a few refs added throughout
2015.2.4

2015/1/28

Supersymmetric Field Theories in 4.5.6 dimensions

@ Kyoto Feb ~~9~~, 10, 12, 13, 2015

Strongly coupled field theories are hard to analyze.

$\swarrow$ Lattice $\searrow$ Assume more symmetry.

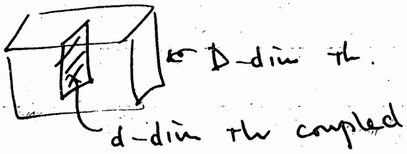
+ conformal : 2d especially powerful.

$\left\{ \begin{array}{l} + \text{super} \\ + \text{superconformal}. \end{array} \right.$

- big developments '94 ~ '97, '02 ~ present
- learned that interrelation of SUSY FT in various dimensions

D-dim th on M_d $\rightsquigarrow$ $(D-d)$ dim th.

or



is important/useful

$\rightsquigarrow$ general overview wouldn't be bad.

Oct 2014 @ Osaka : covered 2 & 4
Feb 2015 @ Kyoto : 4. 5. 6

Traditionally, in 4d and below,
UV gauge th + fermion + boson
$\downarrow$
IR strongly coupled th.

in 5d & 6d
UV strongly coupled
$\downarrow$
IR gauge + f + b

Is there really such a UV f.p. ?
$\rightsquigarrow$ string/M th often gives it.
Once realized, $\exists$ similar UV f.p. in 4d, too.

} I'd like to study them as purely field theoretically as possible, with as little use of string/M as possible.

PLAN:

1st day	2nd day	3rd day
	③ 4d $\mathcal{N}=2,4$ and S-duality.	⑥ 6d $\mathcal{N}=(2.0)$ and 4d $\mathcal{N}=2$.
① SUSY in various dim.	④ 4d $\mathcal{N}=2.4$ and S-duality.	⑦ 5d $\mathcal{N}=1$.
② SUSY in various dim.	⑤ 6d $\mathcal{N}=(2.0)$ and 4d $\mathcal{N}=2$.	⑧ 6d $\mathcal{N}=1$.

①② overlap with Osaka and the same
③-⑥ overlap with Rikkyo but compliment.

いろいろな次元の超対称性. ①

$\{Q_\alpha, \bar{Q}_\beta\} \sim P_\mu \gamma^\mu_{\alpha\beta}$ というのが SUSY. まず スピノルの復習.
cf. Polchinski vol 2
App B

$SO(d-1,1)$ のスピノルを知りたい. $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$.

まず $SO(d)$ を考える.

- $SO(2n)$ $\Gamma^1 \dots \Gamma^{2n}$

  $\Gamma^{2n+1} := \Gamma^1\Gamma^2 \cdots \Gamma^{2n}$ で 自動的に $SO(2n+1)$ のΓ行列.
- $SO(2)$ $\Gamma^1 = \sigma^x$ $\Gamma^2 = \sigma^y$ $\Gamma^3 = \sigma^z$
- $SO(2\tilde{n})$ $\tilde{\Gamma}^1 \dots \tilde{\Gamma}^{2\tilde{n}}$
- $SO(2n+2\tilde{n})$ は $\begin{cases} \hat{\Gamma}^i = \Gamma^i \otimes \mathbb{1} \\ \hat{\Gamma}^{i+2n} = \Gamma^{2n} \otimes \tilde{\Gamma}^i \end{cases}$ と定めればよい.

$\Rightarrow$ $SO(2n)$ の Γ行列は $2^n \times 2^n$ 行列. Dirac sp : 2^n 次元

ローレンツ群の生成子 $M^{\mu\nu} = \frac{1}{2}[\Gamma^\mu, \Gamma^\nu]$ は Γ^{2n} と交換
回転

$\rightsquigarrow$ $\Gamma^{2n} = +1$, -1 に制限できる. Weyl sp : 2^{n-1} 次元.

- $SO(1.1)$ の Γ 行列 : $\underbrace{\Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{実}}$ $\left(\Gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$

とすると. 同様に ±Weyl sp. も実.

$SO(d-2)$ の Γ と $SO(1.1)$ の Γ から $SO(d-1,1)$ の Γ.

$\Rightarrow$ Dirac, Weyl sp. の実性は $SO(d-2)$ $SO(d-1,1)$ で同じ.

