\alpha^a W_\alpha^b$

physical low energy couplings are holomorphic in u.

Determine its behavior in a curve.

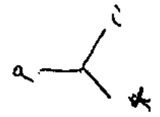
\# U(1) fields $\neq$ \# moduli

- Nothing grande, a fun field theory exercise. Intri-Seiberg 9409?15

Plan

- Q_{ijk} coupled to $N=1$ $SU(2)^3$
- generalization.

$Q_{ai\alpha}$



$M_{ab} = Q_{ai\alpha} Q_{bj\beta} \varepsilon^{ij} \varepsilon^{\alpha\beta}$
M_{ij}
$M_{\alpha\beta}$

$U^{(1)} = M_{ab} M^{ab}$
$U^{(2)} =$
$U^{(3)} =$

classically, $U^{(1)} = U^{(2)} = U^{(3)} \equiv U$. called the hyperdeterminant

$Q_{111} = Q_{222} = \underline{u}$ other components $= 0$

$D_a{}^b =$ traceless part of $Q_{ai\alpha} Q^{*\, bi\beta}$

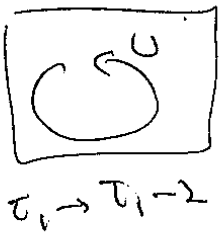
$=$ " of $\delta_a^b u^2$ $= 0$.

$$SU(2)^3 \longrightarrow \overset{\tau_1}{U(1)}\times\overset{\tau_2}{U(1)}\times\overset{\tau_3}{U(1)} \rightarrow U(1)^2.$$

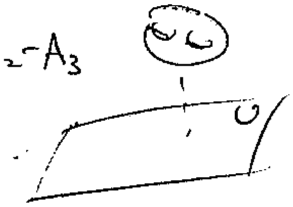
+1 −1
+1 −1
trifund. 1 remaining 1 eaten 1 −1 +1

$8 - 1 = 7$ A_1 A_2 A_3

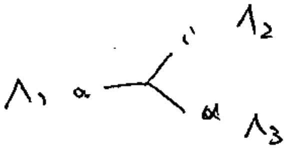
$3\times3 - 2 = 7.$ $\overset{\shortparallel}{-A_2 - A_3}$
$\overset{\uparrow}{SU(2)^3}$ $\overset{\uparrow}{\text{remain.}}$



$\hat{\tau} \rightarrow \hat{\tau} - \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$



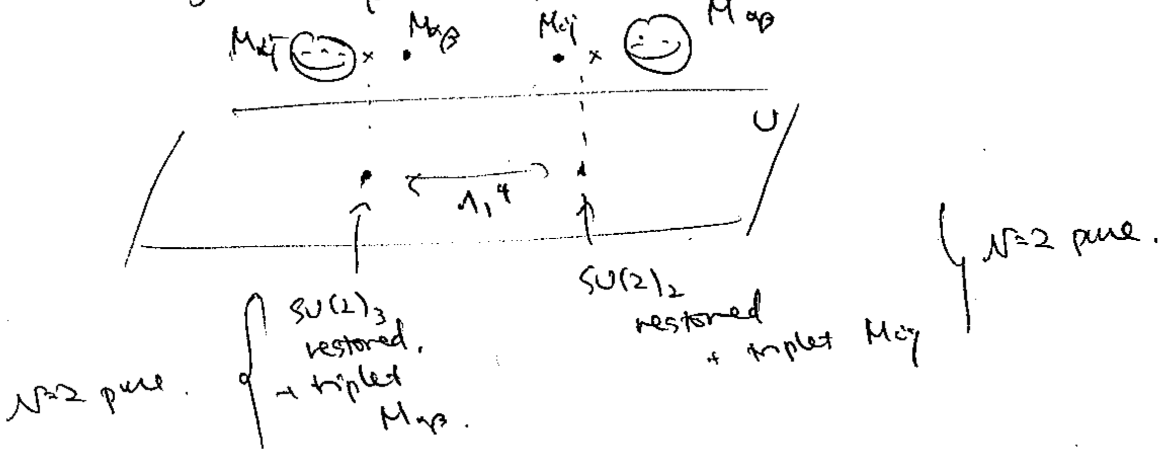
$$\left.\begin{aligned} \tau_1 &\rightarrow \tau_1 - 2 \\ \tau_2 &\rightarrow \tau_2 - 2 \\ \tau_3 &\rightarrow \tau_3 - 2 \end{aligned}\right\}$$

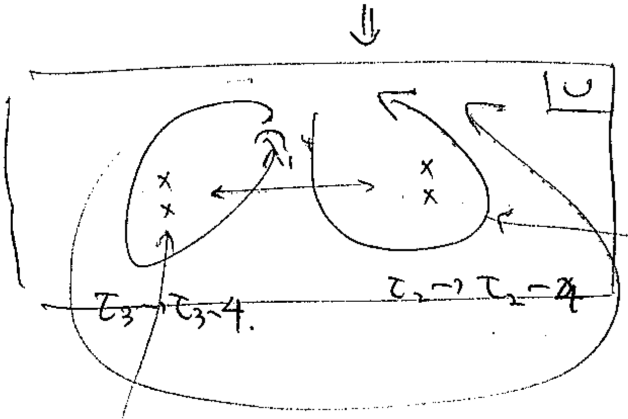


$\Lambda_1 \gg \Lambda_2, \Lambda_3$

↑ SU(2) with 2 flavors ~ deformed moduli.

$(M_{ij})^2 - (M_{\alpha\beta})^2 = \Lambda_1^4$.





