Supersymmetric Dynamics in Various Dimensions @ 大阪大

1.5hr × 8コマ

1コマ ~4ページ?

目的 | Seiberg-Witten 及び Seiberg から丁度20年…
Witten "Phases" からは 21年

超対称理論の研究は随分進んだ。 4d ← 我々が住んでいるから
2d ← 弦の世界面だから

というのに気をとられないなら、他の次元も面白い。

ここ数年は { 局所化による厳密計算
コンパクト化するとどうなるか
0d op だけでなく、1d op / 2d op … はどうか } というのが盛ん。

特定の次元に捉われず、全体を眺めるのは悪くないのでは?

一方で、4d $\mathcal{N}=1$ と言えど完全には分かっていない。
SU(N), N_f flavor はわかっているが
一般に群 G, matter content R, 超ポテンシャル W を与えて、
SUSY vacuum moduli が何か? を一般に答えることは未だ出来ない
(cf: 摂動論は原理的には勝手なラグランジアンで労力のみ。)

- いろんな次元でどういう事があるか? を概観しつつ
- 未解決問題がいろんな所にあることを見ていきたい。

種々の次元の超対称性と超共形対称性 ← ~~1コマ~~ 2コマ　初日

4d $\mathcal{N}=1$: pure gauge
Seiberg duality SO
S^3 index ~~SO~~ } 3コマ

~~3d N=2 ABJM のラグランジアン、等~~ ~~1コマ~~　二日目

2d $\mathcal{N}=(2,2)$ CY/GL 対応
Hori-Tong duality
$\mathcal{N}=(0,2)$ Gadde-Gukov-Putrov の triality } 合計 3コマ　三日目

5d と 6d は ~~11月~~ 2月 に京都で話す予定。

いろいろな次元の超対称性 ①

$\{Q_\alpha, \bar{Q}_\beta\} \sim P_\mu \gamma^\mu_{\alpha\beta}$ というのが SUSY. まず スピノルの復習.
cf. Polchinski vol 2 App B

$SO(d-1,1)$ のスピノルを知りたい. $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$.

まず $SO(d)$ を考える.

- $SO(2n)$ $\Gamma^1 \cdots \Gamma^{2n}$

  $\Gamma^{2n+1} := \Gamma^1 \Gamma^2 \cdots \Gamma^{2n}$ で 自動的に $SO(2n+1)$ の Γ行列.
- $SO(2)$ $\Gamma^1 = \sigma^x$ $\Gamma^2 = \sigma^y$ $\Gamma^3 = \sigma^z$
- $SO(2\tilde{n})$ $\tilde{\Gamma}^1 \cdots \tilde{\Gamma}^{2\tilde{n}}$
- $SO(2n+2\tilde{n})$ は $\begin{cases} \hat{\Gamma}^i = \Gamma^i \otimes 1 \\ \hat{\Gamma}^{i+2n} = \Gamma^{2n} \otimes \tilde{\Gamma}^i \end{cases}$ と定めればよい.

⇒ $SO(2n)$ の Γ行列は $2^n \times 2^n$ 行列. Dirac sp: 2^n 次元

ローレンツ群(回転)の生成子 $M^{\mu\nu} = \frac{1}{2}[\Gamma^\mu, \Gamma^\nu]$ は Γ^{2n} と交換

⇝ $\Gamma^{2n} = +1, -1$ に制限できる. Weyl sp: 2^{n-1} 次元

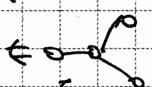
- $SO(1,1)$ の Γ行列: $\Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\left(\Gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$

とすると. 同様に 実. ±Weyl sp. も実.

$SO(d-2)$ の Γ と $SO(1,1)$ の Γ から $SO(d-1,1)$ の Γ.

⇒ Dirac, Weyl sp. の実性 は $SO(d-2)$, $SO(d-1,1)$ で同じ.

		vector charge	spinor charge	
$SO(2) \simeq U(1)$		±1	±1/2	: 複素. ($2 = \bar{2}$ だが $*^2 = -1$)
$SO(3) \simeq SU(2)$			2	: ぎ実.
$SO(4) \simeq SU(2) \times SU(2)$			(2, 2')	ぎ実
$SO(5) \simeq USp(4)$			4	ぎ実
$SO(6) \simeq SU(4)$			$4 \leftrightarrow \bar{4}$	複素.
$SO(7)$			8	正実 ← SO(8)に制限.
$SO(8)$			$8_s, 8_c$	**正実** ← (図) なので…



↑ $SO(d-8)$ のスピノルと $SO(d)$ ($\times SO(8)$) のスピノルの 性質は同じ. (Bott Periodicity)

いろいろな次元の超対称性 ②

SO(1,1)	±Weyl, 正実,	複素 1次元	(実とかす) ⇒ 実 1次元	×	$(\mathcal{N}_+, \mathcal{N}_-)$
SO(2,1)	Dirac 正実	2	⇒ 2次元	×	$\mathcal{N}$
SO(3,1)	±Weyl 複素	2	⇒ 実 4次元	×	$\mathcal{N}$
SO(4,1)	Dirac ギ実	4	8	×	$\mathcal{N}$
SO(5,1)	±Weyl ギ実	4	8		$(\mathcal{N}_+, \mathcal{N}_-)$
SO(6,1)	Dirac ギ実	8	16		$\mathcal{N}$
SO(7,1)	±Weyl 複素	8	16		$\mathcal{N}$
SO(8,1)	Dirac 正実	16	16		$\mathcal{N}$
SO(9,1)	±Weyl 正実	16	16		$(\mathcal{N}_+, \mathcal{N}_-)$
SO(10,1)	Dirac 正実	32	32		$\mathcal{N}$
SO(11,1)	±Weyl 複素	32	64		$\mathcal{N}$
			⋮		

R対称性 … 正実なら $SO(\mathcal{N})$, 複素なら $U(\mathcal{N})$, ギ実なら $Sp(\mathcal{N})$
(但し $Sp(1)=SU(2)$)

massless 表現 ($d \geq 4$)

$p^\mu = (E, E, 0, 0 \cdots 0)$ とする. $p^+ = 2E$, $p^- = 0$, …

supercharge が実で $2n$ コ とする. Q^{+i}, Q^{-i} $i=1\cdots 2n$
それぞれ エルミート

$\{Q^{+i}, Q^{+j}\} = p^+ \delta^{ij}$, $\{Q^{-i}, Q^{-j}\} = p^- \delta^{ij} = 0$

↑ fermionic 生成消滅が n 組.
スピンは $\pm\frac{1}{2}$ あげさげする.

$n=4$ で $-1, -\frac{1}{2}, 0, +\frac{1}{2}, +1$ ← 必ず ベクトル場.

$n=8$ で -2 … 2 ← 必ず 重力.

$n=16$ は -4 … 4 ← が最小. 必ず H.S.

HSのない / 16 supercharges / $d=11$ まで.

⟶ 重力のない理論 は $d=10$ まで.
ゲージ場のないと ∧ $d=6$ まで.
8 supercharges

($d=3$ 以下は スピンはよくわからんので もっとあってもよいかも知れない.)

Q. よく Wick rotate して解析するが. 実性はどうなるの?

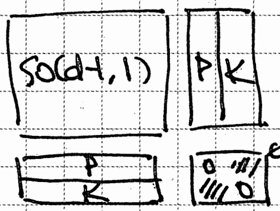
A. Wick rotate した理論で保証されているのは $\phi(x) \to \overline{\phi(x)}$. パリティ flip つき 複素共役の構造は 同じ。

Sheet ______ Date ______

いろいろな次元の超対称性. ③

超共形代数はどうか?

SO(d+1,1) から出発. ⟶ SO(d,2) が共形代数.



← D : dilatation.

さらに SUSY Q_α があるとする. S^α もある.

(Q_α, S^α) が SO(d,2) のスピノルとして変換.

$\{Q_\alpha, S^\beta\} = \cdots + R\text{-symmetry} + \cdots$.

$\underbrace{SO(d,2) \oplus R\text{-sym}}_{\text{even part}} \oplus \underbrace{(Q, S)}_{\text{odd part}}$ が超共形代数.

分類 (Nahm, NPB 135 (1978) 149)

d=2のときは SO(2,2) ≃ SO(1,2)⊕SO(1,2) なので,

d≠2 のときの分類をすることにする. Q,S に作用しない R-sym はのぞく.

すると SO(d,2) は単純. (非自明な代数を含まない).

よって SCFA 自体 (Q,S に作用しない R-sym をのぞけば) 単純.

(∵ Q が入ると S も全部入る. {Q,Q} から P が. よって SO(d,2) も全部. R も全部入る.)

単純 super Lie alg/ℂ の分類 (で boson 部分が reductive なもの)
fin. dim. (Kac, CMP 53 (77) 31)

SU(m|n)	:	SU(m)⊕SU(n)⊕U(1)	⊕	$N\otimes\bar{M} \oplus M\otimes\bar{N}$
OSp(m|2n)	:	O(m)⊕Sp(n)	⊕	M⊗N
D(2,1,α)	:	su(2)⊕su(2)⊕su(2)	⊕	2⊗2⊗2 , α 任意 結合定数
F(4)	:	su(2)⊕o(7)	⊕	2⊗8
G(3)	:	su(2)⊕g_2	⊕	2⊗7 ↑ 不変内積あり
P(n−1)	:	su(n)	⊕	adj なし
Q(n−1)	:	su(n)	⊕	$\mathrm{sym}^2 \oplus \Lambda^2$

even part に SO(d,2) を含み, odd part がスピノル, を探す. 上には O(m) の vector が主.

d は無限にはない!

いろいろな次元の超対称性 ④

- d=2 はヘンテコなのが一杯あるが、どうせさらに Virasoro に拡大するので省略.
- d=3 : so(3,2) ⊕ o(N) ⊕ 4⊗N ← osp(N|4)
- d=4 : so(4,2) ⊕ u(N) ⊕ 4⊗N ⊕ $\bar{4}\otimes\bar{N}$ ← su(4|N)
- d=5 : so(5,2) ⊕ su(2) ⊕ 8⊗2 ← F(4)
- d=6 : so(6,2) ⊕ sp(n) ⊕ 8⊗2n ← osp(8|2n)

可能な超対称性理論（ミンコフスキー d次元 もしくは それをユークリッド化したもの）

d=12 … HS のみ (64 supercharges)

d=11 … 重力のみ (32 supercharges)

d	16 S.C.		8 S.C.	4 SC	
d=10	N=(1,0)				
d=9	N=1				
d=8	N=1				
d=7	N=1				
d=6	N=(1,1)	N=(2,0)	N=(1,0)		
d=5	N=2		N=1		
d=4	N=4	N=3	N=2	N=1	
d=3	N=8	7, 6, 5	4	3, 2	1
d=2	N=(8,8)		N=(4,4)	N=(2,2)	他にも一杯.
d=1	有象無象 …				

場の11次元では閉じない

→ 京都で話す事柄

くりこみ不可能

ゲージ理論 くりこみ可能

赤 : 超共形代数がある.

上のマル囲いをちゃんとやっておけ.

d=10 vector mult : A_μ 10, 8 ; ψ_α M-W. 16, 8 ; on-shell

d						
d=9	ϕ_9, $A_{0\cdots8}$, ψ_8	SO(1)	≃	O(1)		N=1
8	$\phi_{9,8}$ $A_{0\cdots7}$	SO(2)	≃	U(1)		N=1
7	$\phi_{9,8,7}$	SO(3)	≃	Sp(1)		N=1
6	$\phi_{9,8,7,6}$	SO(4)	≃	Sp(1)×Sp(1)		N=(1,1)
5	$\phi_{9,8,7,6,5}$	SO(5)	≃	Sp(2)		N=2
4		SO(6)	≃	SU(4)		N=4
3	$\phi_{9,8,7,6,5,4,3}$	SO(7)	??	SO(8)		N=8
2						???

高い次元の超対称性 ⑤

これらの理論は ゲージ群 G を指定すると決まってしまう

代数 g ?　　$+\frac{trR^4\,trR^2}{552960} + \frac{(trR^2)^3}{1327104}$

d=10　アノマリは

$$\hat{I}_{12} = \frac{1}{1440}\left(-\mathrm{Tr}F^6 + \frac{1}{48}\mathrm{Tr}F^4\,\mathrm{Tr}F^2 - \frac{(\mathrm{Tr}F^2)^3}{14400}\right) \text{あ}$$

$$+(n-496)\left\{\frac{trR^6}{725760} \cdots\right\}_{い} + \frac{Y_4 X_8}{\cancel{384}\ 768}$$

where $Y_4 = trR^2 - \mathrm{Tr}F^2/30$

$$X_8 = trR^4 + \frac{(trR^2)^2}{4} - \frac{1}{30}(\mathrm{Tr}F^2)(trR^2) + \frac{\mathrm{Tr}F^4}{3} - \frac{(\mathrm{Tr}F^2)^2}{900}$$

"Tr" は adj の tr.

これは $g=SO(32)$, $E_8\times E_8$ のとき あ, いが消え

$dH = Y_4$

$\delta S = \int B\wedge X_8$ とすると 相殺可.　⇒ ヘテロ($E_8\times E_8$, $SO(32)$) Type I 超弦に対応.

* 昔から $g = E_8\times U(1)^{248}$, $U(1)^{496}$ でも可なことが知られていた.

これに対応する quantum gr. はあるのか？

Adams-de Wolfe-Taylor 1006.1352:

Bergshoeff-de Roo-de Wit-van Nieuwenhuizen NPB195(82)97

によると, $B_{\mu\nu}-F_{\mu\nu}-F_{\mu\nu}$ 結合は

$g_{\mu\nu}-F_{\mu\nu}-F_{\mu\nu}$ 結合の SUSY partner なので

Y_4 内 $\mathrm{Tr}F^2$ の係数は $1 \neq 0$.

よって相殺できない.

一方で, 昔のヘテロのギロンは $\overset{10}{26}\overset{10}{\searrow}16 \in$ lattice CFT or free fermionic なら $SO(32)$ か $E_8\times E_8$ のみ.

という. 一般に条件をみたす chiral CFT は?

⇒ Dong-Mason math.QA/0203005 で分類. これしかない.

* $E_8\times E_8$ はいいけれど. $SO(32)$ は詳しくは

spin(32) → so(32), spin(32) → spin(32)/$\mathbb{Z}_2$ =: Ss(32)

so(32) → so(32)/±1, spin(32)/$\mathbb{Z}_2$ → so(32)/±1

の四つある. どれか?　Type I / ヘテロ では spin(32)/$\mathbb{Z}_2$.

ヘテロの discrete theta angle は 何か?　Sethi 1304.1557

場の理論的に, spin(32)/$\mathbb{Z}_2$ でないといけない事はわかるか?　} 未解決.