$SO(2) \simeq U(1)$ vector $\overset{\text{charge}}{\pm 1}$ spinor $\overset{\text{charge}}{\pm 1/2}$: 複素.

$SO(3) \simeq SU(2)$ $\mathbf{2}$: 擬実. $\quad$ $\mathbf{2} = \bar{\mathbf{2}}$ だが $*^2 = -1$

$SO(4) \simeq SU(2) \times SU(2)$ $(\mathbf{2}, \mathbf{2}')$ 擬実

$SO(5) \simeq USp(4)$ $\mathbf{4}$ 擬実

$SO(6) \simeq SU(4)$ $\mathbf{4}, \bar{\mathbf{4}}$ ⟷ 複素.

$SO(7)$ $\mathbf{8}$ 正実 ← SO(8)に制限.

$\boxed{SO(8)}$ $\mathbf{8}_s, \mathbf{8}_c$ $\boxed{\text{正実}}$ ←

なので…

↑ $SO(d-8)$ のスピノルと $SO(d)$ ($\times SO(8)$) のスピノルの性質は同じ. (Bott Periodicity.)

いろいろな次元の超対称性 ②

SO(1,1)	±Weyl,	正実,	複素 1次元,	$\overset{実とする}{\Longrightarrow}$	実 1次元	× $(\mathcal{N}_+, \mathcal{N}_-)$
SO(2,1)	Dirac	正実	2	$\Longrightarrow$	2次元	× $\mathcal{N}$
SO(3,1)	±Weyl	複素	2	$\Rightarrow$	実 4次元	× $\mathcal{N}$
SO(4,1)	Dirac	ギ実	4		8	× $\mathcal{N}$
SO(5,1)	±Weyl	ギ実	4		8	$(\mathcal{N}_+, \mathcal{N}_-)$
SO(6,1)	Dirac	ギ実	8		16	$\mathcal{N}$
SO(7,1)	±Weyl	複素	8		16	$\mathcal{N}$
SO(8,1)	Dirac	正実	16		16	$\mathcal{N}$
SO(9,1)	±Weyl	正実	16		16	$(\mathcal{N}_+, \mathcal{N}_-)$
SO(10,1)	Dirac	正実	32		32	$\mathcal{N}$
SO(11,1)	±Weyl	複素	32		64	$\mathcal{N}$
					⋮	

R対称性 … 正実なら $SO(\mathcal{N})$, 複素なら $U(\mathcal{N})$, ギ実なら $Sp(\mathcal{N})$
(但し $Sp(1)=SU(2)$)

massless 表現 $(d \geq 4)$

$p^\mu = (E, E, 0, 0 \cdots 0)$ とする. $p^+ = \cancel{2}E$, $p^- = 0$. $\cdots$

superchage が実で $2n$ コ とする. Q^{+i}, Q^{-i} $i=1\cdots 2n$

それぞれエルミート

$$\{Q^{+i}, Q^{+j}\} = p^+ \delta^{ij}, \quad \{Q^{-i}, Q^{-j}\} = p^- \delta^{ij} = 0$$

↑ fermionic 生成消滅が n 組.

スピンは $\pm\frac{1}{2}$ あげさげする.

$n=4$ で $-1 \rightleftarrows -\frac{1}{2} \rightleftarrows 0 \rightleftarrows +\frac{1}{2} \rightleftarrows +1$ ⇐ 必ずベクトル場.

$n=8$ で $-2 \cdots 2$ ⇐ 必ず重力.

$n=16$ は $\underbrace{-4 \cdots 4}_{d=11まで}$ ⇐ 必ず H.S が最小.

HSのない　16 supercharges

⟶ 重力のない理論は $d=10$ まで.

ゲージ場のないは $d=6$ まで.

8 supercharges

($d=3$以下はスピンはよくわからんので もっとあってもよいかも知れない.)

Q. よく Wick rotate して解析するが. 実性はどうなるの?

A. Wick rotate した理論で保証されているのは $\phi(x) \to \overline{\phi(\Theta x)}$. 複素共役の構造は パリティ flipつき 同じ。

2015. 2.3

Minimally supersymmetric gauge theories

In which dimensions $\exists$ SUSY gauge th with just

$$\underset{\substack{\uparrow\\ d-2}}{A_\mu} \quad \& \quad \underset{\substack{\uparrow\\ \text{some power of } 2}}{\psi} \quad ?$$

on-shell d.o.f.