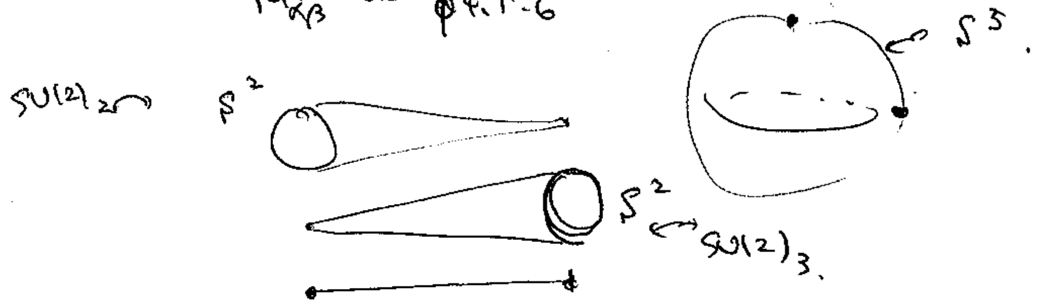
$\hat{\tau} \to \hat{\tau} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. How does off-diag arise?

the monopole becoming massless here is a doublet of SU(2) restored there

$M_{\dot\alpha\dot\beta}$ -- $\Phi_{1,2,3}$

$M_{\alpha\beta}$ -- $\Phi_{4,5,6}$

$|\Phi|^2 = \Lambda_1^4$.



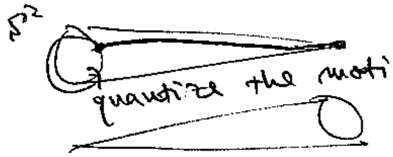
WZW term $\int_{M_5} \phi^* vol(S^5)$ $\partial M_5 = M_4$

$\langle\phi\rangle_\infty$. $SU(2)_3$ broken. $SU(2)_2$ preserved

can consider 't Hooft-Polyakov monopole for $SU(3)$

$\langle\phi\rangle_0$. at the core, $SU(2)_3$ restored.

$SU(2)_2$ broken!



quantize the motion around S^2. ⇒ doublet.

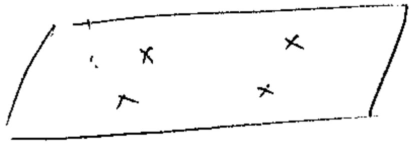
---

genus 2 over U-plane. 4 singularities.

$\boxed{(Q, U)}$

$y^2 = \underline{f_6(x, U)}$.

discriminant ~ deg 10 in U. in



Being in IPMU helps. ask Alexey Bondal (derived categories ...)

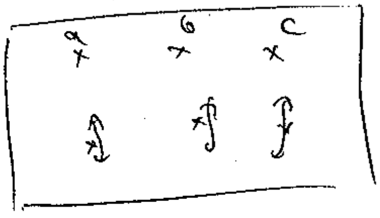
know down-to-earth physics like this.

$$y^2 = Q_3(x)\left(\underline{P_3(x) - U Q_3(x)}\right).$$

$Q_3(x) = (x-a)(x-b)(x-c)$

$P_3(x) = (x-p)(x-q)(x-r)$

$\left.\begin{matrix} a & b & c \\ p & q & r \end{matrix}\right)$ distinct.



$P_3(x) - U \underline{Q_3(x)}$'s zeros

can never be $\underline{Q_3(x)}$'s zero. when

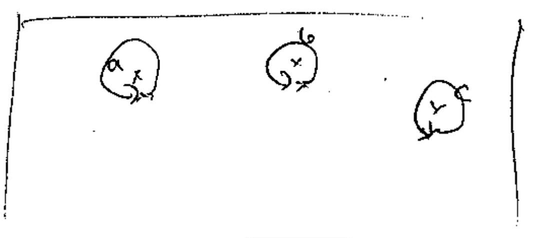
finite.

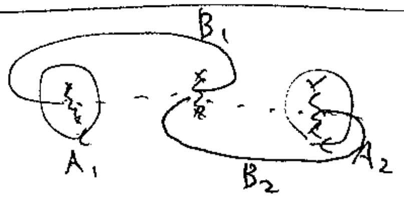
⇒ discriminant comes purely from the la

discriminant ~ deg 4. (in general $2(d-1)$).

When U is large.

$U \to e^{2\pi i} U$





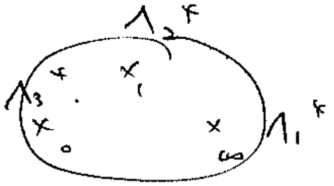
$\tau \to \tau - \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$!

$(a, b, c) \to (0, 1, \infty)$.

three parameters p. q r remain, $\leftrightarrow \Lambda_1^4, \Lambda_2^4, \Lambda_3^4$.

$$\left(\frac{y}{Q_3(x)}\right)^2 + U = \frac{P_3(x)}{Q_3(x)}.$$

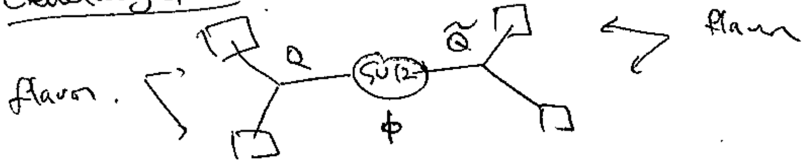
$$v^2 + U = \Lambda_1^4 z + \frac{\Lambda_2^4 z}{z-1} + \frac{\Lambda_3^4}{z}.$$



Note.