高い次元の SUSY ⑥

d=9 どの g が許されるか？ < 場の理論的に。弦/M を知っているとして。

SO(32), $E_8 \times E_8$ を adj. rep. で破った rank 16 のは OK.

D8 N枚で U(N), は transverse に有限キョリで dilaton が diverge して ダメ.

d=8

F-th 7-brane 上のゲージ群は OK.

しかし transverse 有限キョリで やはり dilaton が おかしくなりがち。

高々有限種類. n_{32} ?

d=7. M-th で $\mathbb{C}^2/\Gamma$, $\Gamma \subset SU(2)$

多分 勝手な Γ でよい. → A_n, D_n, E_n, 無限種類.

(M: SO(2n) → SO(2n−7); triples E_8 → F_4, G_2; SO(2n) → USp(2n−8). Witten 9710065)

d=6 さらにこれを S^1 コンパクト化. → B, C, F, G も可.

IIA on $\mathbb{C}^2/\Gamma$ ← $\mathcal{N}=(1,1)$

IIB on $\mathbb{C}^2/\Gamma$ ← $\mathcal{N}=(2,0)$: ヘンテコ理論.

d=5. max susy SYM, かってな g が可.

Sp 群は $\pi_4(Sp(n)) = \mathbb{Z}_2$ に伴い 2種.

d=4 かってな g が可. $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$: 無次元パラメタあり. : SCFT.

これ以外の変てこな $\mathcal{N}=4$ SCFT はあるか？ 未解決.

ラグランジアン $\mathcal{N}=3$ 理論は自動的に $\mathcal{N}=4$ だが, 変てこ $\mathcal{N}=4$ はあるか？ 未解決.

d=3. かってな g が可. IR で強結合になる。恐らく難しかったが...

ここで一段落.

1 $\mathcal{N}$=1 superfields

For this purpose let us quickly recall the $\mathcal{N}$=1 formalism. In this section only, we distinguish the imaginary unit by writing it as i.

An $\mathcal{N}$=1 vector multiplet consists of a Weyl fermion λ_α and a vector field A_μ, both in the adjoint representation of the gauge group G. We combine them into the superfield W_α with the expansion

$$W_\alpha = \lambda_\alpha + F_{(\alpha\beta)}\theta^\beta + D\theta_\alpha + \cdots \tag{1}$$

where D is an auxiliary field, again in the adjoint of the gauge group. $F_{\alpha\beta} = \frac{\mathrm{i}}{2}\sigma^{\mu\beta}{}_{\dot\gamma}\bar\sigma^{\nu\dot\gamma}{}_{\alpha}F_{\mu\nu}$ is the anti-self-dual part of the field strength $F_{\mu\nu}$.

The kinetic term for a vector multiplet is given by

$$\int d^2\theta \frac{-\mathrm{i}}{8\pi}\tau \operatorname{tr} W_\alpha W^\alpha + cc. \tag{2}$$

where

$$\tau = \frac{4\pi\mathrm{i}}{g^2} + \frac{\theta}{2\pi} \tag{3}$$

is a complex number combining the inverse of the coupling constant and the theta angle. We call it the complexified coupling of the gauge multiplet. Expanding in components, we have

$$\frac{1}{2g^2}\operatorname{tr} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{16\pi^2}\operatorname{tr} F_{\mu\nu}\tilde F^{\mu\nu} + \frac{1}{g^2}\operatorname{tr} D^2 - \frac{2\mathrm{i}}{g^2}\operatorname{tr}\bar\lambda\not{D}\lambda. \tag{4}$$

We use the convention that $\operatorname{tr} T^a T^b = \frac{1}{2}\delta^{ab}$ for the standard generators of gauge algebras, which explain why we have the factors $1/(2g^2)$ in front of the gauge kinetic term. The θ term is a total derivative of a gauge-dependent term. Therefore, it does not affect to perturbative computations. It does affect non-perturbative computations, to which we will come back later.

An $\mathcal{N}$=1 chiral multiplet Q consists of a complex scalar Q and a Weyl fermion ψ_α, both in the same representation of the gauge group. In terms of a superfield we have

$$Q = Q\big|_{\theta=0} + 2\psi_\alpha\theta^\alpha + F\theta_\alpha\theta^\alpha \tag{5}$$

where F is auxiliary. The coefficient 2 in front of the middle component is unconventional, but this choice allows us to remove various annoying factors of $\sqrt{2}$ appearing in the formulas later. The chiral multiplet $Q_{1,\ldots}$ can be in an arbitrary complex representation R of the gauge group G. The Lagrangian density is then

$$\int d^4\theta Q^{\dagger j} e^{V^a \rho_a{}^i_j} Q_i + \int d^2\theta W(Q) + cc. \tag{6}$$

where V is the vector superfield, $\rho_a{}^i_j$ is the matrix representation of the gauge algebra, and $W(Q)$ is a gauge invariant holomorphic function of $Q_{1,\ldots}$.

The supersymmetric vacua is obtained by demanding that the supersymmetry transformation of various fields are zero. The nontrivial conditions come from

$$\delta\lambda_\alpha = 0, \qquad \delta\psi_\alpha = 0 \tag{7}$$

which give

$$D_a = 0, \quad F_i = 0. \tag{8}$$

By solving the algebraic equations of motion of the auxiliary fields, we find

$$Q^\dagger_{\bar{j}} \rho_a^{\bar{j}i} Q_i = 0, \quad \frac{\partial W}{\partial Q_i} = 0. \tag{9}$$

2 Renormalization

Recall the one-loop renormalization of the gauge coupling in a general Lagrangian field theory:

$$E\frac{d}{dE}g = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C(\text{adj}) - \frac{2}{3}C(R_f) - \frac{1}{3}C(R_s)\right]. \tag{10}$$

Here, E is the energy scale at which g is measured, and we use the convention that all fermions are written in terms of left-handed Weyl fermions. Then R_f and R_s are the representations of the gauge group to which the Weyl fermions and the complex scalars belong, respectively. The quantity $C(\rho)$ is defined so that

$$\operatorname{tr}\rho(T^a)\rho(T^b) = C(\rho)\delta^{ab} \tag{11}$$

where T^a are the generators of the gauge algebra and $\rho(T^a)$ is the matrix in the representation ρ, normalized so that $C(\text{adj})$ is equal to the dual Coxeter number. For $\mathrm{SU}(N)$, we have

$$C(\text{adj}) = N, \qquad C(\text{fund}) = \frac{1}{2}. \tag{12}$$

In an $\mathcal{N}{=}1$ gauge theory, the equation simplifies to

$$E\frac{d}{dE}g = -\frac{g^3}{(4\pi)^2}\left[3C(\text{adj}) - C(R)\right] \tag{13}$$

or equivalently

$$E\frac{d}{dE}\frac{8\pi^2}{g^2} = 3C(\text{adj}) - C(R), \tag{14}$$

where R is the representation of the chiral multiplet.

In a supersymmetric theory, the coupling g is combined with the theta angle θ and enters in the Lagrangian as

$$\int d^2\theta \frac{-i}{8\pi}\tau \operatorname{tr} W_\alpha W^\alpha + cc. \tag{15}$$

where τ is given by

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}. \tag{16}$$

We call this τ the complexified gauge coupling.

We can consider τ to be an expectation value of a background chiral superfield. There is a renormalization scheme where the superpotential remains a holomorphic function of the chiral superfields, including background fields whose vevs are the gauge and superpotential couplings.

In this scheme, the one-loop running coupling at the energy scale E can be expressed as

$$\tau(E) = \tau_{UV} - \frac{b}{2\pi i}\log\frac{E}{\Lambda_{UV}} + \cdots \tag{17}$$

where b is the rational number appearing on the right hand side of (14) or (**??**). Note that the coupling τ starts from $1/g^2$, and therefore the n loop diagram would have the dependence $g^{2(n-1)}$. The constant shift as in the imaginary part in (17) is then a one-loop effect.

Perturbation theory is independent of the θ angle, since $F_{\mu\nu}\tilde{F}_{\mu\nu}$ is a total derivative, although of a gauge-dependent quantity. Therefore the n loop effect is a function of $(\operatorname{Im}\tau)^{1-n}$, which is not holomorphic unless $n = 1$. We conclude that the running (17) is one-loop exact in the holomorphic scheme. We find that the combination

$$\Lambda^b = E^b e^{2\pi i\tau(E)} \tag{18}$$

is invariant to all orders in perturbation theory. We call this Λ the complexified dynamical scale of the theory.[1] Note that Λ is a complex quantity, and can be considered as a vev of a background chiral superfield.

This one-loop exactness does not necessarily mean that the physical gauge coupling, which controls the scattering process for example, is one-loop exact. In the holomorphic scheme in generic $\mathcal{N}$=1 supersymmetric theories, we have nontrivial wave-function renormalization factors Z_{ij}

$$\int d^4\theta Z^{\bar{i}j}(E) Q_{\bar{i}}^\dagger e^V Q_j \tag{19}$$

which need to be taken into account by a further field redefinition to compute physical scattering amplitudes. This is known to produce further perturbative contributions to the physical running of the gauge coupling. For more on this point, see e.g. [2].

3 Anomalies

3.1 Triangle anomaly

Non-abelian gauge theories have an important source of non-perturbative effects, called instantons. This is a nontrivial classical field configuration in the Euclidean $\mathbb{R}^4$ with nonzero integral of

$$16\pi^2 k := \int_{\mathbb{R}^4} \operatorname{tr} F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{20}$$

[1] A redefinition of the form $\Lambda \to c\Lambda$ by a real constant c corresponds to a redefinition of the coupling of the form $1/g^2 \to 1/g^2 - c'$ where c' is another constant, or equivalently $g^2 \to g^2 + c'g^4 + \cdots$. Therefore this is a redefinition starting at the one-loop order, keeping the leading order definition of g^2 fixed. In this lecture note, we do not track such finite renormalization of the coupling very carefully.

In the standard normalization of the trace for $\mathrm{SU}(N)$, k is automatically an integer, and is called the instanton number. The theta term in the Euclidean path integral appears as

$$\exp\left[i\frac{\theta}{16\pi^2}\operatorname{tr}F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{21}$$

Therefore, a configuration with the instanton number k has a nontrivial phase $e^{i\theta k}$. Note that a shift of θ by 2π does not change this phase at all. Therefore, even in a quantum theory, the shift $\theta \to \theta + 2\pi$ is a symmetry.

Using

$$\operatorname{tr}F_{\mu\nu}F_{\mu\nu} = \frac{1}{2}\operatorname{tr}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2 \mp \operatorname{tr}F_{\mu\nu}\tilde{F}_{\mu\nu} \geq \mp \operatorname{tr}F_{\mu\nu}\tilde{F}_{\mu\nu}, \tag{22}$$

we find that

$$\int d^4x \operatorname{tr}F_{\mu\nu}F_{\mu\nu} \geq 16\pi^2|k| \tag{23}$$

which is saturated only when

$$F_{\mu\nu} + \tilde{F}_{\mu\nu} \propto F_{\alpha\beta} = 0 \quad \text{or} \quad F_{\mu\nu} - \tilde{F}_{\mu\nu} \propto F_{\dot{\alpha}\dot{\beta}} = 0 \tag{24}$$

depending on the sign of k. Therefore, within configurations of fixed k, those satisfying relations (24) give the dominant contributions to the path integral. The solutions to (24) are called instantons or anti-instantons, depending on the sign of k.

In an instanton background, the weight in the path integral coming from the gauge kinetic term is

$$\exp\left[-\frac{1}{2g^2}\int \operatorname{tr}F_{\mu\nu}F^{\mu\nu} + i\frac{\theta}{16\pi^2}\int \operatorname{tr}F_{\mu\nu}\tilde{F}^{\mu\nu}\right] = e^{2\pi i\tau k}. \tag{25}$$

We similarly have the contribution $e^{2\pi i\bar{\tau}|k|}$ in an anti-instanton background. The fact that we have just τ or $\bar{\tau}$, instead of more complicated combinations, is related to the fact that in the instanton background in a supersymmetric theory, $\delta\lambda_{\dot{\alpha}} = F_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}} = 0$ assuming the D-term is also zero, and thus the dotted supertranslation is preserved. Similarly, the undotted supersymmetry is unbroken in the anti-instanton background.

Now, consider charged Weyl fermions ψ_α coupled to the gauge field, with the kinetic term

$$\bar{\psi}_{\dot{\alpha}}D_\mu\sigma^{\mu\dot{\alpha}\alpha}\psi_\alpha. \tag{26}$$

Let us say ψ_α is in the representation R of the gauge group. It is known that the number of zero modes in ψ_α minus the number of zero modes in $\bar{\psi}_{\dot{\alpha}}$ is $2C(R)k$. In particular, the path integral restricted to the k-instanton configuration with positive k is vanishing unless we insert k additional ψ's in the integrand. More explicitly,

$$\langle O_1O_2\cdots\rangle = \int[D\psi][D\bar{\psi}]O_1O_2\cdots e^{-S} = 0 \tag{27}$$

unless the product of the operators $O_1O_2\cdots$ contains $2C(R)k$ more ψ's than $\bar{\psi}$'s. This is interpreted as follows: the path integral measures $[D\psi]$ and $[D\bar{\psi}]$ contain both infinite number of integrations. However, there is a finite number, $2C(R)k$, of difference in the number of integration variables. Equivalently, under the constant rotation

$$\psi \to e^{i\varphi}\psi, \quad \bar{\psi} \to e^{-i\varphi}\bar{\psi}, \tag{28}$$

the fermionic path integration measure rotates as

$$\begin{aligned} [D\psi] &\to [D\psi]e^{+\infty i\varphi + 2C(R)ki\varphi}, \\ [D\bar{\psi}] &\to [D\bar{\psi}]e^{-\infty i\varphi}. \end{aligned} \tag{29}$$

When combined, we have

$$[D\psi][D\bar{\psi}] \to [D\psi][D\bar{\psi}]e^{2C(R)ki\varphi} = [D\psi][D\bar{\psi}] \exp\left[2C(R)\varphi\frac{i}{16\pi^2}\int \mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{30}$$

This can be compensated by a shift of the θ angle, $\theta \to \theta + 2C(R)\varphi$. As we recalled before, the shift $\theta \to \theta + 2\pi$ is a symmetry. Therefore, the rotation of the field ψ by $\exp(\frac{2\pi i}{2C(R)})$ is a genuine, unbroken symmetry.