$\rightsquigarrow$	$d=2+1$		Dirac	2		1
	$2+2$	it's a priori untrivial if a suitable spinor exists, but	Weyl	4	$\longrightarrow$	2
	$2+4$		Weyl	8		4
	$2+8$		Majorana-Weyl	16		8
				on-shell		off-shell.

Consider then

$$L = \mathrm{tr}\left(F_{\mu\nu}F_{\mu\nu} + \bar\psi\,\sigma^M D_M \psi\right)$$

: just gauge f. + minimally coupled adj. f.

[where $\bar{\ }$, σ^M are operations appropriate in each choice of spacetime dimensions.

(On-shell) SUSY tr can only be

$$\delta A_M \propto \bar\varepsilon\,\sigma_M \psi \qquad + \text{c.c.}$$

$$\delta\psi \propto F_{\mu\nu}\bar\sigma^M\sigma^\nu\varepsilon$$

The terms prop. to ψ in δL cancel by choosing the coeff.s in δA_μ & $\delta\psi$ appropriately.

The variation of A_μ in D_M gives the term of the form

$$f_{abc}(\bar\varepsilon\,\sigma_\mu\psi^a)(\bar\psi^b\sigma^M\psi^c)$$

that needs to vanish. From the antisym. of f_{abc} and fermions, we see

$$(\sigma_\mu)^\alpha{}_{(\dot\alpha}(\sigma^M)^\beta{}_{\dot\beta)}$$

should vanish, in $d=3.4.6.10$ respectively.

Brink-Schwarz-Scherk NPB121 (1977) 77

This vanishing is a basic fact also used in the spinor helicity formalism i.e. for <u>bosonic</u> spinor ε, $\bar\varepsilon\sigma^M\varepsilon$ is null.

To see this, note $\varepsilon = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ where $z_i \in \underset{\substack{\mathbb{K}_d\\ \shortparallel}}{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}}$ for $d=3.4.6.10$

and $\sigma^0 = \begin{pmatrix} +1 & \\ & +1 \end{pmatrix}$, $\sigma^1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $\sigma^{1+i} = \begin{pmatrix} & e_i \\ \bar{e_i} & \end{pmatrix}$ where $\{e_i\}$ is an $\mathbb{R}$-basis of $\mathbb{K}_d$

They span 2x2 Hermitean matrices with entries in $\mathbb{K}_d$,

and $\det(v_\mu\sigma^M) = v_0{}^2 - v_1{}^2 - v_3{}^2 - \cdots - v_{d-1}{}^2$.

Now $\bar\varepsilon\cdot\varepsilon$ as a matrix is $\begin{pmatrix} z_1\bar z_1 & z_2\bar z_1 \\ z_1\bar z_2 & z_2\bar z_2 \end{pmatrix}$ and clearly has zero determinant

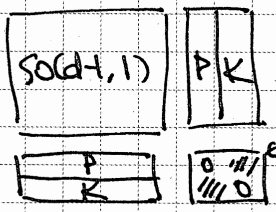
Kugo-Townsend NPB221 (1983) 357

cf. Evans, NPB 298 (1988) 92

いろいろな次元の超対称性．③

超共形代数はどうか？

SO(d-1,1) から出発． ⟶ SO(d,2) が共形代数．



← D : dilatation．

さらに SUSY Q_α があるとする．S^α もある．

(Q_α, S^α) が SO(d,2) のスピノルとして変換．

$\{Q_\alpha, S^\beta\} = \cdots + R\text{-symmetry} + \cdots$ ．

$\underbrace{SO(d,2) \oplus R\text{-sym}}_{\text{even part}} \oplus \underbrace{(Q,S)}_{\text{odd part}}$ が超共形代数．

分類 (Nahm, NPB~~137~~ 135 (1978) 149)

d=2のときは $SO(2,2) \simeq SO(1,2) \oplus SO(1,2)$ なので，

$d \neq 2$ のときの分類をすることにする．Q,S に作用しない R-sym はのぞく．

すると SO(d,2) は単純．(非自明なイデアルを含まない)．

よって SCFA 自体 (Q,S に作用しない R-sym をのぞけば) 単純．

(∵ Q が入ると S も全部入る．{Q,Q} から P が．よって SO(d,2) も全部．R も全部入る．)

単純$_{\wedge}$ super Lie alg/$\mathbb{C}$ の分類 (で boson 部分が reductive なもの)
fin. dim.
(Kac, CMP 53 (77) 31)