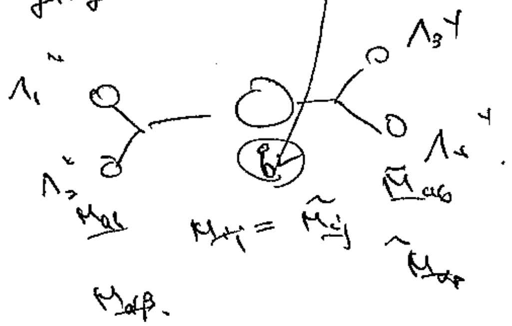
1. → simple poles.
2. functions, not quadr. diff.
3. Qeije gets over, no Φ (so far).
"Higgs branch".

Generalization



$$W = \phi Q Q + \phi \hat{Q} \tilde{Q}.$$

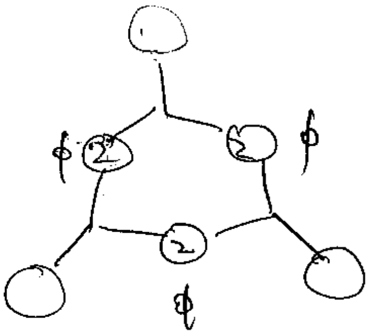
1. add $m^2 \phi^2$ by $\underline{\underline{\mathcal{N}=1}}$ SU(2) mult.
no ϕ.
2. gauge external SU(2)'s



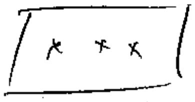
$\Rightarrow$ only one U.

$$\boxed{v^2 + U = \Lambda_1^4 z + \frac{\Lambda_2^4 z(z-c)}{z-1} + \frac{\Lambda_3^4 z(z-1)}{z-c} + \frac{\Lambda_4}{z}.}$$

One final interesting effect.



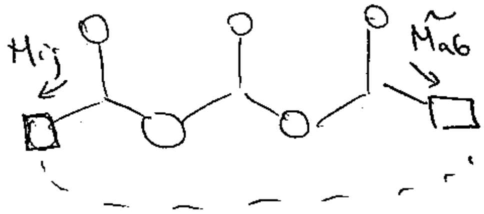
$\rightsquigarrow$



$$v^2 + U = \underset{\uparrow}{F(z)}$$

poles with residues $\Lambda_1^4, \Lambda_2^4, \Lambda_3^4$.

SW curve can be written only when $\underline{\Lambda_1^4+\Lambda_2^4+\Lambda_3^4=0}$ ... why?



- $(M_{ij})^2 - (\tilde{M}_{ab})^2 = \Lambda_1^4+\Lambda_2^4+\Lambda_3^4$.

- $W = \phi M + \phi\tilde{M} + m\phi^2 + X(M^2 - \tilde{M}^2 + \Sigma\Lambda^4)$.

$$= \phi(\underbrace{M+\tilde{M}}_{V}) + m\phi^2 + X[\underbrace{(M-\tilde{M})}_{\langle U\rangle}\underbrace{(M+\tilde{M})}_{V} + X\Sigma\Lambda^4.$$

$$W \overset{?}{=} \frac{(\Sigma\Lambda^4)^2}{m\langle U\rangle^2}$$

let $\langle U\rangle$ big.

integrate out X, ϕ, V.

$$\Sigma\Lambda - X - V - \phi - \phi - U - X - \Sigma\Lambda.$$

$$\langle U\rangle \quad 1 \quad m \quad 1 \quad \langle U\rangle$$

More on $\mathcal{N}=2$

①

SU(2) with one quark $Q, \tilde{Q}$

A_μ | Φ $\qquad Q^\dagger e^V Q$ and $W = Q\Phi\tilde{Q}$. $\qquad$ mass term $\mu Q\tilde{Q}$.

. $\Phi = \mathrm{diag}(a, -a)$ with $Q = \tilde{Q} = 0$ still vacua.

$SU(2) \to U(1)$. $\qquad (Q_1\ Q_2)\begin{pmatrix} a & \\ & -a \end{pmatrix}\begin{pmatrix} \tilde{Q}_1 \\ \tilde{Q}_2 \end{pmatrix} + \mu (Q_1\ Q_2)\begin{pmatrix} \tilde{Q}_1 \\ \tilde{Q}_2 \end{pmatrix}$

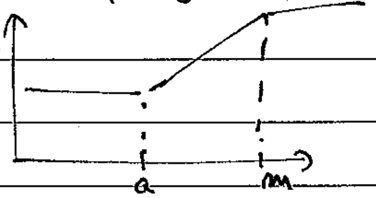
$\Rightarrow$ mass $= |\pm a \pm \mu|$.

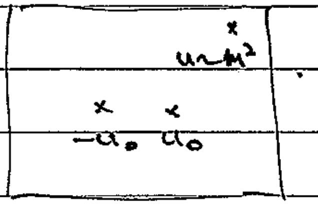
in general, mass $\geq |ea + m a_D + f\mu|$.

$\uparrow$ flavor charge.

Consider two extreme cases ① m very big. ② $m=0$.

① m very big.





② $m = 0$.

$$\begin{array}{cc} R=0 & A \\ 1 & \lambda\ \lambda \\ 2 & \phi \end{array} \qquad \begin{array}{cc} & R \\ \psi & -1 \\ Q\ \tilde{Q} & 0 \\ \bar{\psi} & 1 \end{array}$$

$\lambda \to e^{i\varphi}\lambda$

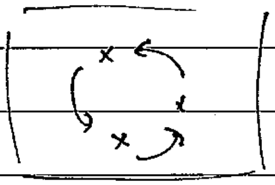
$\theta \to \theta + 6\varphi$

$\boxed{\varphi = \frac{2\pi}{6}}$ is a symmetry.