3.2 Global anomaly

One needs to be careful about Witten's global anomaly [3]. It is known that a Weyl fermion in the doublet of gauge $\mathrm{SU}(2)$ is anomalous, due to the following fact. When we perform the path integral of this system, we first need to consider

$$Z[A_\mu] = \int [D\psi_{\alpha i}][D\bar{\psi}_{\dot{\alpha} i}]e^{-\int \bar{\psi}D_\mu\sigma^\mu\psi} \tag{31}$$

where $i = 1, 2$ is the $\mathrm{SU}(2)$ doublet index. To perform a further integration over A_μ consistently, we need

$$Z[A_\mu] = Z[A_\mu^g], \quad A_\mu^g = g^{-1}A_\mu g + g^{-1}\partial_\mu g. \tag{32}$$

for any gauge transformation $g : \mathbb{R}^4 \to \mathrm{SU}(2)$. These maps are characterized by $\pi_4(\mathrm{SU}(2))$. It is known that

$$\pi_4(\mathrm{SU}(2)) = \pi_4(S^3) = \mathbb{Z}_2. \tag{33}$$

Let $g_0 : \mathbb{R}^4 \to \mathrm{SU}(2)$ be the one corresponding to the nontrivial element in this $\mathbb{Z}_2$. Then it is known that

$$[D\psi_{\alpha i}][D\bar{\psi}_{\dot{\alpha} i}] \xrightarrow{g_0} -[D\psi_{\alpha i}][D\bar{\psi}_{\dot{\alpha} i}] \tag{34}$$

resulting in

$$Z[A_\mu^{g_0}] = -Z[A_\mu], \tag{35}$$

thus making the path integral over A_μ inconsistent.

In general $\pi_4(G) = \mathbb{Z}_2$ if $G = \mathrm{Sp}(n)$, and $\pi_4(G) = 1$ otherwise. Therefore Witten's global anomaly can be there only for Weyl fermions in a representation R under gauge $\mathrm{Sp}(n)$. A short computation reveals that there is an anomaly only when $C(R)$ is half-integral.

Witten's anomaly is always $\mathbb{Z}_2$ valued in four dimensions. Therefore full hypermultiplets are always free of Witten's global anomaly. The danger only exists for half-hypermultiplets of gauge $\mathrm{Sp}(n)$. For example, one cannot have odd number of half-hypermultiplets in the doublet representation of gauge $\mathrm{SU}(2)$, or more generally, one cannot have half-hypermultiplets in a pseudo-real representation R of gauge $\mathrm{Sp}(n)$ such that $C(R)$ is half-integral.

4 $\mathcal{N}$=1 pure Yang-Mills

4.1 Confinement and gaugino condensate

As an example of the application of what we learned in this section, let us consider the $\mathcal{N}$=1 pure supersymmetric Yang-Mills theory with gauge group $\mathrm{SU}(N)$. The content of this section will not be used much in the rest of the lecture note.

This theory has just the vector multiplet, with the Lagrangian

$$L = \int d^2\theta \frac{-i}{8\pi}\tau \operatorname{tr} W_\alpha W^\alpha + cc., \qquad W_\alpha = \lambda_\alpha + F_{\alpha\beta}\theta^\beta + \cdots \tag{36}$$

The one-loop running of the coupling is given by

$$E\frac{\partial}{\partial E}\tau(E) = 3N, \tag{37}$$

and therefore we define the dynamical scale Λ by the relation

$$\Lambda^{3N} = e^{2\pi i \tau_{UV}}\Lambda_{UV}^{3N}. \tag{38}$$

We assign R-charge zero to the gauge field, and R-charge 1 to the gaugino λ_α. The phase rotation $\lambda_\alpha \to e^{i\varphi}\lambda_\alpha$ is anomalous, and needs to be compensated by $\theta \to \theta + 2N\varphi$. The shift of θ by 2π is still a symmetry, therefore the discrete rotation

$$\lambda_\alpha \to e^{\pi i/N}\lambda_\alpha, \qquad \theta \to \theta + 2\pi \tag{39}$$

is a symmetry generating $\mathbb{Z}_{2N}$. Note that under this symmetry, Λ defined above has the transformation

$$\Lambda \to e^{2\pi i/(3N)}\Lambda. \tag{40}$$

This theory is believed to confine, with nonzero gaugino condensate $\langle \lambda_\alpha \lambda^\alpha \rangle$. What would be the value of this condensate? This should be of mass dimension 3 and of R-charge 2. The only candidate is

$$\langle \lambda_\alpha \lambda^\alpha \rangle = c\Lambda^3 \tag{41}$$

for some constant c. The symmetry (40) acts in the same way on both sides by the multiplication by $e^{2\pi i/N}$. Assuming that the numerical constant c is non-zero, this $\mathbb{Z}_{2N}$ is further spontaneously broken to $\mathbb{Z}_2$, generating N distinct solutions

$$\langle \lambda_\alpha \lambda^\alpha \rangle = c e^{2\pi i \ell/N}\Lambda^3 \tag{42}$$

where $\ell = 0, 1, \ldots, N-1$. Unbroken $\mathbb{Z}_2$ acts on the fermions by $\lambda_\alpha \to -\lambda_\alpha$, which is a 360° rotation. This $\mathbb{Z}_2$ symmetry is hard to break.

It is now generally believed that this theory has these N supersymmetric vacua and not more. For other gauge groups, the analysis proceeds in the same manner, by replacing N by the dual Coxeter number $C(\text{adj})$ of the gauge group under consideration. For example, we have $N-2$ vacua for the pure $\mathcal{N}$=1 $\mathrm{SO}(N)$ gauge theory.

4.2 The theory in a box

It is instructive to recall another way to compute the number of vacua in the $\mathcal{N}$=1 pure Yang-Mills theory with gauge group G, originally discussed in [4]. We put the system in a spatial box of size $L \times L \times L$ with the periodic boundary condition in each direction. We keep the time direction as $\mathbb{R}$. By performing the Kaluza-Klein reduction along the three spatial directions, the system becomes supersymmetric quantum mechanics with infinite number of degrees of freedom.

The box still preserves the translation generators P^μ and the supertranslations Q_α unbroken. We just use a linear combination $\mathcal{Q}$ of Q_α and $Q^\dagger_\alpha$, satisfying

$$H = P^0 = \{\mathcal{Q}, \mathcal{Q}^\dagger\}. \tag{43}$$

We also have the fermion number operator $(-1)^F$ such that

$$\{(-1)^F, \mathcal{Q}\} = 0. \tag{44}$$

Consider eigenstates of the Hamiltonian H, given by

$$H|E\rangle = E|E\rangle. \tag{45}$$

In general, the multiplet structure under the algebra of $\mathcal{Q}$, $\mathcal{Q}^\dagger$, H and $(-1)^F$ is of the form

$$\begin{array}{ccccc} & \swarrow\!\!\!\nearrow & \mathcal{Q}^\dagger|E\rangle & \leftrightarrow & (\mathcal{Q}^\dagger\mathcal{Q} - \mathcal{Q}\mathcal{Q}^\dagger)|E\rangle \\ |E\rangle & \leftrightarrow & \mathcal{Q}|E\rangle & \swarrow\!\!\!\nearrow & \end{array} \tag{46}$$

involving four states. When $\mathcal{Q}|E\rangle = 0$ or $\mathcal{Q}^\dagger|E\rangle = 0$, the multiplet only has two states. If $\mathcal{Q}|E\rangle = \mathcal{Q}^\dagger|E\rangle = 0$, the multiplet has only one state, and E is automatically zero due to the equality

$$E\langle EE\rangle = \langle E|H|E\rangle = \langle E|(\mathcal{Q}\mathcal{Q}^\dagger + \mathcal{Q}^\dagger\mathcal{Q})|E\rangle = |\mathcal{Q}|E\rangle|^2 + |\mathcal{Q}^\dagger|E\rangle|^2. \tag{47}$$

We see that a bosonic state is always paired with a fermionic state unless $E = 0$.

This guarantees that the Witten index

$$\operatorname{tr} e^{-\beta H}(-1)^F = \operatorname{tr}\big|_{E=0}(-1)^F \tag{48}$$

is a robust quantity independent of the change in the size L of the box: when a perturbation makes a number of zero-energy states to non-zero energy $E \neq 0$, the states involved are necessarily composed of pairs of a fermionic state and a bosonic state. Thus it cannot change $\operatorname{tr}(-1)^F$.

Therefore, we can compute the Witten index in the limit where the box size L is far smaller than the scale Λ^{-1} set by the dynamics. Then the system is weakly coupled, and we can use perturbative analysis. To have almost zero energy, we need to have $F_{\mu\nu} = 0$ in the spatial directions, since magnetic fields contribute to the energy. Then the only low-energy degrees of freedom in the system are the holonomies

$$U_x, U_y, U_z \in \mathrm{SU}(N), \tag{49}$$

which commute with each other. Assuming that they can be simultaneously diagonalized, we have

$$U_x = \mathrm{diag}(e^{i\theta_1^x}, \ldots, e^{i\theta_N^x}), \tag{50}$$
$$U_y = \mathrm{diag}(e^{i\theta_1^y}, \ldots, e^{i\theta_N^y}), \tag{51}$$
$$U_z = \mathrm{diag}(e^{i\theta_1^z}, \ldots, e^{i\theta_N^z}). \tag{52}$$

together with gaugino zero modes

$$\lambda_1^{\alpha=1}, \ldots, \lambda_N^{\alpha=1}, \quad \lambda_1^{\alpha=2}, \ldots, \lambda_N^{\alpha=2} \tag{53}$$

with the condition that

$$\sum_i \theta_i^x = \sum_i \theta_i^y = \sum_i \theta_i^z = 0, \quad \sum_i \lambda_i^{\alpha=1} = \sum_i \lambda_i^{\alpha=2} = 0. \tag{54}$$

The wavefunction of this truncated quantum system is given by a linear combination of states of the form

$$\lambda_{i_1}^{\alpha_1} \lambda_{i_2}^{\alpha_2} \cdots \lambda_{i_\ell}^{\alpha_\ell} \psi(\theta_i^x; \theta_i^y; \theta_i^z) \tag{55}$$

which is invariant under the permutation acting on the index $i = 1, \ldots N$. To have zero energy, the wavefunction cannot have dependence on $\theta_i^{x,y,z}$ anyway, since the derivatives with respect to them are the components of the electric field, and they contribute to the energy. Thus the only possible zero energy states are just invariant polynomials of λs. We find N states with the wavefunctions given by

$$1,\ S,\ S^2,\ \ldots,\ S^{N-1} \tag{56}$$

where $S = \sum_i \lambda_i^{\alpha=1} \lambda_i^{\alpha=2}$. They all have the same Grassmann parity, and contribute to the Witten index with the same sign. Thus we found N states in the limit of small box, too.

The construction so far, when applied to other groups, only gives $1 + \mathrm{rank}\, G$ states. For example, let us consider for $G = \mathrm{SO}(N)$ for $N > 4$. Then the method explained so far only gives $\lfloor N/2 \rfloor + 1$ states

$$1,\ S,\ S^2,\ \ldots,\ S^{\lfloor N/2 \rfloor}, \tag{57}$$

and does not agree with $C(\mathrm{adj}) = N - 2$ when $N \geq 7$. This conundrum was already pointed out in [4] and resolved later in the Appendix I of [5] by the same author.[2] What was wrong was the assumption that three commuting matrices $U_{x,y,z}$ can be simultaneously diagonalized as in (52). It

[2] It is a sad state of affairs that a problem reported in such an important paper as [4] was not resolved for 15 years by any other physicist. It seems that people in our field rely too much on the author of [4, 5].

is known that there is another component where they cannot be simultaneously diagonalized into the Cartan torus. For $\mathrm{SO}(7)$, an example is given by the triple

$$U_x^{(7)} = \mathrm{diag}(++---+-), \tag{58}$$
$$U_y^{(7)} = \mathrm{diag}(+-+-+--), \tag{59}$$
$$U_z^{(7)} = \mathrm{diag}(-+++---). \tag{60}$$

These three matrices might look diagonal, but not in the same Cartan subgroup. This component adds one supersymmetric state. Then, in total, we have $(\lfloor 7/2 \rfloor + 1) + 1 = 5 = 7 - 2$, reproducing $C(\mathrm{adj})$.

For larger N, one can consider $U_{x,y,z}$ given by the form

$$U_x = U_x^{(7)} \oplus U'_x, \quad U_y = U_y^{(7)} \oplus U'_y, \quad U_z = U_z^{(7)} \oplus U'_z, \tag{61}$$

where $U'_{x,y,z}$ are in the Cartan subgroup of $\mathrm{SO}(N-7)$. Applying the analysis leading to (56) in both components, i.e. in the component where $U_{x,y,z}$ are in the Cartan subgroup of $\mathrm{SO}(N)$, and in the component where $U_{x,y,z}$ has the form (61), we find in total

$$(\lfloor N/2 \rfloor + 1) + (\lfloor (N-7)/2 \rfloor + 1) = N - 2 \tag{62}$$

zero-energy states, thus reproducing $C(\mathrm{adj})$ states. This analysis has been extended to arbitrary gauge groups.

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Perturbative non-renormalization ①

usual global symmetry ~ flavor symmetry

$$\Phi(y,\theta) = \phi(y) + \psi_\alpha(y)\theta^\alpha + F(y)\theta\theta$$

$$\Phi \to e^{i\varphi}\Phi \qquad \phi \to e^{i\varphi}\phi \qquad \psi \to e^{i\varphi}\psi \qquad F \to e^{i\varphi}F$$

R-symmetry ... $\theta^\alpha \to e^{i\varphi}\theta^\alpha$.

$$\Phi \to e^{i r\varphi}\Phi(y, e^{i\varphi}\theta)$$

$$\phi \to e^{ir\varphi}\phi, \quad \psi \to e^{i(r-1)\varphi}\psi, \quad F \to e^{i(r-2)\varphi}F.$$

For vec superf, $W_\alpha = \lambda_\alpha + F_{\alpha\beta}\theta^\alpha\theta^\beta + D(\theta\theta) + \cdots$

~~R charge 1~~ ~~R charge 0~~

$$\int d^2\theta \left(W(\Phi) + \tau \, \mathrm{tr} W^\alpha W_\alpha \right)$$

~~note that this is~~ $\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^2}$. needs to be R-charge 2.

shift symmetry $\tau \to \tau +$ real number . adds $\theta F\tilde{F}$

WZ-model $\Phi_1 \cdots \Phi_n$ $\mathcal{L} = \int d^4\theta\, K(\Phi,\bar\Phi) + \int d^2\theta f(\Phi) + c.c.$ (arb. complicated)

think of it as $= \int d^4\theta\, K(\Phi,\bar\Phi) + \int d^2\theta\, Y(y,\theta) f(\Phi)$ (R charge 2, 0)

Y: external chiral bkgd superfield. later we set it to 1.

↓ perturbative calc.

$$= \int d^4\theta \, [\,///\,] + \int d^2\theta\, Y(y,\theta)\, g(\Phi).$$

you can only use Y^n. R-charge.

Let Y very small. $f = g$. done.

IT NEEDS TO BE EMPHASIZED THAT IT'S NOT THAT low energy is easy! It's just that there's a scheme, in holomorphic. IR result is a func. of $f(\Phi)$ and Λ.