$SU(m|n)$: $SU(m) \oplus SU(n) \oplus U(1)$ $\oplus$ $N \otimes \overline{M} \oplus M \otimes \overline{N}$

$OSp(m|2n)$: $O(m) \oplus Sp(n)$ $\oplus$ $M \otimes 2N$

$D(2,1,\alpha)$: $SU(2) \oplus SU(2) \oplus SU(2)$ $\oplus$ $2 \otimes 2 \otimes 2$, α 任意，結合定数

$F(4)$: $SU(2) \oplus O(7)$ $\oplus$ $2 \otimes 8$

$G(3)$: $SU(2) \oplus \mathfrak{g}_2$ $\oplus$ $2 \otimes 7$ ↑ 不変内積あり

- ↓

$P(n-1)$: $SU(n)$ $\oplus$ adj なし

$Q(n-1)$: $SU(n)$ $\oplus$ $\mathrm{sym}^2 \oplus \Lambda^2$

even part に SO(d,2) を含み，odd part がスピノル，を探す．上には O(m) の vector が主．

d は無限にはない！

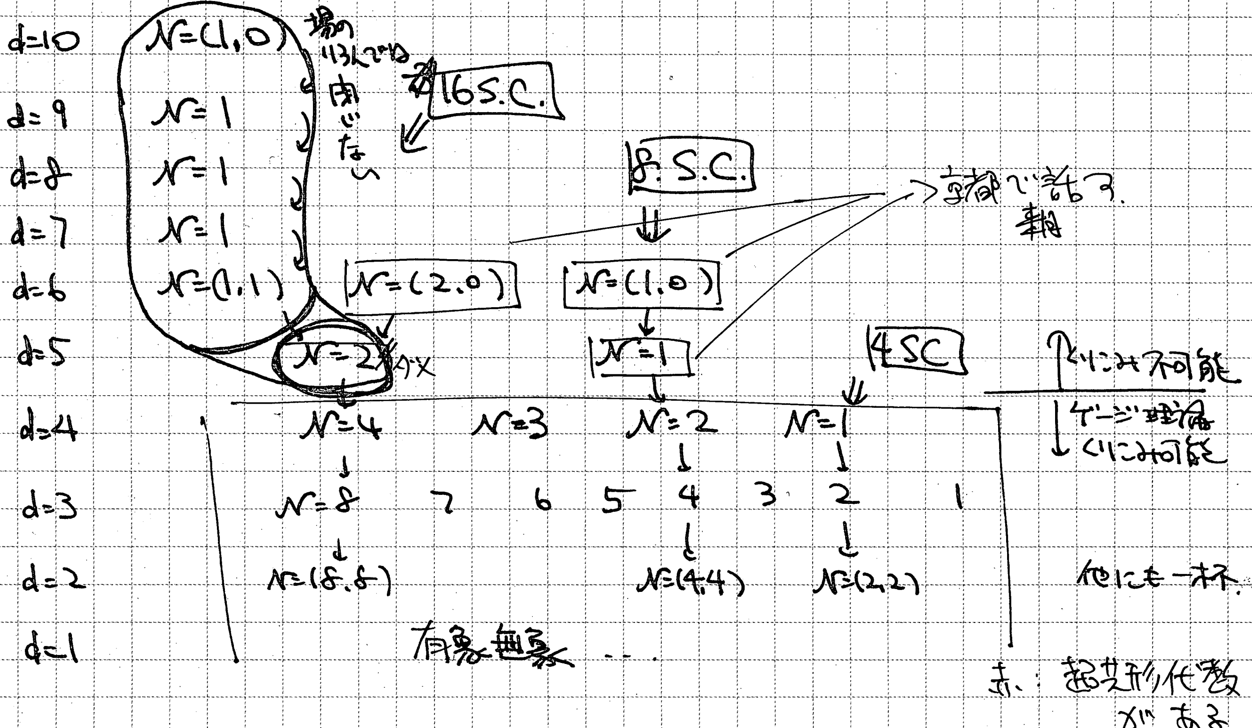
いろいろな次元の超対称性 ④

- $d=2$ はヘンテコなのが一杯あるが、どうせさらに V.rasoro に拡大するので省略.
- $d=3$: $so(3,2) \oplus o(N) \oplus 4 \otimes N \leftarrow osp(N|4)$
- $d=4$: $so(4,2) \oplus u(N) \oplus 4 \otimes N \oplus \bar{4} \otimes \bar{N} \leftarrow su(4|N)$
- $d=5$: $so(5,2) \oplus su(2) \oplus 8 \otimes 2 \leftarrow F(4)$
- $d=6$: $so(6,2) \oplus sp(n) \oplus 8 \otimes 2n \leftarrow osp(8|2n)$

可能な超対称理論（ミンコフスキ d次元 もしくはそれをユークリッド化したもの.）

$d=12$ … HS のみ. (64 supercharges)

$d=11$ … 重力のみ. (32 supercharges)



上のマル囲いをちょっとだけ.