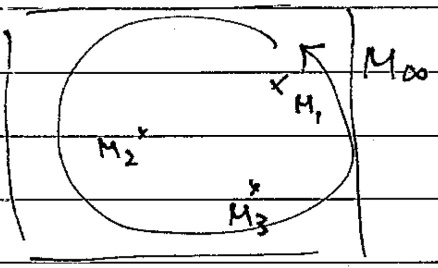
$\theta \to \theta + 2\pi$

$\phi \to e^{2\pi i/3}\phi$

$u \to e^{4\pi i/3} u$.



$\begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix}$.



$M_\infty : (a, a_D) \to (-a, 3a - a_D)$

$$\begin{cases} M_\infty = M_3 M_2 M_1 \\ M_2 = T M_3 T^{-1} \\ M_1 = T^2 M_3 T^{-2} \end{cases}$$

$\Rightarrow M_1 = STS^{-1}$.

$\begin{matrix} a' \\ a_D' \end{matrix} = \begin{matrix} -a_D \\ a \end{matrix} \qquad (a', a_D') \to (a, a_D + 1)$

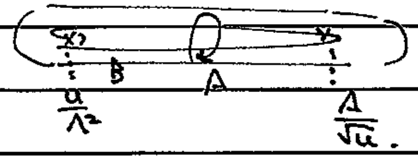
: U(1) with one massless hyper.

The curve is

$$\begin{cases} \Lambda^2 z + \dfrac{2\Lambda(x+\mu)}{z} = x^2 - u \\ \lambda = x\dfrac{dz}{z}. \end{cases}$$

More on $\mathcal{N}=2$ ②

some checks. at $\mu=0$ $\quad \Lambda^2 z + \frac{2\Lambda x}{z} = x^2 - u$



$$\frac{1}{2\pi i}\oint_A x\frac{dz}{z} = \sqrt{u}$$

correct ~~monodromy~~ (β func.)

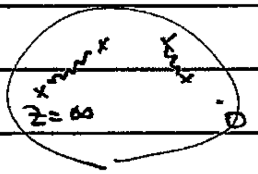
$$\frac{1}{2\pi i}\int_B x\frac{dz}{z} = -\frac{6}{2\pi i}\, a \log\frac{a}{\Lambda}.$$

$$\therefore \frac{\partial a_D}{\partial a} = -\frac{6}{2\pi i}\log\frac{a}{\Lambda}.$$

with $\mu\neq 0$, close to $z\sim 0$, $\frac{2\Lambda(x+\mu)}{z} = x^2$. $\rightarrow x \sim 2\Lambda/z + \mu + O(z)$

$\therefore \frac{1}{2\pi i}\oint_{z=0} x\frac{dz}{z} \sim \pm\mu.$ $\quad\sim\quad -\mu + O(z)$

not a branch pt.



where are branch pts.? $\quad x^2 - u = \Lambda^2 z + \frac{2\Lambda(x+\mu)}{z}$ has double roots

$$\hookrightarrow z^3 + \frac{uz^2}{\Lambda^2} + \frac{2\mu z}{\Lambda} + 1 = 0.$$

~~when do the branch pt collide?~~

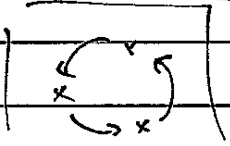
$$u^3 - \mu^2u^2 - 9\Lambda^3\mu u + \frac{27}{4}\Lambda^6 + 8\Lambda^3\mu^3 = 0.$$

$\mu=0$ $\quad u^3 + \frac{27}{4}\Lambda^6 = 0.$

$\rightsquigarrow u = c\Lambda^2$

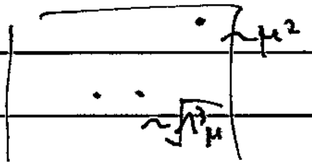
$\omega c\Lambda^2$

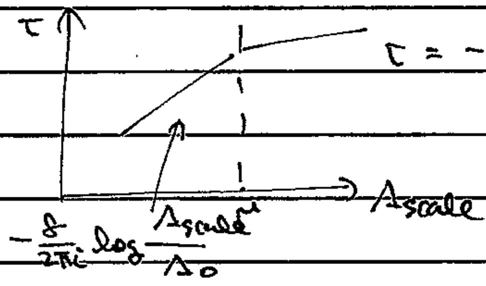
$\omega^2 c\Lambda^2$



$\mu \gg \Lambda$. $\quad u^3 - \mu^2u^2 = 0 \quad \rightsquigarrow \quad u = \mu^2$

$-\mu^2u^2 + 8\Lambda^3\mu^3 = 0 \quad \rightsquigarrow \quad u = \pm\sqrt{8\Lambda^3\mu}.$





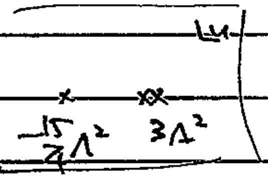
$$\tau = -\frac{6}{2\pi i}\log\frac{\Lambda_{scale}}{\Lambda_1}$$

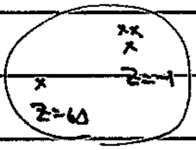
$$-\frac{8}{2\pi i}\log\frac{\Lambda_{scale}}{\Lambda_0}$$

$$\Lambda_0^4 = \mu\Lambda_1^3.$$

~~when two of these points collide?~~ $\quad \mu^3 - \frac{27}{8}\Lambda^3 = 0$

$\mu = \frac{3}{2}\Lambda.$





$a = a_D = 0 \rightarrow$ massless monopole elecm.