Similarly, with gauge fields. UV $\int d^2\theta \, (\tau)\, \mathrm{tr} W_\alpha W^\alpha$ (R-charge 2.)

think of it as a superf.

τ: $\frac{i}{g^2}(\Lambda_{UV'}) + \frac{\theta}{8\pi}$ — Λ_{UV}, $\Lambda_{UV'}$

$$= \frac{i}{g^2}(\Lambda_{UV}) + \frac{\theta}{8\pi} + i b \log\frac{\Lambda_{UV'}}{\Lambda_{UV}} \left(\frac{i}{g^2} + \frac{\theta}{8\pi}\right) + O(1) + g^2 + g^4 + \cdots$$

(1-loop, 2-loop, 3-loop ... can't appear in $\int d^2\theta\, (\tau-\bar\tau)^{-1}$, $(\tau-\bar\tau)^{-2}$ ...)

$$\Lambda^b \equiv (\Lambda_{UV'})^b e^{-\frac{1}{g^2}(\Lambda_{UV'}) + i\theta} = (\Lambda_{UV})^b e^{-\frac{1}{g^2}(\Lambda_{UV}) + i\theta}$$

⇐ complex! backgd superfield.

SUPER QCD

$3N_c$ free ① $\to N_f$ $b = 3N_c - N_f$

$SO(N)$

what happens at We learned $N_f = 0$; when $N_f = 3N_c$, one loop $= 0$, two loop decreases g. what happens in between? — draw a graph and fill them gradually

A_μ			
λ_α	$\psi_\alpha^{\alpha i}$ $\psi_{\alpha\, a i}$	$a = 1 \cdots N_c^N$	
	Q^{ai} $\tilde{Q}_{ai}$	$i = 1 \cdots N_f$.	

symmetry: $SU(N_f) \curvearrowright Q^i$ $U(1) \curvearrowright Q^i \to e^{i\varphi} Q^i$ $\Lambda^b \to e^{i\varphi N_f} \Lambda^b$

$SU(N_f) \curvearrowright \tilde{Q}_i$ $U(1) \curvearrowright \tilde{Q}_i \to e^{i\varphi} Q_i$ $\Lambda^b \to e^{i\varphi N_f} \Lambda^b$

R-sym $Q: 0$, $\tilde{Q}: 0$ $A: 0$ $\lambda \to \lambda e^{i\varphi}$, $\psi \to \psi e^{-i\varphi}$, $\tilde{\psi} \to \tilde{\psi} e^{-i\varphi}$ $\Lambda^b \to \Lambda^b e^{i\varphi(\frac{N_c}{2} - \frac{N_f}{2})}$

gauge inv chiral op $M^i{}_j = Q^{ia} \tilde{Q}_{ja}$ $\Rightarrow$ $\det M$: inv under $SU(N_f) \times SU(N_f)$

$\Lambda^b / \det M$: inv under $U(1) \times U(1)$.

$C_{N_f} \int d^2\theta \left(\Lambda^b / \det M \right)^{\frac{1}{N_c - N_f}}$: R-charge 2.

sensible when $N_c > N_f$. $N_c = N_f \Rightarrow$ ✗!! $N_c < N_f$: $e^{+\frac{1}{g^2}}$ instead of $e^{-\frac{1}{g^2}}$. ✗!!

Affleck-Dine-Seiberg superpotential C_{N_f} believed to be non-zero.

e.g. $N_f = 1$ $W = \left(\frac{\Lambda^{3N_c - 1}}{Q\tilde{Q}}\right)^{\frac{1}{N_c - 1}}$. add the mass term $c\left(\frac{\Lambda^{3N_c-1}}{Q\tilde{Q}}\right)^{\frac{1}{N_c-1}} + m(Q\tilde{Q})_M$

$c(\Lambda^{3N_c-1})^{\frac{1}{N_c-1}} (Q\tilde{Q})^{\frac{-N_c}{N_c-1}} = m^{N_c-1}$ $\Rightarrow$ $(Q\tilde{Q}) = c(\quad)^{\frac{1}{N_c}}$

note that with $m=0$ there's no vacuum. "runaway" $\rightsquigarrow$ N_c vacuum. if $c \neq 0$.

$N_f = N_c - 1$ $W = \left(\frac{\Lambda^{3N_c - N_c + 1}}{\det M}\right)^1$: 1-instanton factor.

consider n_f random vectors in $\mathbb{R}^{N_c}$ --> $SO(N_c - N_f)$ unbroken.

similarly, $SU(N_c - N_f)$ unbroken. $\Rightarrow$ if $N_f = N_c - 1$, no unbroken gauge

$\Lambda_A^{3N_c - N_f} = e^{-\frac{8\pi^2}{g^2}(\Lambda') + i\theta} \Lambda'^{3N_c - N_f}$ → weakly coupled → calculable. (ADS)

A: $SU(N_c)$ N_f

- - - - - - - - at scale $\Lambda_{\text{transition}} \sim (\det M)^{\frac{1}{2N_f}}$

B: $SU(N_c - N_f)$ $N_f = 0$ pure.

$\Lambda_B^{3N_c - 3N_f} = e^{-\frac{8\pi^2}{g^2}(\Lambda'') + i\theta} \Lambda''^{3N_c - 3N_f}$. should be equal $\sim \det M^{1/2N_f}$.

$\therefore \Lambda_A^{3N_c - N_f} = \Lambda_B^{3N_c - 3N_f} (\det M)$.

B's superpotential $= \Lambda_B^3 = \left(\frac{\Lambda_A^{3N_c - N_f}}{\det M}\right)^{\frac{1}{N_c - N_f}}$.

~~what happens when~~ $N_f = N_c$?

$B = \epsilon_{a_1 \cdots a_N} Q^{a_1 i_1} Q^{a_2 i_2} \cdots Q^{a_N i_N}$ } automatically antisym in i's. $SU(N_f)$-inv

$\tilde{B} =$

classically, $\det M = B\tilde{B}$. $\epsilon^{a_1 \cdots a_N} \epsilon_{b_1 \cdots b_N} = \delta^{a_1}_{[b_1} \delta^{a_2}_{b_2} \cdots \delta^{a_N}_{b_N]}$.

SUSY QCD ②.

QM'lly, $\det M - B\tilde{B} = \Lambda^{3N_c-N_c}$: mass dim, R-sym, U(1)-charges all OK.

1-instanton effect

SU(2) → calculable. (Beasly-Witten 0409149)

you might say how can a relation like

$\varepsilon^{\cdots}\varepsilon_{\cdots} = \delta_{\cdot}\cdots\delta_{\cdot}$ be modified ?

operators can't be at a same point. $Q^{ia}(x)\exp\left(\int_x^y A_\mu dx^\mu\right)\tilde{Q}_{jb}(y)$

$x \to y$.

What happens when $N_f \lessapprox 3N_c$?

two-loop fixed point at $g^2 \sim O(\varepsilon)$, $\varepsilon = 3 - \frac{N_f}{N_c}$.

very weakly coupled SCFT. (~~Banks-Zaks~~)

How is it compatible with "one-loop exactness" of the running of $\tau = i\frac{4\pi}{g^2} + \frac{\theta}{2\pi}$?

$$\int \begin{matrix}Q^\dagger e^V Q \\ + \tilde{Q}^\dagger e^{V} \tilde{Q}\end{matrix} + \int \left(\frac{4\pi}{g^2}(\Lambda') + \frac{\theta}{2\pi}\right) \operatorname{tr} W^\alpha W_\alpha$$

$$\to \int Z\begin{matrix}(Q^\dagger e^V Q) \\ + \tilde{Q} e^V \tilde{Q}\end{matrix} + \int \left(i\frac{4\pi}{g^2_{hol}}(\Lambda'') + \frac{\theta}{2\pi}\right)\operatorname{tr} W^\alpha W_\alpha \qquad \frac{4\pi}{g^2_{hol}}(\Lambda'') = \frac{4\pi}{g^2}(\Lambda') + \frac{b}{2\pi}\log\frac{\Lambda''}{\Lambda'}$$

$1 + \left(\log\frac{\Lambda''}{\Lambda'}\right)\gamma(g^2)$, $\gamma = kg^2 + \cdots$

There's a relation $\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}(Q^\dagger e^V Q) = k \operatorname{tr} W^\alpha W_\alpha$. ~~"Konishi anomaly"~~

$\partial_\mu(\bar\psi\gamma^\mu\gamma_5\psi) = \operatorname{tr} F_{\mu\nu}\tilde{F}_{\mu\nu}$ standard anomaly

also $\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}(\tilde{Q}^\dagger e^{-V}\tilde{Q}) = \operatorname{tr} W^\alpha W_\alpha$

so $\underbrace{\int d^4\theta}_{d^2\theta\, d^2\bar\theta} \log\frac{\Lambda''}{\Lambda'}\gamma(g^2)\, Q^\dagger e^V Q = \int d^2\theta\, \underset{\left(\log\frac{\Lambda''}{\Lambda'}\right)\gamma(g^2)}{\bar{D}^2}(Q^\dagger e^V Q) = \int d^2\theta \left(\log\frac{\Lambda''}{\Lambda'}\right)\underset{-k\gamma(g^2) = k^2g^2 + \cdots}{\operatorname{tr} W^\alpha W_\alpha}$

Then

$$\to \int Q^\dagger e^V Q + \int\left(i\frac{4\pi}{g^2_{phys}}(\Lambda'') + \frac{\theta}{2\pi}\right)\operatorname{tr} W^\alpha W_\alpha \qquad \text{can cancel.}$$

where $\frac{4\pi}{g^2_{phys}}(\Lambda'') = \frac{4\pi}{g^2}(\Lambda') + \boxed{\frac{b}{2\pi}}\left(\log\frac{\Lambda''}{\Lambda'}\right) + \boxed{\underset{-2N_f k g^2 + \cdots}{\gamma(g^2)}}\left(\log\frac{\Lambda''}{\Lambda'}\right)$

note that this is not holomorphic in τ. dep. on $\tau - \bar\tau$

γ is called the anomalous dimension 'cause

$\underbrace{\int d^4x\, d^4\theta}_{\text{dim } -2.}\ \underbrace{Q^\dagger e^V Q}_{\text{dim } 2.}$ $\quad x \to e^{\lambda} x \quad \theta \to e^{\lambda/2}\theta \quad Q \to e^{\lambda} Q$ classically.

$\left(1 + \frac{\gamma}{2}\log\frac{\Lambda''}{\Lambda'}\right)$ $\quad \Lambda' \to e^{\lambda}$ flow Λ'' $\quad Q \to e^{\lambda + \frac{\gamma}{2}\lambda} Q$.

λ $\quad x \to e^{-\lambda} x$ $\quad$ Q's dimension is $1 + \frac{\gamma}{2}$.

← compensate $Q \to Q e^{-\lambda\frac{\gamma}{2}}$ $\qquad 1 - \frac{3N_c - N_f}{2N_f} + O(g^2) +$

a 為に 一寸と不達.

SQCD (3)

$\gamma(g^*)$ at the conf. point can be exactly determined!

note $\gamma(g^2)$ is a perturbative series in g,
~~g^* itself is given by solving~~
$\beta(g)=0$, where $\beta(g)$ is a perturbative ...
both β & γ are scheme dependent, but $\gamma(g^*)$ is not

SCFT | CFT

Q_α $Q^\dagger_{\dot\alpha}$ | P_μ K_μ
S_α $S^\dagger_{\dot\alpha}$ | D

$I: x^\mu \to \frac{x^\mu}{|x^\mu|^2}$ $\quad K_\mu = I \cdot P_\mu \cdot I$.

$S_\alpha = I \cdot Q_\alpha \cdot I$ superconformal $\leftrightarrow$ any R-symmetry

R "R-symmetry" $[R, Q] = -Q$, $[R, S] = +S$.

$\{Q^\dagger_{\dot\alpha}, S^\dagger_{\dot\beta}\} = \varepsilon_{\dot\alpha\dot\beta}(2D - 3R) + M_{\dot\alpha\dot\beta}$

~~iD の係数はゴツカる.~~

now an operator is chiral $\leftrightarrow$ annihilated by $Q^\dagger_{\dot\alpha}$.
scalar $\leftrightarrow$ $M_{\dot\alpha\dot\beta}$ is zero. $\quad \theta \to e^{D(\theta)}\theta$

$\Rightarrow 2D - 3R = 0$. $\quad \theta \to e^{iR(\theta)}\theta$

$\Rightarrow R(\theta) = \frac{2}{3}D(\theta)$.

e.g. a free chiral field. Φ: dimension 1. $\quad \int d^2\theta\, \Phi^3$
R-charge 2. $\quad \Phi$: R-charge $\frac{2}{3}$.

~~superconformal R-sym is anomaly free.~~

A_μ
λ_α $\quad \psi$ $\quad \tilde\psi$
$\quad Q$ $\quad \tilde Q$

$\lambda_\alpha \to e^{i\varphi}\lambda_\alpha$
$\psi_\alpha \to e^{-ia\varphi}\psi_\alpha$
$\tilde\psi_\alpha \to e^{-ia\varphi}\tilde\psi_\alpha$

$\theta +$
$\theta \to 2N_c\varphi - a 2N_f\varphi$.

$\therefore a = \frac{N_c}{N_f}$.

$\therefore Q \to e^{(1-\frac{N_c}{N_f})i\varphi} Q$ $\quad R(Q) = 1 - \frac{N_c}{N_f}$.

~~note that when $N_f \lesssim 3N_c$,~~ $\leftarrow$ ~~$D(Q) = \frac{3}{2} - \frac{3N_c}{2N_f}$~~
$\frac{N_c}{N_f} \gtrsim \frac{1}{3}$.

$\therefore D(Q) \lesssim 1$.

$D(M) = 3 - 3\frac{N_c}{N_f}$ becomes 1 at $N_f = \frac{3}{2}N_c$.

gauge inv scalar op has $D(M) \geq 1$. $\rightsquigarrow$ $N_f < \frac{3}{2}N_c$ can't be SCFT.
use unitarity. note that gauge dof. Hilb. space is not unitary

Seiberg duality when $N_f \geq N_c + 2$ $\quad$ ($N_f = N_c + 1$ はあとまわし.)

$SU(N_c)$ with N_f $\quad \overline{\quad}$ $\quad SU(N_f - N_c)$ with N_f
$Q^{ai}, \tilde Q_{ai}$ $\quad$ at IR $\quad q_{ai}\, \tilde q^{ai}$ plus $M^i{}_j$
with $W = M^i{}_j\, q_i \tilde q^j$.

$M^i{}_j = Q^{ai}\tilde Q_{aj} \longleftrightarrow M^i{}_j$. $\quad$ (note $q_i \tilde q^j = 0$ due to $\frac{\partial W}{\partial M}$).