$d=10$ vector mult: A_μ ψ_α

10 M-W. 16

8 8 on-shell

$d=9$	ϕ_9, $A_{0\cdots 8}$. ψ_8	$SO(1)$	$\simeq$	$O(1)$		$\mathcal{N}=1$
8	$\phi_{9.8}$ $A_{0\cdots 7}$	$SO(2)$	$\simeq$	$U(1)$		$\mathcal{N}=1$
7	$\phi_{9.8.7}$	$SO(3)$	$\simeq$	$Sp(1)$		$\mathcal{N}=1$
6	$\phi_{9.8.7.6}$	$SO(4)$	$\simeq$	$Sp(1)\times Sp(1)$		$\mathcal{N}=(1,1)$
5	$\phi_{9.8.7.6.5}$	$SO(5)$	$\simeq$	$Sp(2)$		$\mathcal{N}=2$
4		$SO(6)$	$\simeq$	$SU(4)$		$\mathcal{N}=4$
3	$\phi_{9.8.7.6.5.4.3}$	$SO(7)$	??		$SO(8)$	$\mathcal{N}=8$
2		???				

高い次元の超対称性 ⑤

これらの理論は ゲージ群 G を指定すると きまってしまう

代数 $\mathfrak{g}$? $+\frac{\mathrm{tr}R^4\,\mathrm{tr}R^2}{552960}+\frac{(\mathrm{tr}R^2)^3}{1327104}$

d=10. アノマリは

$$\hat{I}_{12}=\frac{1}{1440}\underline{\left(-\mathrm{Tr}F^6+\frac{1}{48}\mathrm{Tr}F^4\,\mathrm{Tr}F^2-\frac{(\mathrm{Tr}F^2)^3}{14400}\right)}$$ あ

$$+(n-496)\left\{\frac{\mathrm{tr}R^6}{725760}\right\}+\frac{Y_4X_8}{\cancel{384}\,768}$$

where $Y_4=\mathrm{tr}R^2-\mathrm{Tr}F^2/30$

$$X_8=\mathrm{tr}R^4+\frac{(\mathrm{tr}R^2)^2}{4}-\frac{1}{30}(\mathrm{Tr}F^2)(\mathrm{tr}R^2)+\frac{\mathrm{Tr}F^4}{3}-\frac{(\mathrm{Tr}F^2)^2}{900}$$

これは $\mathfrak{g}=SO(32)$, $E_8\times E_8$ のとき あ、いが消え   "Tr" は adj の tr.

$dH=Y_4$

$\delta S=\int B\wedge X_8$ とすると 相殺可. ⇒ ヘテロ ($E_8\times E_8$, $SO(32)$) Type I 超弦に対応.

※ 昔から $\mathfrak{g}=E_8\times U(1)^{248}$, $U(1)^{496}$ でも可なことが知られていた.

これに対応する quantum gr. はあるのか?

Adams-de Wolfe-Taylor 1006.1352:

Bergshoeff-de Roo-de Wit-van Nieuwenhuizen NPB195(82)97

によると, $B_{\mu\nu}-F_{\mu\nu}-F_{\mu\nu}$ 結合は

$g_{\mu\nu}-F_{\mu\nu}-F_{\mu\nu}$ 結合の SUSY partner なので

Y_4 内 $\mathrm{Tr}F^2$ の係数は $\neq$ ゼロ.

よって 相殺できない.

一方で, 両者のヘテロのギロニは 10 10 26 2 16 ∈ lattice CFT or free fermionic なら SO(32) か $E_8\times E_8$ のみ.

という. 一般に条件をみたす chiral CFT は?

⇒ Dong-Mason math.QA/0203005 で分類, これしかない.