N=2 more

③

$~~\delta\mu~~\ \mu = \frac{3}{2}\Lambda + \delta\mu$

$z = -1 + \delta z$ $\qquad \lambda = x\frac{dz}{z} \sim (\delta x)\, d(\delta z)$.

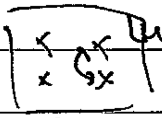
$x = \Lambda + \delta x$

$\int\lambda$: particle mass $\Downarrow$ dimension 1.

curve $\longrightarrow$ $\underset{\overset{\uparrow}{3/5}}{(\delta x)^2} + \underset{\overset{\uparrow}{6/5}}{\delta u} = \underset{\overset{\uparrow}{2/5}}{\delta z^3} + \delta\mu\, \delta z$

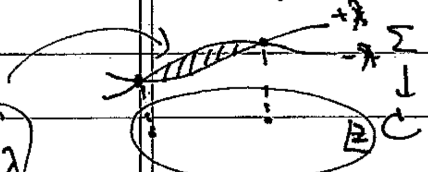
$\longrightarrow$ $\mathrm{tr}\,\phi^2$ has dimension $6/5$. (note it's 2 in UV.)

happens in $SU(2)^3$, $\mathcal{N}=1$ (2, 2, 2) too!



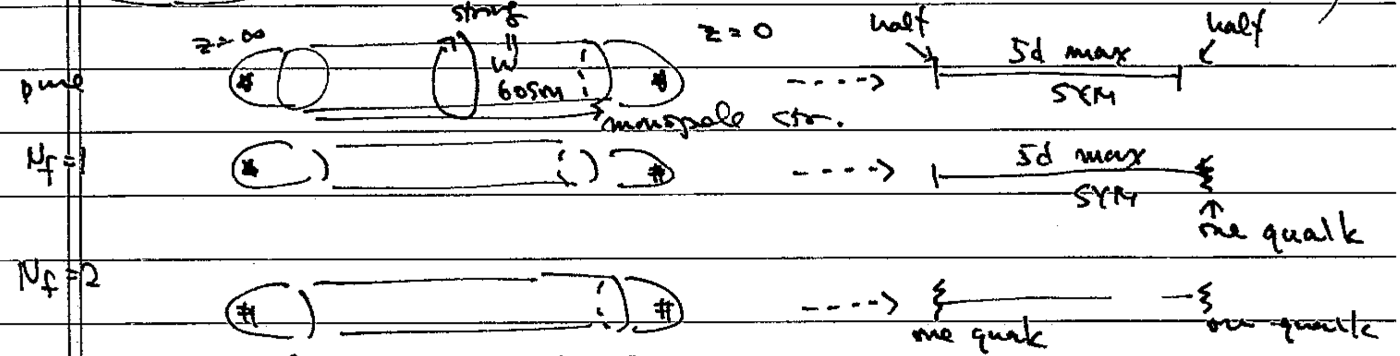
$\mathcal{N}=2$ SU(2) pure $\Lambda^2 z + \frac{\Lambda^2}{z} = x^2 - u$

w/ a quark $\Lambda^2 z + \frac{2\Lambda(x+\mu)}{z} = x^2 - u$ $\Rightarrow \lambda^2 = \phi_2(z)$ $\qquad \lambda = x\frac{dz}{\Lambda z + c}$



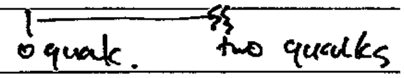
$\boxed{2\int\lambda}$

$$\left(\begin{array}{ll} \text{pure} & \phi_2(z) = \left(\Lambda^2 z + u + \frac{\Lambda^2}{z}\right)\frac{dz^2}{z^2} \\ N_f = 1 & \phi_2(z) = \left(\Lambda^2 z + u + \frac{2\Lambda\mu}{z} + \frac{\Lambda^2}{z^2}\right)\frac{dz^2}{z^2}. \end{array}\right)$$

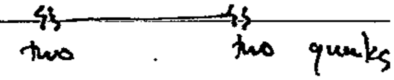


$2\Lambda(x+\mu')z + \frac{2\Lambda(x+\mu)}{z} = x^2 - u$.

$z' = \frac{2(x+\mu')z}{\Lambda}$ $\qquad$ ~~$2\Lambda(x+$~~ $\Lambda^2 z' + \frac{~~2\Lambda~~(x+\mu)(x+\mu')}{z'} = x^2 - u$



$N_f = 4$. $\qquad (x+\mu_1)(x+\mu_2)z + \frac{f}{z}(x+\mu_2)(x+\mu_3) = x^2 - u$



↑ 1 loop $\beta = 0$. marginal coupling.

$\lambda = x\frac{dz}{\Lambda z + c}$ is more important.

$\left(1 - z - \frac{f}{z}\right)x^2 - ▨\, x - ▨ = 0$. $\quad \Big\} \times \frac{1}{1 - z - \frac{f}{z}}$

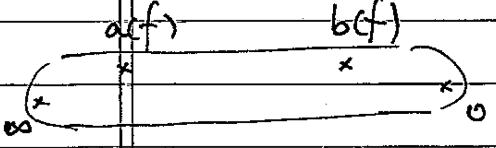
$x^2 - ▨\, x - ▨ = 0$

$\tilde{x}^2 - ▨ = 0$. $\quad \Big\}$

$\lambda^2 - ▨\frac{dz^2}{z^2} = 0$. $\qquad$ 4 mass param + u.