$\binom{N_f}{N_f - N_c}$ $\quad B^{i_1 \cdots i_N} = \varepsilon^{a_1 \cdots a_N} Q^{a_1 i_1} \cdots Q^{a_N i_N} \longleftrightarrow B_{i_1 \cdots i_{N_f - N_c}}$

SQCD ④.

't Hooft anomaly matching. $\mathrm{tr}_{UV}\left(\gamma^5\ \varepsilon\right) = \mathrm{tr}_{IR}\left(\ \varepsilon\right)$

't Hooft's argument : gauge it. you can't.

add a spectator fermion which cancels it.

Coleman–Grossman : more operator-ish.

$\sum_{UV_1} = \mathrm{IR}\left(\ \varepsilon\ \right) = \sum_{UV_2}$ because subscript

eg. $SU(N_f)^3_L$ Q^{ac} : N_c mult. in $\triangleleft$ $\longleftrightarrow$ q_{ab} : $N_f - N_c$ mult. in $\triangleleft$

$M^{c}{}_{j}$: $+N_f$

~~you can check others.~~

< など $N_f = N_c + 1$ など...

at one extreme, $SU(N_c)$ with $N_f = N_c + 1$ $=$ $SU(1)$ with $N_f = N_c + 1$ $q_i\ \tilde{q}$

$M^{c}{}_{j}$ $\longleftrightarrow$ plus $M^{c}{}_{j}$.

$B^{c_1 \cdots c_N}$ $\longleftrightarrow$ q_i. $W = q_i M^{c}{}_{j} \tilde{q}^{j}$

it just says you have superp. $+\det M$

$W = B_i M^{i}{}_{j} \tilde{B}^{j} + \det M$ why do we have it?

~~using this,~~ add $m Q_{N_c} Q_{N_c+1} \longrightarrow m M^{N_c+1}{}_{N_c+1}$.

$\longrightarrow B\tilde{B}^{N_c+1}_{N_c+1} + \det \hat{M} = m\,(\Lambda)^{\#}$. → reproduces mod.

to understand the appearance of $\det M$, consider $N_f = N_c + 2$ の $N_c = 3$ から mass deform する. 7/3から5

$SU(2)'$ with $q_i\ \tilde{q}^{i}\ M^{i}{}_{j}$, $N_f' = N_f + 1$, $W = q_i \tilde{q}^{j} M^{i}{}_{j}$.

~~give a vev to the last~~ $m M^{N_f+1}{}_{N_f+1}$ add q_{N_f+1} $\tilde{q}^{N_f+1}$ have vevs → break $SU(2)$. ~~become massive.~~

$\left(\Lambda^{3\cdot 2 - (N_f+1)}\right)\det M_{N_f \times N_f}$ M^2 ⇐ mass dim 3? R-charge 2??

N_f M q N_f+1 N_f+1 lowest factor. $U(1)_L$: $q \to e^{i\varphi} q$ $M \to e^{-i\varphi} q$

$b = N_f - 1$ ~~mass dim~~ requires. $\Lambda^{b_0} \det M$.

~~and~~ 4 $+N_f$ R charge m: 0 $q, \tilde{q} = 0$ M: 2

$+2$ $\Lambda^{3\cdot 2 - (N_f+1)}$: $4 - 2(N_f+1)$.

$= \frac{3}{2}N_c \Leftrightarrow SU(\frac{N_f}{2})$ $\frac{3N_f}{2}$: upper bound! $\det M$: $2N_f$.

2.

in addition to chiral ops & anomalies, we can now compare part. func. on $S^1 \times S^3$.

also called "supercnf. index". Dolan–Osborn 0801.4947 Römelsberger 0707.3702

Spiridonov–Vartanov Rastelli 1011.5278

~~take whatever dual pair in the literature, check it,~~ and write a paper.

Super"conformal" index ①

pure SYM を T^3 にして より深いチェックをしたように、

SQCD の Seiberg duality も何か出来ないか？

T^3 はなされていない。 $S^3 \times \mathbb{R}_t$ を考える。　Römelsberger 0510060, Festuccia-Seiberg 1105.0689

bosonic sym は $SU(2)_l \times SU(2)_r \times U(1)$ ← time translation.

$\mathcal{N}=1$ susy にするれば、先の議論より、(空間の boson が single だから)

$$SU(2|1) \times SU(2)_r \; \text{~~}\times U(1)\text{~~}$$ にするしかない。

↑ ~~$U(1)_R$ を含む ← 独立な R-sym が 必要~~

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^0_{\alpha\dot\beta} P + \frac{2}{r}\sigma^i_{\alpha\dot\alpha} J^i_l$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

$$[P_0, Q_\alpha] = \frac{1}{r} Q_\alpha$$ ← supercharge は時間方向に回転する。

$S^3 \times S^1$ にして、周期な S^1 の半径で SUSY を保つには、回転を compensate しないといけない。

continuous unbroken
⇒ $U(1)_R$ 対称性の存在を必要とする。

$[R, Q_\alpha] = -Q_\alpha$ として $H = P_0 + \frac{1}{r}R$ を考える。

$[H, Q_\alpha] = 0$.

$$Z_{S^3\times S^1} = \mathrm{tr}\,(-1)^F e^{-\beta(P_0 + \frac{1}{r}R) + \mu_i J_i}$$

(J_i ← take a flavor symmetry 及び J_R：右角運動量; μ_i ← chemical potential)

寄与する short rep. は 上記 $\{Q_\alpha, \bar{Q}_{\dot\beta}\}$ commutator より $P_0 = \frac{2}{r}J^3_l$ のもの.

$$= \mathrm{tr}|_{short}(-1)^F e^{-\frac{\beta}{r}(2J^3_l + R) + \mu_i J_i}$$

ラグランジアンは書け、SUSY がある。しかし使わなくてもよい。

(主な変更点 (w.r.t. flat sp) $\partial_t \to \partial_t + \frac{iq}{r}$　where q is the R-charge of the field.)

計算法：Witten index だから、理論の詳細によらない。
S^3 上なので gap があるから、特に安全。

S^3 - index ②

$\Phi \to \phi$ S^3 上のスカラー場は $SU(2)_L \times SU(2)_R$ のもとでは $\phi \bigoplus_j V_j \otimes V_j$ と分解.

$\Psi \to \psi$ フェルミオン場

ϕ : q $\quad$ $\bar\phi$: $-q$

ψ : $q-1$ $\quad$ $\bar\psi$: $-q+1$

$\psi^\dagger$: $P_0 = 2j+1$ → $\oplus (V_{j+\frac12} \otimes V_j \oplus V_j \otimes V_{j+\frac12})$ ← $P_0 = 2j+2$

ψ_α, ψ_β ; top cmp. だけのこる.

$P_0 = 2j+2$ → $V_j \otimes V_j \oplus V_{j+\frac12} \otimes V_{j+\frac12}$ ← $P_0 = 2j+1$

$\bar\phi$ $\quad$ ϕ : top. cmp だけのこる.

$2(j+\frac12) - q + 1$

残る mode :

$j=0$, $j=\frac12$, $j=1$ → m, n

$\bar\psi$

ϕ

$j=0$, $j=\frac12$, $j=1$

N.B. ゼロ点 energy.

$$Z = \prod_{m,n \ge 0} \frac{1 - t^{m+n-q+2}\, y^{m-n}/z}{1 - t^{m+n+q}\, z\, y^{m-n}}$$

但し $t = e^{-\frac{\beta}{r}}$, $y = e^{-ML}$, $z = e^{-M_{U(1)}}$ chg

elliptic gamma func :

$$\Gamma_{p,q}(z) := \frac{\prod (1 - z^{-1} p^{i+1} q^{j+1})}{\prod (1 - z\, p^i q^j)}$$

より $Z_{\text{chiral}} = \Gamma_{ty,\, t/y}(t^q z)$ である.

$q = 2/3$ だと $\phi^{>>}$ $\psi^\dagger$

$1 + z t^{2/3} + (z^{+2} - z^{-1}) t^{4/3} + (y + y^{-1}) z t^{5/3} + \cdots$

ϕ with angl. mom.

cnst mode of ϕ

誘導表現の言葉

$P_0 = 2j,\ R = q$ ϕ : $V_j \otimes V_j$ ココ $\quad \oplus \quad$ $\phi^\dagger$: $V_j \otimes V_j$ $\quad$ $P_0 = 2j+2 \quad R = -q$ ← $D = 2j+1$

$P_0 = 2j+1,\ R = q-1,\ \psi$: $V_{j-\frac12} \otimes V_j$ $\quad$ $\bar\psi$: $V_{j+1/2} \otimes V_j$ ココ が残る. $\quad$ $P_0 = 2j+1$, $R = -q+1$

$SU(2|1)$ の表現. $\quad$ $D = 2j + \frac32$

$V_j \otimes V_j$

bc $\quad$ $V_{j+1/2} \otimes V_j \oplus V_j \otimes V_{j+1/2}$

λ $\quad$ $V_{j+1} \otimes V_j \oplus V_j \otimes V_j \oplus V_j \otimes V_{j+1}$

A

$A_0 = $ const.

$$\boxed{P_0 = D - \frac32 R_{UV}}$$

$A_0 = \mathrm{diag}(a_1, a_2, \dots, a_N)$ として $z_i = e^{-\beta a_i}$

同様に vect. mult. を考えると $SU(N)$ では

分子は ghost を表す.

$$Z = \prod_{i \ne j} \frac{1 - t^{m+n}\, y^{m-n}\, z_i/z_j}{1 - t^{m+n+2}\, y^{m-n}\, z_j/z_i} = \frac{1}{\prod_{i \ne j} \Gamma_{ty,\, t/y}(z_i/z_j)} \left(\frac{1}{\Gamma_{ty,\, t/y}(\cdots)} \right)$$

但し $m=n=0$, $i=j$ は除く.

S^3 index ③

SU(2) で N_f flavor がある。 Q の $R = 1 - \frac{N_c}{N_f}$ だから

$z_1, z_2 = z_1^{-1}$ (z)　　$\mu_1 \cdots \mu_{N_f}$　ν

$\tilde\mu_1 \cdots \tilde\mu_{N_f}$　ν として　$\prod \mu = \prod \tilde\mu = 1$

$$\underbrace{\frac{1}{\Gamma'(1)\prod_{\pm}\Gamma(z^{\pm2})}}_{\text{vector}} \underbrace{\prod_{\pm,i}\Gamma(t^{1-\frac{2}{N_f}} z^{\pm1}\mu_i\nu)}_{Q} \underbrace{\prod_{\pm,i}\Gamma(t^{1-\frac{2}{N_f}} z^{\pm1}\tilde\mu_i\nu^{-1})}_{\tilde Q}$$

SU(2) gauge inv part に proj. out.

‖

$$\begin{pmatrix} z & \\ & z^{-1}\end{pmatrix} = e^{-i\beta\begin{pmatrix}A_0 & \\ & -A_0\end{pmatrix}}$$ と書いたときの A_0 の path integral

単に $\oint \frac{dz}{2\pi i z}$ すればよい。

(SU(2) を U(1) に gauge fix するときの Vandermonde det. は既に $\frac{1}{\prod\Gamma(z_i/z_j)}$ の $m=n=0$ の項に $\prod_{i\neq j}\left(1-\frac{z_i}{z_j}\right)$ として含まれている。)

$N_f=3$ だと、 low energy desc dual は $M = Q_i\tilde Q_j$ (R=2/3) ~~…~~ $B^k = Q_iQ_j\epsilon^{ijk}$, $\tilde B^k = \tilde Q\tilde Q$, がある。

$$\overset{?}{=} \prod\Gamma(t^{2/3}\mu_i^{-1}\nu^{2}) \prod\Gamma(t^{2/3}\tilde\mu_i^{-1}\nu^{-2}) \prod\Gamma(t^{2/3}\mu_i\tilde\mu_j)$$

べき展開して、下から確認可能。

UV: SU(2), $N_f=3$
↓
IR: M, B^k, $\tilde B^k$

$N_f=4$

UV$_1$ ↘ IR ↙ UV$_2$

S^1/S^3 index は IR では同じはず。
⇒ UV でも同じはず。

$$\oint\frac{dz}{2\pi i z}\frac{1}{\Gamma'(1)\prod\Gamma(z^{\pm2})}\prod\Gamma(t^{1/2}z^{\pm1}\mu_i\nu)\prod\Gamma(t^{1/2}z^{\pm1}\tilde\mu_i\nu^{-1})$$

$$\overset{?}{=}\left[\oint\frac{dz}{2\pi i z}\frac{1}{\Gamma'(1)\prod\Gamma(z^{\pm2})}\prod\Gamma(t^{1/2}z^{\pm1}\mu_i^{-1}\nu)\prod\Gamma(t^{1/2}z^{\pm1}\tilde\mu_i^{-1}\nu^{-1})\right]\prod\Gamma(t\mu_i\tilde\mu_j)$$

S^3-index ④

一般に $SU(N_c)$ N_f Q $\tilde{Q}$ ↔ $SU(N_f-N_c)$ N_f q $\tilde{q}$ $+M$

のS.D.は elliptic Γの積の(積分の)とんでも無い公式を導く.

Rains, math.QA/0309252 で証明されている
(その当時は S.D. との関係は未知).

その他沢山 Seiberg duality は知られている
→ 沢山の Γ の等式. 証明されているものは少ない

例 このRainsの論文で $SU(2)$ $N_f=4$ をちょっと弄ると $SU(8) \to E_7$ を示している.

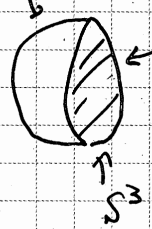
例 5d $\mathcal{N}=1$ on $S^4 \times S^1$ でも同様のことが出来る.

物理的には: free hyper $Z = \left| \frac{1}{(tz)_{ty,t/y}} \right| \left| \frac{1}{(t/z)_{ty,t/y}} \right|$ ← Y $(z)_{p,q} = \prod_{i,j\geq 0}(1-z p^i q^j)$

S^4半分

Dimofte Gaiotto 1209.1434

5d $\mathcal{N}=1$



5d hyper → 4d chiral (X, Y) R, $\bar{R}$

boundary superp. $\begin{cases} W = XO \\ Y|_{bndry} = O \end{cases}$ は half-BPS condition.

hemisphere index

$$= \frac{1}{(tz)_{ty,t/y}} \times (S^3\text{-index}).$$

$SU(2)$ 4 flavor Q_i $\tilde{Q}^i$ → $q_{i=1\cdots 8}$ と思っておく. fugacity $\mu_1 \cdots \mu_8$ $\prod \mu = 1$

$M_{[ij]} = q_i q_j$ ← 28コ (ちょっと前は M_{16}, B_6, $\tilde{B}_6$ と書いた)

5d bulk に X^{ij}, Y_{ij} を導入, $\begin{cases} W = XM \\ Y| = M \end{cases}$ とする.