※ $E_8\times E_8$ はいいけれど. SO(32) は群は

$$\begin{tikzcd} & \mathrm{spin}(32) \arrow[dl] \arrow[dr] & \\ \mathrm{so}(32) \arrow[dr] & & \mathrm{spin}(32)/\mathbb{Z}_2 =: Ss(32) \arrow[dl] \\ & \mathrm{so}(32)/\pm1 & \end{tikzcd}$$

の四つある. どれか? Type I / ヘテロ では $\mathrm{spin}(32)/\mathbb{Z}_2$

ヘテロの discrete theta angle は 何か? Sethi 1304.1557

場の理論的に, $\mathrm{spin}(32)/\mathbb{Z}_2$ でないといけない事はわかるか? } 未解決.

Sheet Date

高い次元の SUSY ⑥

d=9 どの g が許されるか? < 場の理論的に.
弦/M を知っているとして.

SO(32), $E_8 \times E_8$ を adj. rep. で破った rank 16 のは OK.

D8 N枚で U(N), は transverse に有限キョリで
dilaton が diverge して ダメ.

d=8

F-th 7-brane 上のゲージ群は OK.

しかし transverse 有限キョリで やはり dilaton が おかしく
なりがち。

高々有限種類.

432 ?

SO(2n)
↓
SO(2n−7)

triples
E_8 ↓
F_4, G_2

d=7. M-th で $\mathbb{C}^2/\Gamma$, $\Gamma \subset SU(2)$.

~~多分~~ 勝手な Γ でよい. ⟶ A_n, D_n, E_n, 無限種類. $M \frac{}{SO(2n)}$
↓
$USp(2n-8)$

Witten
9710065.

d=6 さらにこれを S^1 コンパクト化.

⟶ B, C, F, G も可.

IIA on $\mathbb{C}^2/\Gamma$ ← $\mathcal{N}=(1,1)$.

IIB on $\mathbb{C}^2/\Gamma$ ← $\mathcal{N}=(2,0)$: ヘンテコ理論.

d=5. max susy SYM, かってな g が可.

Sp 群は $\pi_4(Sp(n)) = \mathbb{Z}_2$ に伴い 2種.

d=4 かってな g が可. $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$: 無次元パラメタあり. : SCFT.

これ以外の変てこな $\mathcal{N}=4$ SCFT はあるか? 未解決.

ラグランジアン $\mathcal{N}=3$ 理論は 自動的に $\mathcal{N}=4$ だが,
変てこ $\mathcal{N}=4$ はあるか? 未解決.

d=3. かってな g が可. IRで強結合になる。 恐らく難しかったが…

ここで一段落.

2015/11/21

4d supersymmetric Lagrangians ①

$\mathcal{N}=1$ vect. mult.

$$W_\alpha = \lambda_\alpha + F_{(\alpha\beta)}\theta^\beta + D\theta_\alpha + \cdots$$

$\uparrow$ $F_{[\mu\nu]}$ is in $\underset{(\alpha\beta)}{3\otimes 1} \oplus \underset{(\dot\alpha\dot\beta)}{1\otimes 3}$

$$\int d^2\theta \, \frac{-i}{8\pi}\tau \, \mathrm{tr}\, W^\alpha W_\alpha + c.c. \qquad \text{where} \qquad \tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$$

$$\Rightarrow \frac{1}{2g^2}\mathrm{tr}\, F_{\mu\nu}F_{\mu\nu} + \frac{\theta}{16\pi^2}\mathrm{tr}\, \underbrace{F_{\mu\nu}\tilde{F}^{\mu\nu}}_{\text{instanton density.}} + \frac{1}{g^2}\mathrm{tr}\, D^2 - \frac{2i}{g^2}\mathrm{tr}\, \bar\lambda \not{D}\lambda$$

standard Lag. of gauge + adj ferm.

$\int \frac{1}{16\pi^2}\mathrm{tr}\, F\tilde{F} \in \mathbb{Z}$ $\rightsquigarrow$ θ is 2π periodic.

$\mathcal{N}=1$ chiral mult.

$$Q = Q|_{\theta=0} + \psi_\alpha\theta^\alpha + F\theta_\alpha\theta^\alpha \quad : \text{in same rep of } G.$$

$Q \to e^{g\Lambda} Q$ $\quad \Lambda$: chiral

$\int d^4\theta \, Q^\dagger e^{gV} Q$ $\qquad V \to V - \Lambda - \Lambda^\dagger$: vector superf.