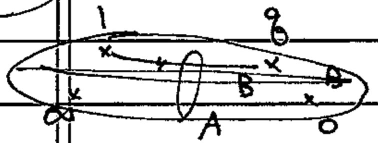
$\uparrow$

$\frac{▨ z^4 + ▨ z^3 + ▨ z^2 + ▨ z + ▨ 1}{z^2 (z - a(f))^2 (z - b(f))^2} dz^2$.



$\phi_2(z) \sim \frac{dz^2}{(z - z_0)^2}$ $\quad$ e.g. at $\overset{1/z}{\overset{\|}{w}} = 0$ $\cdot$ $\frac{▨ z^4}{z^6}dz^2 \sim \frac{▨\, dw^2}{w^2}$.

More $\mathcal{N}=2$ ④



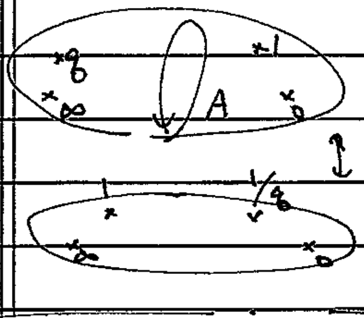
$\int_A \lambda \sim a$

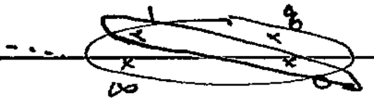
$\int_B \lambda \sim \frac{a}{2\pi i} \log q$ } ratio = τ '' $\frac{i}{g^2}+\theta$

$q \sim e^{2\pi i \tau}$.

weak coupling : $|q| \ll 1$.

$|q| \gg 1$ by $z' = z/q$,





: originally a monopole.

again, weakly coupled!
SU(2) with 4 flavors.

$\text{Im}\,\tau \int d^4\theta\, \Phi^\dagger \Phi + \int d^2\theta\, \tau\, \text{tr} W^\alpha W_\alpha + \underline{Q\Phi\tilde{Q}} + \underline{m\Phi\Phi}$.

$\mathcal{N}=2$ dual

$\int d^2\theta\, \tau\, \text{tr} W^\alpha W_\alpha + \text{tr}(Q\tilde{Q})^2$
$\int d^2\theta\, \tau'\, \text{tr} W^\alpha W_\alpha + \text{tr}(\tilde{Q}'\tilde{Q}')^2$ } $\mathcal{N}=1$ seiberg dual

$\text{Im}\,\tau' \int d^4\theta\, \Phi'^\dagger \Phi' + \int d^2\theta\, \tau'\, \text{tr} W'^\alpha W'_\alpha + Q'\Phi'\tilde{Q}' + m\Phi'\Phi'$.

recall $\underline{\mathcal{N}=1}$ $SU(N_c)$ N_f $\longleftrightarrow$ $SU(N_f-N_c)$ N_f

when $2N_c = N_f$, $Q^i\ \tilde{Q}_j$, $q_i\ \tilde{q}^j\ M^i{}_j$

$D(Q)=\frac{3}{4}$. $D(M)=\frac{3}{2}$ $W = q\tilde{q}\underline{M}$

can add $\int d^2\theta (M^i{}_j)^2$ $+ \text{tr}\,\underline{M^2}$

$\shortparallel$

$\int d^2\theta\; Q\tilde{Q}$ $= \int d^2\theta\, (q\tilde{q})^2$.

Study a bit more about flavor symmetries.

in $\mathcal{N}=2$ theory, mass comes from flavors.

$Q^{ia}\Phi_a{}^b \tilde{Q}_{jb} + m_i{}^j Q^{ia}\tilde{Q}_{jb}$ $m_i{}^j = \text{diag}(m_1 \cdots m_{N_f})$

$\uparrow$ consider it a vev of Φ : adj. scalar of $U(N_f)$ gauge f.

When $N_c = 2$, Q^a and $\tilde{Q}_b$ both $\mathbf{2}$. (no distinc. with $\bar{\mathbf{2}}$)

$\mathbb{Q}_{xa} = (Q^{i=1} \cdots Q^{i=N_f}\ \tilde{Q}_{j=1} \cdots \tilde{Q}_{j=N_f})_{a=1,2}$ $x = 1 \cdots 2N_f$

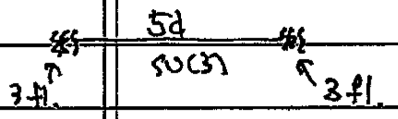
mass term $m_{xy}\, \mathbb{Q}^x_a\, \mathbb{Q}^y_b\, \varepsilon^{ab}$

$\uparrow$ antisymmetric. $SO(2N_f)$.

More $\mathcal{N}=2$

⑥

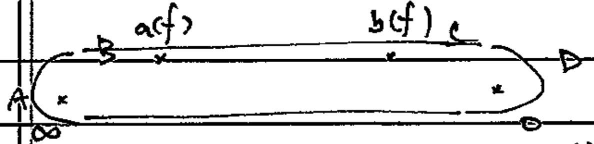
What about SU(3) $N_f = 6$? (in general, SU(N) $N_f > 2N_c$)



ϕ^2 ϕ^3

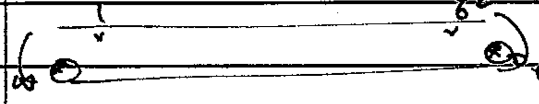
$$z(x+\mu_1)(x+\mu_2)(x+\mu_3) + \frac{f}{z}(x+\mu_4)(x+\mu_5)(x+\mu_6) = x^3+ux+v.$$

$$\lambda^3 + \phi_2(z)\lambda + \phi_3(z) = 0$$



$$\underbrace{\left(1-z-\frac{f}{z}\right)}_{\text{divide, shift }\phi_1,\ ,}x^3 + \phi_1 x^2 + \phi_2 x + \phi_3 = 0$$