乙 $J(\mu_1 \cdots \mu_8)$ ~~index~~ $= \frac{1}{\prod_{i<j}(t\mu_i^{-1}\mu_j^{-1})_{ty,t/y}} \oint \frac{dz}{2\pi i z} \frac{1}{\Gamma(1)} \frac{1}{\prod \Gamma(z^{\pm 2})} \prod_{i=1}^{8} \Gamma(t^{1/2} z^{\pm} \mu_i)$

(このprefactorをつければよい)
↓
次回につづいた後

~~積分から $SU(8)$ 対称性を持つ~~

~~数学者 ⇒ E_7 sym がある (上記 Rains の論文)~~ ~~index の関係~~

S^3 - index ⑤

もう少し説明すると. SU(2) $N_f=4$ の index を $I(\mu_1 \cdots \mu_8)$ と書くと.
Seiberg duality は

$$I(\mu_1 v, \mu_2 v, \mu_3 v, \mu_4 v, \tilde\mu_1 v^{-1}, \tilde\mu_2 v^{-1}, \tilde\mu_3 v^{-1}, \tilde\mu_4 v^{-1})$$
$$= \frac{(t\mu_i^{-1}\tilde\mu_j^{-1})_{ty,t/y}}{(t\mu_i\tilde\mu_j)_{ty,t/y}} I(\mu_1^{-1}v, \cdots \mu_4^{-1}v; \tilde\mu_1^{-1}v^{-1} \cdots \tilde\mu_4^{-1}v^{-1})$$

両辺に $\frac{1}{(t\mu_i^{-1}\tilde\mu_j^{-1})} \frac{1}{(t\mu_i^{-1}\mu_j^{-1}v^{-2})(t\tilde\mu_i^{-1}\tilde\mu_j^{-1}v^{+2})}$ をかけると

i) ✡ $J(\mu_1 v \cdots \mu_4 v, \tilde\mu_1 v^{-1} \cdots \tilde\mu_4 v^{-1}) = J(\mu_1^{-1}v \cdots \mu_4^{-1}v, \tilde\mu_1^{-1}v^{-1} \cdots \tilde\mu_4^{-1}v)$

iii) ✡ $J(M_1 \cdots M_8)$ は $\mathfrak{S}_8$ で不変.

$SU(2) = Sp(1)$ と思うと. $Sp(N_c)$ N_f flavor $q_1 \cdots {}_{2N_f}$
$\Updownarrow$
$Sp(N_f - N_c - 3)$, N_f flavor. Q^i + meson.

ii) ✡ $J(M_1 \cdots M_8) = J(M_1^{-1} \cdots M_8^{-1})$.

はじめの cility : 1つ
i) は ${}_8C_4 = 70$ とおり.
ii) は 1つ
} 72つ. $\frac{W(E_7)}{W(A_7)} = 72$.

この prefactor $\frac{1}{(tM_i^{-1}M_j^{-1})_{t/y,ty}}$ は何か? → 問 に戻る

[J] $W = X^{ij} M_{ij}$, $Y_{ij}| = M_{ij}$ において Seiberg dual をとると

一般の M は boundary elementary field になる.

$\begin{cases} W = X^{z+1} M^{z-1} \\ {}^{z}Y| = M \end{cases}$ をとるのは 単に $X| = \frac{\partial W}{\partial M} = 0$, $Y = M$: neumann と等価.

index の level では

$$\frac{1}{(tz)_{ty,t/y}} \cdot \Gamma_{ty,t/y}(tz^{-1}) = \frac{1}{(tz^{-1})_{ty,t/y}}$$

↙ Mは別に用は bdry に いない.

⤳ IR では E_7-inv. bdry condition になるのであろう.

(X, Y)
28つ 28つ ← E_7 の half-hyper.

E_7 のはず.
SU(8) SU(8)
$SU(4) \times SU(4) \times U(1)$

Seiberg-Dual では
$W = c(X\overset{6}{B} + \tilde X\overset{6}{\tilde B})$
$+ c' YM$
↑ 12
Cの関数.

$W = cXM$ (28 28) → Seib. dual

$c = c'$ になるところまで来ると,
~~$M_i v, \mu_i$~~ dual frame で
$SU(4) \times SU(4) \times U(1)$ → SU(8)

$\mathcal{N}=(2,2)$

Sheet ______ Date ______

2d SUSY theories ① $\mathcal{N}=(2,2)$ Lagrangian

・2d は ゲージ場を使わないでも strongly coupled でいろいろ出来る.
 (3d でも. しかし,僕はやった事が無いので,森山 or 奥山さんに)
・scale-inv になると, Virasoro 代数という無限次元対称性.
 → さらに良くわかる.
・勿論 ゲージ場を足しても面白い.

4d $\mathcal{N}=1$ superfield formalism Witten 9301042

$x^M,\ \theta_\alpha,\ \bar\theta_{\dot\alpha}$

次元還元 → $x^{0,1},\ \theta^{\pm},\ \bar\theta^{\pm}$ $\mathcal{N}=(2,2)$ SUSY in 2d.

superspace 積分として

$\int d^2\theta\, d^2\bar\theta$ (勝手な実の超場)

$\int d^2\theta$ (勝手な chiral 超場) + c.c. i.e. $\bar D_\pm \Phi = 0$

だけでなく

$\int d\theta^+ d\bar\theta^-$ (twisted chiral) + c.c. i.e. $\bar D_+ \Sigma = D_- \Sigma = 0$

なものも考えられる.

chiral superfield $\Phi = \phi + \theta^\alpha \psi_\alpha + F\theta\theta$: complex boson, complex fermion (both ±)

$\int d^2\theta d^2\bar\theta\, K(\Phi^\dagger, \Phi) + \int d^2\theta\, W(\Phi) + c.c.$ ↑ spectrum は non-chiral じゃないの...

vector superfield $e^V \to e^{\bar\Lambda} e^V e^\Lambda$

4d では $W_\alpha = D_\alpha \bar D\, e^{-V} \bar D e^V$ が chiral gauge-covariant adjoint

kin. term $= \int d^2\theta_{(\tau)}\, W_\alpha W^\alpha + c.c.$ だった.

2d では $\Sigma := D_- e^{-V} \bar D_+ e^V$ の段階で

twisted chiral adjoint で変換. $\Sigma = \sigma - \theta^+ \bar\lambda_+ - \bar\theta^- \lambda_- + \theta^+\bar\theta^- (D - iF_{01}) + \cdots$

・kin. term $= \frac{1}{g^2}\int d^2\theta d^2\bar\theta\, \mathrm{tr}\, \bar\Sigma \Sigma$

U(1) のばあいは Σ は gauge invariant

→ $\int d\theta^+ d\bar\theta^-\, t\Sigma + c.c.$ も許される.

$t = i\xi + \frac{\theta}{2\pi}$ とすると FI term, 2d theta angle

$= \xi D + \frac{\theta}{2\pi} F_{01}$ 4d の τ のように振舞う.

Sheet ______ Date ______

2d SUSY theories ② LG ① ($\mathcal{N}=(2,2)$)

まずは ゲージ場の無いモデルを考える.

$$\int d^2\theta d^2\bar\theta\, X^\dagger X + \int d^2\theta\, X^k + c.c.$$

ポテンシャルは $V=|X^{k-1}|^2$ ⇝ low energy で面白い理論?

CFT になるとすると $\left(\begin{array}{c}\mathcal{N}=2\\ \text{super Virasoro}\end{array}\right)_L \times \left(\begin{array}{c}\mathcal{N}=2\\ \text{super Virasoro}\end{array}\right)_R$ であると思われる.

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots$$

$\mathcal{N}=2$ ↕ SO(2) R-sym. = U(1)

$$T(z)G^\pm(w) = \frac{3/2}{(z-w)^2}G^\pm(w) + \frac{\partial G(w)}{z-w} + \cdots$$

$$T(z)J(w) = \frac{J(w)}{(z-w)^2} + \frac{\partial J}{z-w} + \cdots$$

$$G^+(z)G^-(w) = \frac{2c/3}{(z-w)^3} + \frac{2J}{(z-w)^2} + \frac{2T+\partial_w J}{z-w} + \cdots$$

$$J(z)G^\pm(w) = \pm\frac{G^\pm(w)}{z-w} + \cdots$$

$$J(z)J(w) = \frac{c/3}{(z-w)^2} + \cdots$$

Left-moving U(1) current J が $JJ = \frac{k}{(z-w)^2} + \cdots$

⟷ U(1) の 't Hooft アノマリー

$\mathrm{tr}\, QQ = k$ (カッパ)
left-moving fermion

X^k : left moving $R=1$, right moving $R=1$ ⇝ X : left, right moving $R = 1/k$

	L-R	r-R
ψ_+	$1/k-1$	$1/k$
ψ_-	$1/k$	$1/k-1$

⇝ $\text{アノマリー}_k = \left(\frac{1}{k}-1\right)^2 - \frac{1}{k} = \frac{k-2}{k}$.

⇝ $c = 3\cdot\frac{k-2}{k}$. ← Lerche Vafa Warner NPB 324 (1989) 427 にこの計算は無い気がするが…

$\mathcal{N}=2$ super Virasoro の表現論:

- ユニタリで $c<3$ なら $\exists k$: 整数 >2, $c=3\cdot\frac{k-2}{k}$.

k=3: ボゾン1コ 半径$\sqrt{3}$

- 左右はりあわせて modular invariant SCFT をつくると $c<3$ では

$c=3\cdot\frac{k-2}{k}$ の diagonal invariant A_{k-1} 型.

$c=3\cdot\frac{2n-2}{2n}$ の invariant D_{n+1} 型 ← $k=2n$

$c=3\cdot\frac{5}{6}, 3\cdot\frac{8}{9}, 3\cdot\frac{14}{15}$ の 〃 $E_{6,7,8}$ 型 ← $k=12,18,30$ で尽きる.

(k は dual coxeter.)

2d $\mathcal{N}=(2,2)$ SUSY th ③ LG ② 新

$c = 3\cdot\frac{k-2}{k}$ のときのユニタリ表現は l, m でラベルされる. Gepner NPB 296 (88) 757

$l = 1, \dots, k-1$; $m = -l, 2-l, \dots, l$

primary : $\begin{cases} h = \frac{l^2-1}{4k} - \frac{m^2}{4k} + \frac{1}{8} \\ Q = \frac{m}{k} - \frac{1}{2} \end{cases}$ in the R-sector. ~~(± で spectral flow した)~~

G_0^-, G_0^+ : $Q+1$, Q ; $\frac{m}{k} - \frac{1}{2}$

chiral primary : + で $m = l$ をとるから $\rightarrow h = \frac{1}{8}\frac{k-2}{k} = \frac{c}{24}$

modular invariant in the RR-sector (with $(-1)^{F_L+F_R}$)

$$Z_{\text{full part}} = \frac{1}{2}\sum N_{l,\bar{l}}\, I^{l}_{m}(\tau, z)\, \bar{I}^{\bar{l}}_{m}(\bar{\tau}, \bar{z}) \quad \leftarrow$$

ここの m の summation は?

ここで $N_{l,\bar{l}}$ は $SU(2)_{k-2}$ の modular invariant であればよい.

注意 : $I^{k-l}_{m} = -I^{l}_{m+k}$ で $m \in \mathbb{Z}_{2k}$ に拡げた.

A_{k-1} 型 : $\sum_{l=1}^{k-1} |\chi_l|^2$

D_{n+1} 型 : $k = 2n$, $|\chi_1 + \chi_{2n-1}|^2 + |\chi_3 + \chi_{2n-3}|^2 + \cdots + |\chi_{n-2} + \chi_{n+2}|^2 + 2|\chi_n|^2$ if n odd,

$|\chi_1|^2 + |\chi_3|^2 + \cdots + |\chi_{2n-1}|^2 + |\chi_n|^2 + \chi_2\bar{\chi}_{2n-2} + \chi_4\bar{\chi}_{2n-4} + \cdots \bar{\chi}_2\chi_{2n-2}$ if n even.

E_6 型 : $k = 12$ $\quad |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2$

E_7 型 : $k = 18$ $\quad |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + |\chi_9|^2 + (\chi_3 + \chi_{15})\bar{\chi}_9 + cc.$

E_8 型 : $k = 30$ $\quad |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2$.

これが LG で再現できる, というのをどれくらいできそうか?

full の part. func は むつかしいが. antihol. 側を完全に Witten index にしてしまうと robust.

$I^{l}_{m}(\bar{\tau}, 0)$ は $l = m$ で $+1$, $l = -m$ で -1, その他で 0 なので

$Z_{ell}(\tau, z) = \sum N_{l\bar{l}}\, I^{l}_{l}(\tau, z)$ となる。これは LG で再現可のはず.

$$Z_{ell}(\tau, z) := \mathrm{Tr}_{\text{RR sect.}} (-1)^{F_R} e^{2\pi i z J_L} e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{2\pi i \bar{\tau} (\bar{L}_0 - \frac{c}{24})}$$

I^{l}_{m} の無限積表示は Y. Matsuo PTP 77 (87) 793.

型の名の由来 : k が丁度 dual coxeter. $\chi_l \bar{\chi}_{\bar{l}}$ とでてくる l が丁度 exponents.

2d $N=(2,2)$ SUSY +4 ⊕ LG ③ 続

UVで R-charge r の chiral mult Φ
(left-mov.)

	left	right
boson	r	r
fermion	$r-1$	r

(right は 相殺.)

Witten 9304026

すると $Z_{ell}(z,\tau) = -\dfrac{\theta_1(y^{r-1},q)}{\theta_1(y^r,q)}$

ただし $\theta_1(y,q) := -i q^{1/8} y^{1/2} \prod (1-q^k)(1-yq^k)(1-y^{-1}q^{k-1})$

$W = X^k$ ならば X の R-charge は $1/k$.

$$Z_{ell,UV}(z,\tau) = +\frac{\theta_1(y^{1-1/k},q)}{\theta_1(y^{1/k},q)}$$

} 一致するか?

IRでは
$$Z_{ell,IR}(z,\tau) = \sum_{l=1}^{k-1} I^l_l(y,q)$$

di Francesco-Yankielowicz 9305037
T.Kawai-Yamada-Yang 9306096
T.Kawai 0909.1879

- 楕円関数 (Jacobi form) としての変換性も同じ
- c が等しい
- $q\to 0$ 極限が一致すれば一致

$I^l_{l'}(y,q) \to \begin{cases} y^{l/k-1/2} & \text{if } l=l' \\ 0 & \text{otherwise} \end{cases}$

$\theta_1(y,q) \to y^{1/2}-y^{-1/2}$ なので, $t := y^{1/k}$ として

$$\frac{t^{\frac{k-1}{2}} - t^{\frac{1-k}{2}}}{t^{\frac12}-t^{-\frac12}} = t^{\frac{k}{2}-1} + t^{\frac{k}{2}-2} + \cdots + t^{1-\frac{k}{2}}$$ という具合.