$\qquad W_\alpha = D_\alpha \bar{D}^2 V$

$+ \int d^2\theta \, W(Q) + c.c.$

$$\supset \partial_\mu Q^\dagger \partial^\mu Q + \bar\psi \not{D}\psi + |F|^2 + (g\, Q^\dagger Q)D$$

$$+ F\frac{\partial}{\partial Q}W + c.c.$$

$\rightsquigarrow$ $\bar{F}_Q = \frac{\partial}{\partial Q}W$, $\quad D = \sum_i g_i Q^\dagger_i Q_i$

$$V = |F|^2 + g^2|D|^2$$

SUSY unbroken : $\delta\psi = 0 \Rightarrow F = 0$, $\delta\lambda = 0 \Rightarrow D = 0$ $\Big\}$ $V = 0$

$\mathcal{N}=2$ vect. mult

W^α : SU(N) adjoint $\qquad \lambda^\alpha \quad F_{\alpha\beta}$

Φ : SU(N) adjoint $\qquad \Phi \quad \psi^\alpha_\Phi$

$$\int d^2\theta \, \tau \, \mathrm{tr}\, W^\alpha W_\alpha + \frac{1}{g^2}\int d^4\theta \, \mathrm{tr}\, \Phi^\dagger e^{\mathrm{ad} V}\Phi$$

: if written in components. has $\mathbb{Z}_2$ sym exchanging $\lambda^\alpha \leftrightarrow \psi^\alpha_\Phi$.

automatically $\Rightarrow \mathcal{N}=2$!

$SU(2)_R$.

2015/1/21

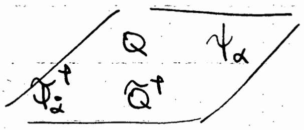
4d supersymmetric Lagrangians ②

$\mathcal{N}=2$ hyper mult.

$$Q = Q + \psi_\alpha\theta^\alpha + F\theta_\alpha\theta^\alpha \qquad \text{in rep } R \text{ of } G$$
$$\tilde{Q} = \tilde{Q} + \tilde{\psi}_\alpha\theta^\alpha + \tilde{F}\theta_\alpha\theta^\alpha \qquad \text{in rep } \bar{R} \text{ of } G$$

$$\int d^4\theta \; Q^\dagger e^V Q + \tilde{Q} e^V \tilde{Q}^\dagger$$
$$+ \int d^2\theta \; \tilde{Q}\Phi Q + c.c.$$

$\downarrow$ super partner.



$\mathcal{N}=4$ vect. mult.

Just add

$$\begin{array}{lllll} W^\alpha & & \lambda^\alpha & F_{\alpha\beta} & \left.\right\} \mathcal{N}=2 \text{ vect} \\ \Phi & \Phi & \psi^\alpha & & \\ A & A & \psi_A^\alpha & & \left.\right\} \mathcal{N}=2 \text{ hyp.} \\ B & B & \psi_B^\alpha & & \end{array}$$

$$W = \mathrm{tr}\, \Phi[A, B].$$

$\uparrow$ 6 of $SU(4)_R \simeq SO(6)_R$ $\qquad$ $\uparrow$ 4 of $SU(4)_R$

$\hookrightarrow$ $V = \sum_{i,j} \mathrm{tr}\, [\phi_i, \phi_j]^2$, $i = 1 \cdots 6$

$\mathrm{Re}\,\Phi$, $\mathrm{Im}\,\Phi$, $\mathrm{Re}\,A$ $\mathrm{Im}\,A$ $\mathrm{Re}\,B$ $\mathrm{Im}\,B$
ϕ_1 ϕ_2 ϕ_3 ϕ_4 ϕ_5 ϕ_6

$\Rightarrow$ automatically $\mathcal{N}=4$.

When $\mathcal{N}=2$, there are two obvious ways to have susy vacua:

① set all $Q=0$, $\Rightarrow$ only constraint is to set $[\Phi, \Phi^\dagger]=0$

$\rightsquigarrow$ $\Phi = \mathrm{diag}(a_1 \cdots a_N)$, up to gauge tr.

'Coulomb branch' 'cause $U(1)^{N-1}$ unbroken

② set $\Phi=0$. $\Rightarrow$ $\left\{ \begin{array}{l} \tilde{Q}Q=0 \\ \tilde{Q}^\dagger\tilde{Q} - QQ^\dagger = 0 \end{array} \right\}$ / up to gauge tr.

'Higgs branch'

In this lecture series we only consider Coulomb branches.