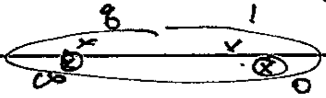
$\frac{\mu_1+\mu_2+\mu_3}{3}(1,1,-2)$ ← U(1)



← $(\mu_1,\mu_2,\mu_3) - \frac{\mu_1+\mu_2+\mu_3}{3}(1,1,1)$ ← SU(3)

$U(6)$
$\cup$
$U(3)\times U(3)$
$=$
$U(1)_B\times SU(3)_A \times U(1)_C\times SU(3)_D$.

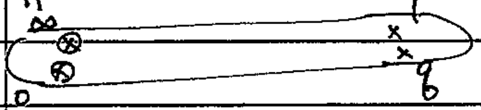
you can still exchange



$U(6)$
$\cup$
$U(3)\times U(3)$
$=$
$U(1)_C\times SU(3)_A\times U(1)_B\times SU(3)_D$

⇒ $\mathcal{N}=1$ Seiberg self-dual SU(3) $N_f=6$

what happens when

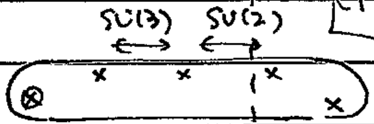


??

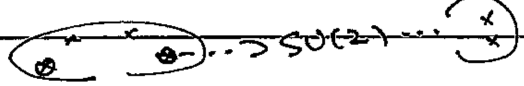
It's hard to explain without doing a bit more... flavor sym U(1).

second

(3,3) (3,2)
i.e. $SU(3)_F\times SU(3)\times SU(2)\times U(1)_F$
$\boxed{\begin{matrix}3\\1\end{matrix}}$ $U(1)'_F$ (2, 1)

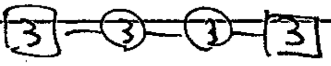


=



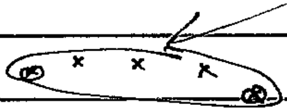
← represents 1 flavor.

first



$SU(3)_F$ $SU(3)\times SU(3)$ $SU(3)_F$
(3,3)
(3, 3)
(3, 3)

SU(3) U(1) SU(3)
(3, 3)
(3, 3)



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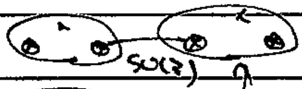
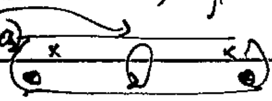
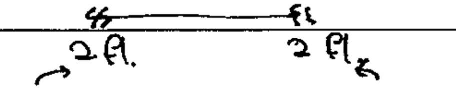


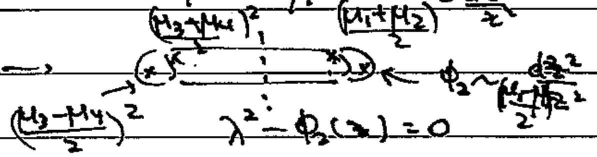
poles $\sim (a_1, a_2, -a_1-a_2)$



$\boxed{\mathcal{N}=2 \text{ more}}$ ⑤

so, $\mathcal{N}=2$ SU(2) $N_f=4$ has SO(8) flavor symmetry.





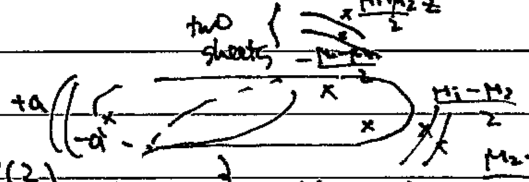
$\lambda^2 - \phi_2(x) = 0$

SO(4) SO(4)

$\mu_3 Q_3\tilde{Q}_3$ $\mu_1 Q_1\tilde{Q}_1$

$\mu_4 Q_4\tilde{Q}_4$ $\mu_2 Q_2\tilde{Q}_2$

$(x+\mu_1)(x+\mu_2)z + f\frac{(x+\mu_3)(x+\mu_4)}{z} = x^2 - u$



} residue theorem 2

$SO(4) \simeq SU(2)\times SU(2)$

$\begin{pmatrix}\mu_1 & & & \\ & -\mu_1 & & \\ & & \mu_2 & \\ & & & -\mu_2\end{pmatrix}$ $\begin{pmatrix}\frac{\mu_1+\mu_2}{2} & \\ & -\frac{\mu_1+\mu_2}{2}\end{pmatrix}, \begin{pmatrix}\frac{\mu_1-\mu_2}{2} & \\ & -\frac{\mu_1-\mu_2}{2}\end{pmatrix}$

$\int\lambda = \pm a \pm \frac{\mu_1+\mu_2}{2} \pm \frac{\mu_1-\mu_2}{2}$

$= a+\mu_1$, $a-\mu_2$ etc.

these "SU(2) masses."