$D_{n+1}, E_{6,7,8}$ 特異点は

		a	b	c	h	r-charge は
		整数				
D_{n+1}	$X^n+XY^2+Z^2$	2,	$n-1$,	n	$2n$	a/h
E_6	$X^4+Y^3+Z^2$	3	4	6	12	b/h
E_7	X^3Y+Y^3 ~~$+XY^3$~~ $+Z^2$	4	6	9	18	
E_8	$X^5+Y^3+Z^2$	6	10	15	30	c/h
A_{n-1}	X^n+YZ	1	b	$n-b$	n	

$a+b+c=h+1$

$t := y^{1/h}$ として

$$\frac{\theta_1(t^{h-a})}{\theta_1(t^a)}\frac{\theta_1(t^{h-b})}{\theta_1(t^b)}\frac{\theta_1(t^{h-c})}{\theta_1(t^c)} \overset{?}{=} \sum N_{l\bar{l}}\, I^l_{\bar{l}}(t^h)$$

$q\to 0$ 極限を調べればよく,

$$\frac{\sin(h-a)u}{\sin au}\cdot\frac{\sin(h-b)u}{\sin bu}\cdot\frac{\sin(h-c)u}{\sin cu} = \sum_l \cos(h-2l)u$$

という式に帰着. (斎藤 数学38 (1986) 夏号・秋号)

$\mathcal{N}=(2,2)$

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2d SUSY th. ⑤ CY/LG ①

弦理論では worldsheet 理論は 2d. NSR 弦は $\mathcal{N}=(1,1)$ が必要.

$\mathbb{R}^{3,1}\times X$ で spacetime SUSY を出すには w.s で (0,2) か (2,2) が必要.

X : ケーラー多様体. $\Phi^{i=1,2,\dots,d}$: local coordinate.

$\int K(\bar\Phi^i,\Phi^i)\, d^4\theta\, d^2\sigma$: $\mathcal{N}=(2,2)$ non-linear σ-model on X.

くりこみ : $\frac{d}{d\log\Lambda} g_{i\bar j} + R_{i\bar j} + \cdots$ 簡単な形の Einstein $g_{\mu\nu}=\lambda^{-1}R_{\mu\nu}$ となる.

$\uparrow$ 正

・球面 等, positively curved だと $\lambda>0$

$\rightsquigarrow$ Λ を下げると 小さくなってつぶれる.

・conformal … $R_{i\bar j}=0$ (to leading order).

Ricci flat & Kähler $\Rightarrow$ Calabi Yau.

これをどう調べるか? ① semiclassical method.

$c=3d$. (superp $=0$ なので. R-charge がある.

$\uparrow$ 複素次元. Φ の R-charge はゼロ fermion の R charge $=1$.)

補正なし. bos & fer

elliptic genus もわかる: X を動くゼロモードと オイラー形式 curvature の式 $\rightsquigarrow x_a$.

$$Z(y,q)=\int_X \prod_a \frac{x_a\,\theta_1(y^{-1}e^{x_a},q)}{\theta_1(e^{x_a},q)}$$

但し $x_{a=1,\dots,d}$ は $T_{\mathbb{C}}X$ の Chern-root.

$y\to1$: $\int_X \prod_a x_a = \chi(X)$: オイラー数.

$q\to0$: $\int_X \prod_a x_a \frac{y^{-\frac12}e^{x_a/2}-y^{\frac12}e^{-x_a/2}}{e^{x_a/2}-e^{-x_a/2}} =: \chi_y(X)$ χ_y-genus

$= y^{-\frac{d}{2}}\sum_{p,q}(-y)^p(-1)^q \dim H^{p,q}(X)$

$TM\oplus\mathbb{C}=L^{\oplus N}$

$TX\oplus N=TM$

$L^{\oplus d}$ i.e. d 次式 $X\subset\mathbb{CP}^{N-1}$, $M=\mathbb{CP}^{N-1}$

$c(TX)=\frac{c(TM)}{c(N)}=\frac{(1+H)^N}{1+dH}=1+(N-d)H+\cdots$

$d<N$: +ve, $d=N$: Ricci flat, $d>N$: −ve

quintic CY: $\mathbb{CP}^4$. $z_1\cdots z_5$ up to $\mathbb{C}^\times$ mult.

の中で $f(z_1,z_2,\dots,z_5)=0$ 5次式

Hodge diamond

```
      1
     0 0
    0 1 0
  1 101 101 1
    0 1 0
     0 0
      1
```

$\chi_y = (y^{-\frac32}+y^{\frac12})\times(-100)$

$\mathcal{N}=(2,2)$

2d SUSY th. ⑥ CY/LG ②

Martinec ; Vafa-Warner ; Greene … CY/LG 対応.

$W = X^5$ のモデルは $c = 3\cdot\frac{3}{5}$ の $\mathcal{N}=(2,2)$ 理論. 5つコピーを取って.

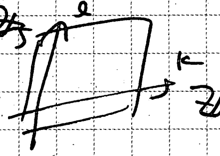
$W = X_1^5 + \dots + X_5^5$ を考えると $c = 9$. 似ている.

R-charge が $\frac{\square}{5}$ のものが混ざっている (X_i の R-charge が $\frac{1}{5}$ なので).

・geometric なものは R-charge は整数

$X_i \to e^{\frac{2\pi i}{5}} X_i$ という non-R $\mathbb{Z}_5$ sym があるので

この $\mathbb{Z}_5$ を「ゲージ化する」($\mathbb{Z}_5$ で orbifold する.) と, R-charge はみな整数



susy T^2 part. func (ell. genus) は について

足しあわせて

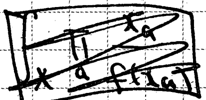
$$\frac{1}{5}\sum_{k,l=0}^{4}\left[y^{-\frac{4}{5}}\frac{\theta_1(y^{-4/5}e^{2\pi i(k+l\tau)/5}, q)}{\theta_1(y^{1/5}e^{2\pi i(k+l\tau)/5}, q)}\right]^5$$

← Berglund-Henningson 9401029

$$\xrightarrow{q\to 0} -100(y^{-1/2}+y^{1/2}).$$

(直接計算して, C がじ続/違い ことなるに 決まで 一致. 必要か?

実は この等価性は 以下のように示せる。 $X \subset \mathbb{CP}^4$ は ある line bundle L^5 の zero section.



$$= \int_X \varphi(T_{\mathbb{C}}X) = \int_X \frac{\varphi(T_{\mathbb{C}}M)}{\varphi(L)}$$

$T_{\mathbb{C}}M \oplus \mathbb{C} = L \oplus \dots \oplus L$ (5)

$$\frac{1}{\varphi^{\prime}(0)}\int_M \frac{e(L^5)}{\varphi(L^5)}\varphi(L)^5.$$

taut.

$$\left(= \int_M e(L^5)\wedge\frac{\varphi(T_{\mathbb{C}}M)}{\varphi(L^5)}\right.$$

$\varphi(L)^{5} = \varphi(T_{\mathbb{C}}M)\varphi$

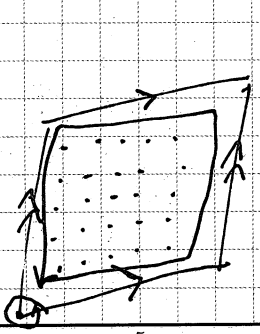
$\mathbb{CP}^4$ 上 line の class を H と書くと

$$= \frac{\theta_1^{\prime}(0)}{\theta_1(y^{-1})}\int_M \frac{\theta_1(e^{\pm 5H})}{\theta_1(e^{5H}y^{-1})}\cdot\left[\frac{H\theta_1(e^{H}y^{-1})}{\theta_1(e^{H})}\right]^5.$$

$$\int_{\mathbb{CP}^4} H^n = \begin{cases}1 & n=4\\ 0 & \text{otherwise}\end{cases}$$

← 実質 Kawai-Mohri 9402148 にある. Ma-Zhou math/0411081

so $\int_{\mathbb{CP}^4} H^5 g(H) = \oint_{z=0} \frac{dz}{2\pi i} g(z)$



$$= \oint_{z=0} \frac{\theta_1(e^{-5z})}{\theta_1(e^{-5z}y^{-1})}\cdot\left[\frac{\theta_1(e^{z}y^{-1})}{\theta_1(e^{z})}\right]^5$$

$\wedge$ $\frac{\theta_1^{\prime}(0)}{\theta_1(y^{-1})}$

$\mathcal{N}=(2,2)$

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2d SUSY th ⑦ CY/LG ③

これをもうちょっと physical に理解できないか? Witten 9301042

2d $\mathcal{N}=(2,2)$ U(1) gauge th. charge +1 の ch.supf $X_1 \cdots X_5$ ← R charge 0
-5 P ← R charge 1

$$W = P f(X_1, X_2 \cdots X_5)$$
で f は 斉次5次式.

$\int d\theta^+ d\bar\theta^- \, t\Sigma$ も足しておく. $t = i\xi + \frac{\theta}{2\pi}$

* left-moving, right-moving $\}$ R は古典的にはある. $U(1)_R$ ⊖ $U(1)_{gauge}$

~~は $\sum$ +1 +1 ⋯ +1 −5 = 0~~

+1 +1 ⋯ +1 −5 = 0.
X の left mover, P の right mover

one-loop でも大丈夫.

$U(1)_{LR} - U(1)_{LR}$ は
+1 + ⋯ + 1 ← X の left mover
−1 ← P の right mover
−1 ← λ の right mover
= 3.
⤳ $c = 9$

$\phi \to \psi_+, \psi_-$; $\lambda_+ \to \sigma$, $\lambda_- \to$

* ポテンシャルは 古典的には
$$V = |F_i|^2 + D^2 + |\sigma|^2 \left(\sum_i Q_i^2 |\phi|^2\right).$$
で $D = \sum Q_i |\phi_i|^2 - \xi$.

これより, $\sigma \sim \mu \neq 0$ では $\xi_{eff} \sim \xi + \frac{1}{2\pi} \sum Q_i \log \frac{|\sigma|}{\mu}$

とわかる. $\left(|\phi_i|^2 \sim \int \frac{d^2k}{(2\pi)^2} \left(\frac{1}{k^2 + Q^2\sigma^2} - \frac{1}{k^2 + Q^2\mu^2} \right) \sim \frac{1}{2\pi} \log \frac{|\sigma|}{|\mu|} \text{ より} \right)$.

$\xi \gg 0$ なら. $Q_i > 0$ の場 i.e. $X_1 \cdots X_5$ が vev.
$$\sum |X_i|^2 = \xi, \quad X_i \sim e^{i\theta} X_i \quad \Big\} \; \mathbb{CP}^4.$$
P の F-term: $f(X_1, \cdots X_5) = 0$ $\Big\}$ quintic CY の σ-model.

$\xi \ll 0$ なら, $Q_i < 0$ の場 i.e. P が vev.
charge 5 なので $\mathbb{Z}_5$ unbroken.
⟶ LG model $X_1^5 + \cdots X_5^5 = 0$ with $\mathbb{Z}_5$ gauged.

∴ ξ を変化させて滑らかに移行できる.

* elliptic genus は? Benini-Eager-Hori-YT 1305.0533, 1308.4896

$$\oint \underbrace{\frac{\theta_1'(1)}{\theta_1(y^{-1})}}_{U(1)\text{ vector}} \underbrace{\frac{\theta_1(e^{-5z})}{\theta_1(e^{-5z} y)}}_{P} \underbrace{\left[\frac{\theta_1(e^z y^{-1})}{\theta_1(e^z)} \right]^5}_{X_1 \cdots X_5} dz$$
← 積分路の導出が大変!
[$Q>0$ のまわりを回るか)
[$Q<0$ のまわりを回るか.

$\mathcal{N}=(0,2)$ in 2d ①

(2,2) chiral $\quad (\phi,\ \psi_+)\ \ \psi_-,\ E$ ← (0,2) Fermi mult.

(2,2) vector $\quad (\sigma,\ \bar\lambda_-)\ \ \lambda_+,\ D+iF_{01}$ ← (0,2) chiral

superspace $x^{0,1}\ \theta^+\ \bar\theta^+$

(0,2) chiral Φ : $\bar D_+\Phi=0 \Rightarrow \Phi=\phi+\theta^+\psi_+ + \cdots$

fermi Ψ_- : $\bar D_+\Psi_- = E(\Phi) \Rightarrow \Psi_- = \psi_- + \theta^+ E(\Phi)+\cdots$

vectorから作った $\Upsilon_- = \lambda^+ + \theta^+(D-iF_{01})+\cdots$

kin $\int d^2y\, d\theta^+ d\bar\theta^+\ \bar\Phi D \Phi + \int d^2x\, d\theta^+ d\bar\theta^+ (\bar\Psi_-\Psi_- + \bar\Upsilon_-\Upsilon_-)$

interaction $\int d^2x\, d\theta^+\ \Psi_-^i J_i(\Phi)$ + c.c.

これのSUSY変換は $\bar D_+(\Psi_-^i J_i(\Phi)) = \underbrace{E^i(\Phi)J_i(\Phi)}$ required to be 0.

$\int d^2x\, d\theta^+\, t\,\Upsilon_-$ = FI term および θ角.

Gadde-Gukov-Putrov の triality (とそれに関連する僕の若干) を紹介

1310.0818, 1404.5314

2d $\mathcal{N}=(0,2)$ で 非可換ゲージ理論で何か考えてみよう.

$U(n_3)$ vector mult.

N_1 chiral mult. in fund. P

N_2 in anti-fund Φ

N_3 fermi in fund. Ψ

はじめはあけておく [2 fermi in det 表現 $\Omega_{1,2}$

potential: $D^2+|E|^2+|J|^2$.

$\Psi_- \leftrightarrow \bar\Psi_-$

$E \quad \bar E = J$

$J \quad \bar J = E$

等価.

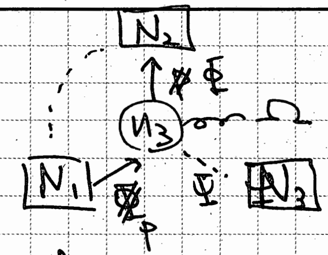
$SU(n_3)^2$ のアノマリ : $\frac{N_1}{2}+\frac{N_2}{2}-\frac{N_3}{2} - n_3 = 0 \Rightarrow n_1 = N - N_1$

where $N=\frac{N_1+N_2+N_3}{2}$.