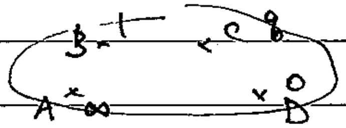
$SO(8) \leftarrow \mu_1, -\mu_1, \mu_2, -\mu_2, \mu_3, -\mu_3, \mu_4, -\mu_4.$

$\cup$

$SO(4)\times SO(4)$

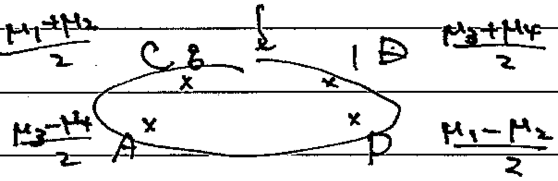
$\shortparallel$

$SU(2)_A \times SU(2)_B \times SU(2)_C \times SU(2)_D$



$\Downarrow$

in the dual, B & C are interchanged.



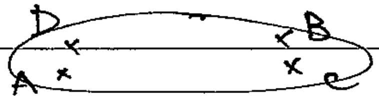
The masses are

$\pm\frac{\mu_1+\mu_2+\mu_3-\mu_4}{2}, \pm\frac{\mu_1+\mu_2-\mu_3+\mu_4}{2}, \pm\frac{\mu_1-\mu_2+\mu_3+\mu_4}{2}, \pm\frac{-\mu_1+\mu_2+\mu_3+\mu_4}{2}$

this is the positive chiral spinor representation of the original SO(8) flavor sym.

$\Gamma^1 \cdots \Gamma^8 \Rightarrow 4$ fermionic creation annihilation $\Rightarrow$ 16 states.

$\Gamma^9 = \Gamma^1 \cdots \Gamma^8 \longrightarrow$ 8 states, 8 states.



is also possible $\Rightarrow$ negative chirality spinor!

$Q_{xa}\Phi^{ab}Q_{xb} \longleftrightarrow q_{\dot{x}\dot{a}}\Phi^{(\dot{a}\dot{b})}q_{\dot{x}\dot{b}}$

vector of SO(8), SU(2) — spinor of SO(8), dual SU(2)

does the set of the gauge invariant operators agree?

$Q_{xa}Q_{yb}\epsilon^{ab}$ — asym in x & y

$q_{\dot{x}\dot{a}}q_{\dot{y}\dot{b}}\epsilon^{\dot{a}\dot{b}}$ — asym in $\dot{x}$ & $\dot{y}$

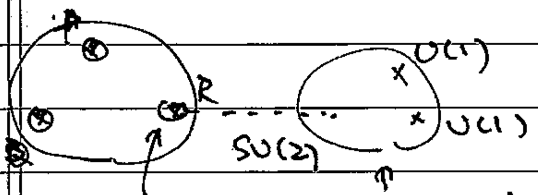
both are in adj of SO(8). Nati told me that this motivated Seiberg duality

More $\mathcal{N}=2$ ⑦

ref. 菅野修論
or Gaiotto $\mathcal{N}=2$ duality
0904.2715

assuming it comes from 6d physical theory,

so,



$SU(3)_F$. represents 1 flavor. of SU(2).

⇑ what's this theory ?

originally,

$SU(2)$, $N_f=1$. + mysterious. ⟷ $SU(3)$ with $N_f=6$.

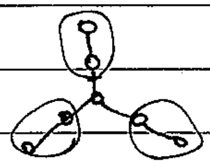
$\mathrm{tr}\,\varphi^2$. u_3 : dim 3. $\mathrm{tr}\,\phi^2$ $\mathrm{tr}\,\phi^3$. $U(1)_F \times SU(6)_F$

$SO(2N_f) = U(1)$.

$SU(3)^3 = SU(3)_P \times SU(3)_Q \times SU(3)_R$. → SU(6), SU(6), SU(6)

$SU(3)_P \times SU(3)_Q \subset SU(6)_F$.

is it possible ? YES! open slansky. one finds



$E_6 \supset SU(3) \times SU(3) \times SU(3)$

$\underline{27}$ (3 , 3)
(3, 3)
(3 , , 3)

~~so, we found a mysterious theory~~

$\underline{\mathcal{N}=2}$.
- dim 3 operator. on the ~~Coulomb~~ vector side 9608047 M.N.
- E_6 flavor symmetry.

⇒ can't be a Lagrangian theory (if so, $\mathrm{tr}\,\varphi^2$: dim-2 vector mult op. flavor sym is always $U(N_f)$, SO or Sp)

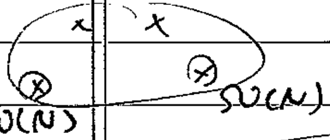
non-Lagrangian theory.

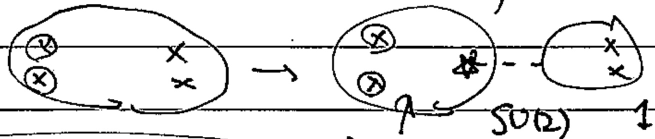
$\mathcal{N}=2\,SU(3)$ $N_f=6$ $\xrightarrow{\Delta W = m\,\mathrm{tr}\,\varphi^2}$ $\mathcal{N}=1$ $SU(3)$ $N_f=6$ $(Q\tilde{Q})^2$

$\mathcal{N}=2$ $SU(3)'$ $N_f=6$ ⟷ $\mathcal{N}=2$ $SU(2)$, $N_f=1$ + myst. $\mathcal{N}=1$ $SU(3)$ $N_f=6$ $(Q\tilde{Q})^2$ ⟷ $\mathcal{N}=1$ $SU(2)$ $N_f=1$ + myst. $(q\tilde{q}+X)^2$.

Is it an exception ? Yes and No.

$\mathcal{N}=2$ $SU(N_c)$ $N_f=2N_c$ $N_c>3$ flavor sym $SU(2N_c) \times SU(2)$.





SU(2) 1 flavor.

NON-Lagrangian theory is everywhere. always non-Lagrangian !