$U(1)^2$ のアノマリ : $(N_1+N_2-N_3)n_3$

$-2n_3^2 = 0$

$\mathcal{N}=(0,2)$ in 2d ②

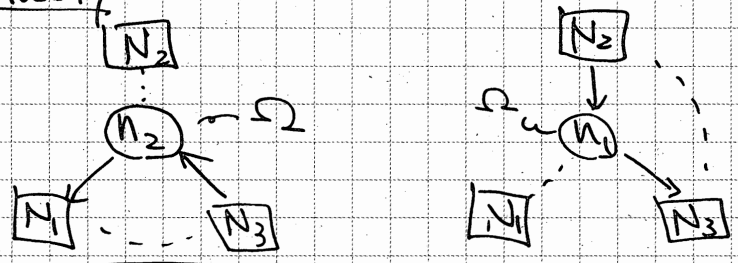


更に $N_1 N_2$ Fermi M を入れて

$\int d\theta^+ M\,\bar{\Phi} P$ を入れたモデルを

彼らはいろんな動機があって考えた。

triality



を主張。

$\left(n_i = \frac{N_1+N_2+N_3}{2} - N_i \right)$

チェック (※ flavor anom のための frame で)

$SU(N_1)^2$ アノマリー : $\frac{n_3}{2} - \frac{N_2}{2} = -\frac{n_1}{2}$

$SU(N_2)^2$: $\frac{n_3}{2} - \frac{N_1}{2} = -\frac{n_2}{2}$

$SU(N_3)^2$: $-\frac{n_3}{2}$

$U(N_i)$ の自然な subgroup $U(1)_i$ は $U(1)$ gauge とアノマリーする。
→ Ω の charge で打ち消せる。

	Φ	Ψ	P	M	Ω_1	Ω_2
$U(1)_1$	0	0	1	−1	$-N_1$	
$U(1)_2$	−1	0	0	1	$-N_2$	
$U(1)_3$	0	1	0	0	0	N_3

と取ると
$$\begin{cases} U(1)_i - U(1)_i \text{ アノマリー} & -\frac{N}{2} N_i \\ U(1)_i - U(1)_j \text{ アノマリー } (i\neq j) & 0 \end{cases}$$
となる

c_L & c_R

left moving R-対称性は $R_P, R_\Phi, R_\Psi, R_M, R_{\Omega_1}, R_{\Omega_2}$

constraint ① $R_P + R_\Phi + R_M = 1$

② $(R_P - 1)\cdot N_1 - (R_\Phi - 1)\cdot N_2 - R_\Psi N_3 - R_{\Omega}\cdot n_3 = 0$

6つ変数 → 4つ。
2つ式
→ 1つの自由度は ゲージ charge.
もう3つ : $U(1)_{1,2,3}$.

1302.4451
1211.4030
Benini-Bobev

IR で SCFT になるとして 正しい R-charge は？

$U(1)_R - U(1)_R$ アノマリー を極大化するもの (c-extremization)

proof $T, G^{\pm}, J$ に加え、独立な $U(1)$ カレント J_F で $J_F \cdot G \sim 0$ とする
$J_F \cdot J \sim 0$ が従う。☑

$\mathcal{N}=(0,2)$ in 2d ③

計算すると, $R_P = \frac{n_2}{N}$, $R_\Psi = \frac{n_1}{N}$, $R_M = \frac{n_3}{N}$, $R_\Phi = R_{\Omega_{1,2}} = 0$.

$$\Rightarrow \frac{c_L}{3} = n_3\left((R_P-1)^2 N_1 + (R_\Phi-1)^2 N_2 - R_\Phi^2 N_3\right) - R_M N_1 N_2 - 2R_\Omega^2 - n_3^2$$

$$= \frac{n_1 n_2 n_3}{N}$$

} cyclic!

$$c_L - c_R = n_3(N_1 + N_2 - N_3) - N_1 N_2 - 2 - n_3^2$$

$$= \frac{1}{4}\left(N_1^2 + N_2^2 + N_3^2 - 2N_1N_2 - 2N_2N_3 - 2N_3N_1\right) - 2$$

Index elliptic genus

fermi mult : $\frac{\theta_1(e^z,q)}{\eta(q)}$ chiral $\frac{\eta(q)}{\theta_1(e^z,q)}$ vector $(\eta(q)^2)^{\mathrm{rank}\,G} \prod_\alpha \frac{\theta_1(e^{\alpha(z)})}{\eta(q)}$

$$\rightsquigarrow Z = \frac{1}{n_3!}(\eta(q)^2)^{n_3} \oint \frac{dz_i}{2\pi i} \underbrace{\prod_{i\neq j} \frac{\theta_1(e^{z_i-z_j})}{\eta(q)}}_{\text{vec}} \underbrace{\prod \frac{\eta(q)}{\theta_1(e^{a_i-z_j})}}_{P}$$

$$\underbrace{\prod \frac{\eta(q)}{\theta_1(e^{z_i-b_j})}}_{\Phi} \underbrace{\prod \frac{\theta_1(e^{z_i-c_j})}{\eta(q)}}_{\Psi} \underbrace{\prod \frac{\theta_1(e^{a_i-b_j})}{\eta(q)}}_{M}$$

$$\underbrace{\left[\frac{\theta_1(e^{z_1+\cdots+z_{n_3}})}{\eta(q)}\right]^2}_{\Omega}$$

P ... positive poles

Φ -- negative poles

結局, $\{z_i\} \subset \{a_i\}$ を取る事になる. ($n_3!$ を打ち消して)

⇝ θ の積の形で triality のもとで一致 (異なる θ の等式をつかわなくても, 項の並べかえでよい)

もっと凄い話

IR で SCFT だとする. (right moves SUSY) $SU(N_i)^2$ のアノマリ ... $-\frac{n_i}{2}$

⇒ $SU(N_i)$ level n_i が left にある.

$U(1)_{1,2,3}$ もある.

Sugawara $c = \frac{n_i(N_i^2-1)}{N_i+n_i}$

何と

$$3 + \sum_i \frac{n_i(N_i^2-1)}{N_i+n_i} = c_L$$

→ left-mover は current alg. で書ける.

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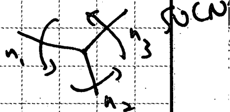
$\mathcal{N}=(0,2)$ in 2d ④

これと全うに結合できる左側の SUSY chiral CFT は? Gaiotto 1306.5661 にも登場.

$\dfrac{U(N)_N}{\prod U(n_i)_N}$ Kazama-Suzuki coset (super) が丁度よい.



super G_k … bosonic $G_{k-h^\vee}$ + free fermion in $\mathfrak{g}$

$U(N)_N$ … ~~N^2 free fermions~~ … $c=\dfrac{N^2-1}{2}+\left(1+\dfrac{1}{2}\right)$

$U(n_i)_N$ … bosonic $U(n_i)_{N-n_i = N_i}$ + n_i^2 free ferm.

level-rank duality を使って right-moving current alg と比べる.

$$c=\left(\frac{N^2}{2}+1\right)-\sum_i\left(\frac{n_i(n_i^2-1)}{N}+1+\frac{n_i^2}{2}\right)=\frac{3n_1n_2n_3}{N}\ !$$

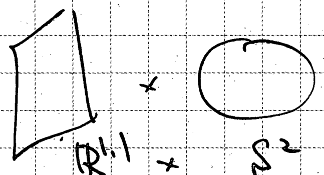
⇒ candidate IR 2d $\mathcal{N}=(0,2)$ CFT の完全なスペクトル!

elliptic genus も評価可能 ⇒ 展開するとゲージ理論と合う.

LG X^k ↔ minimal model の計算のスゴい版.

この triality (すごい) は 4d Seiberg duality から導出できる (未発表 後述の理由により)

4d $\mathcal{N}=1$ 理論で $U(1)_R$ があるものを考える.



$\mathbb{R}^{1,1}\times S^2$

そのままでは SUSY は無いが $U(1)_R$ flux を S^2 に入れると spinor bundle のねじれが半分とれて SUSY を半分保つ. ⇒ 2d $\mathcal{N}=(0,2)$

S^2 上の charge q 4d Weyl fermion

⇝ $|q|$ right-moving ($q>0$) / left-moving ($q<0$) 2d fermions (massless) ← 4d supercharge

これより 4d chiral multiplet, R-charge q

⇝ $|q-1|$ fermi ($q>1$)

$|q-1|$ chiral ($q<1$)

$\mathcal{N}=(0,2)$ in 2d ⑤

4d vector $\longrightarrow$ 2d $\mathcal{N}=(0,2)$ vector 1つ.
+ KK tower.

さて、4d $SU(N_c)$ で N_f flavor の理論を考える。 Q, $\tilde{Q}$

non-anomalous $U(1)_R$, superconformal は $1-\frac{N_c}{N_f}$, $1-\frac{N_c}{N_f}$.

$U(N_c)$ ゲージ理論にしたい.

$U(1)-U(1)-U(1)_R$ アノマリがある: $-N_c^2 - N_c^2$.

$\Rightarrow$ det 表現 (charge N_c) と $\det^{-1}$ の $\Omega, \tilde{\Omega}$ を入れる.
R-charge 2.

これを S^2 上におく: $U(1)_R$ のもとでの場の charge は整数でないと困る.

$\Rightarrow$ superconformal R と flavor sym をまぜよう.

$U(n_3)$: Q $r_1 \cdots r_{N_1}$, $\tilde{Q}$ $\tilde{r}_1 \cdots \tilde{r}_{N_1}$ $\qquad N_c = n_3$, $N_f = N_1$ として

$\Sigma(r_i + \tilde{r}_i) = 2N_f - 2N_c$ でなければならない.

$U(n_3)$: Q $[0 \cdots 0]$, $\tilde{Q}$ $[1-N_2 \,|\, 1+N_3, 1, 1, \cdots 1]$. $\quad M^1{}_1 Q_1 \tilde{Q}_1$

$N_1 - N_2 + N_3 = 2N_1 - 2n_3$

$\Rightarrow n_3 = \frac{N_1+N_2-N_3}{2}$.

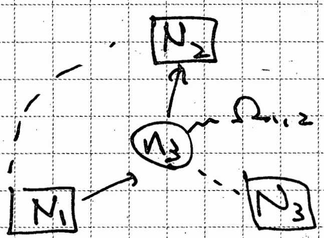
$+ \Omega_2, \tilde{\Omega}_2$

Seiberg dual

$U(N_1-n_3)$ ($= n_2$): q $[0 \cdots 0]$, $\tilde{q}$ $[1+N_2 \,|\, 1-N_3, 1, 1, \cdots 1]$ $\quad M_{ij} q^i \tilde{q}^j$, $j \neq 1$

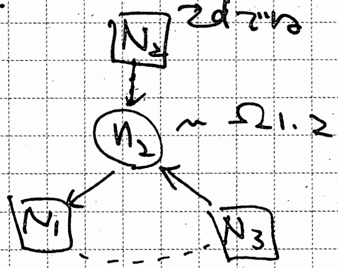
$+\Omega, \tilde{\Omega}$

2d では



triality!

2d では



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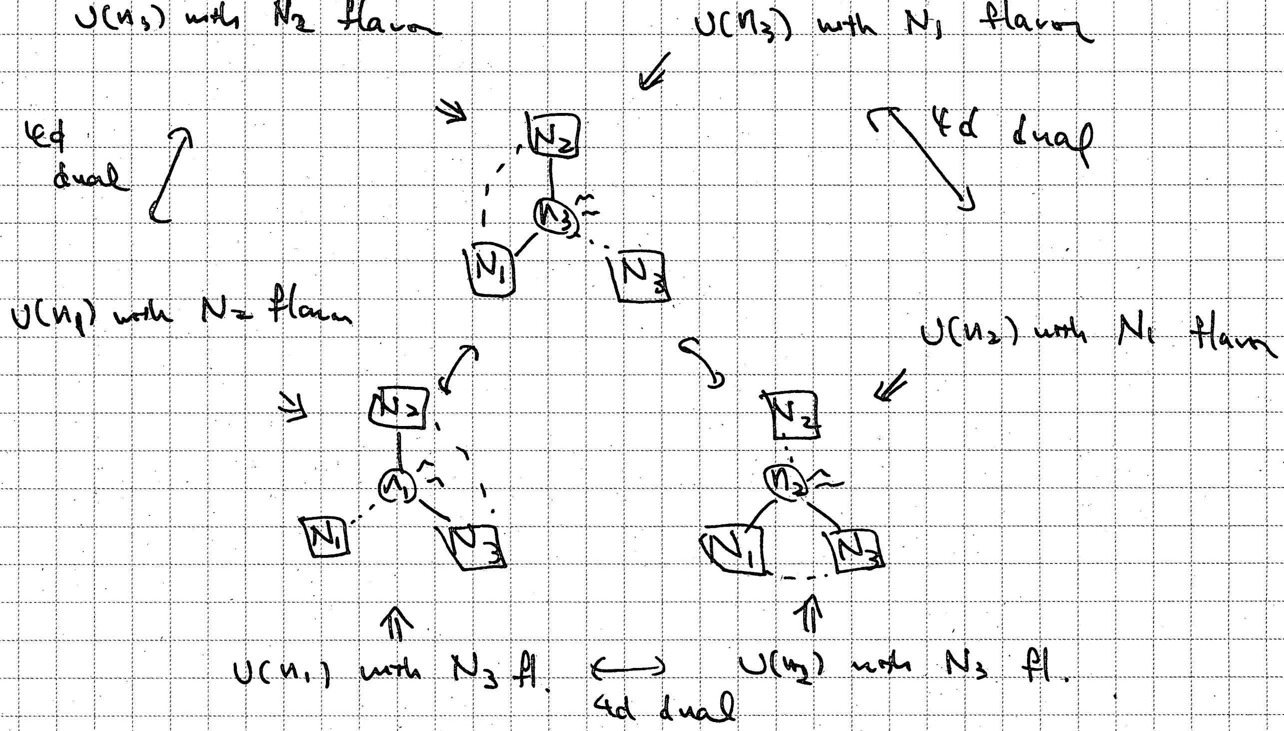
$\mathcal{N}=(0,2)$ in 2d, ⑥

何故 duality から triality が出るのか？



何故論文にしないのか？

以上の事柄は、4d で U(n) ならずとも、SU(n) ならでやって良いように思える。しかし、2d で GGP triality で U を SU に替えると成立しない。

- index が合わない。
- 'branch' が合わない（2d では branch は 厳密には ないが…）

gauge-inv op は SU(N_i) のバージョンは

$\det P$, $\det \Phi$, 及び $P\Phi$ ← superp. で殺されている。

同時には 1 ヶ ゼロにならない。 chem. pot の

⤳ index は SU(N_1), SU(N_2) の func として pole がある。

- U(n) の場合は $\det P$, $\det \Phi$ も殺されていて non-cpt branch は無い。

おそらく 2d Kähler pot の補正により、cigar 上になって non-holomorphic modular になるのだと思われる。

cf. Harvey-Lee-Murthy 1406.6342