For simplicity just consider $\overset{\mathcal{N}=2 \text{ pure}}{\vee}$ $SU(2)$ gauge th. ~~with some matter field~~

$$\Phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \quad \text{breaks} \quad SU(2) \to U(1). \quad \text{when } a \neq 0.$$

$SU(2)$ gauge boson $\rightsquigarrow$ massive, mass $= |2a|$ $\Leftarrow$ massive vector mult.

't Hooft - Polyakov monopole $\rightsquigarrow$ massive, mass $= |2\tau a|$

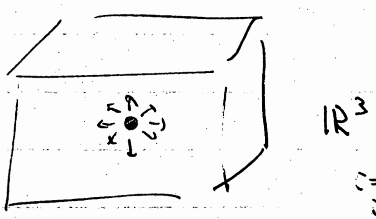
$\Uparrow$ which multiplet?

2015 / 11 / 21

Semi classical analysis of monopoles

① ~~③~~

cf. Harvey, hepth/9603086



$$\Phi(\vec{x},t) = \Phi_0(\vec{x}) + \sum_n \delta\Phi^{(n)}(\vec{x}) \cancel{e^{i\omega_n t}}\ a^{(n)}(t)$$

$$A_\mu = \cdots$$

$$\lambda_\alpha^{(i)} = \sum_n \delta\lambda_\alpha^{(n)}(\vec{x}) \cancel{e^{i\omega_n t}}\ b_{(i)}^{(n)}(t)$$

i=1,2 → is for $SU(2)_R$

~~$\psi_\alpha = \sum_n \delta\psi_\alpha^{(n)}(\vec{x}) e^{i\omega_n t}\ b_{(2)}^{(n)}(t)$~~

Expand the full action ~~in terms of~~ around the static solution

$$\rightsquigarrow \quad \mathcal{L} = \mathcal{L}_0 + \sum_n \left[|\dot{a}^{(n)}|^2 + m^{(n)\,2} |a^{(n)}|^2 \right]$$

$$+ \sum_{n,i} \left[b_{(i)}^{(n)}(t)^\dagger \partial_t b_i^{(n)}(t) + \mu^{(n)} b_i^{(n)}(t)^\dagger b_{(i)}^{(n)}(t) \right]$$

+ interactions.

where $(D_i)^2 \delta\Phi^{(n)}(\vec{x}) = m^{(n)} \delta\Phi(\vec{x})$

$$\cancel{D}\, \delta\lambda_\alpha^{(n)}(\vec{x}) = \mu^{(n)} \delta\lambda_\alpha^{(n)}(\vec{x}) \quad \text{etc.}$$

when $m^{(n)}$ is nonzero, $a^{(n)}$ (or $b^{(n)}$) is just an oscillator.

m is zero, called <u>zero modes</u>. needs to be treated carefully.

obvious ones: $\Phi_0(\vec{x}+\delta x)$ is also a sol'n.

$$\overset{\shortparallel}{\Phi_0(\vec{x}) + \underbrace{\delta\Phi^{(0)}}_{c}(\vec{x})} \quad m^{(0)} = 0.$$

<u>Fermionic zero modes</u>. ~~Callias' index thm~~:

Explicit computation shows that the zero modes in $\lambda_\alpha^{(i)}$ has the form

$$\mathcal{L} \supset \epsilon_{ij}\, \epsilon_{\alpha\beta}\, b_\alpha^i \partial_t b_\beta^j$$

α,β : doublet index of SO(3) little group

$$(b_\alpha^i)^\dagger = \epsilon_{ij}\, \epsilon^{\alpha\beta}\, b_\beta^j .$$

i,j : doublet of $SU(2)_R$

$\{,\}$

"quantize"

$$\{b^A, b^B\} = \delta^{AB}$$

A=1,2,3,4 : vector of $SO(4)_X \simeq SO(3)_{\text{little}} \times SU(2)_R$

$\{,\}_b$

act on 'Dirac spinor' of $SO(4)_X$

$= \underset{\chi}{}$ ~~spinor of~~ $(\mathbf{1},\mathbf{2}) \oplus (\mathbf{2},\mathbf{1})$ of $SO(3)$ little $+ SU(2)_R$.

massive hypermultiplet !

'tH-P monopole is <u>half BPS</u>

$P_\mu \curvearrowright \Phi_0(\vec{x})$: $P_{\mu=0}$: conserved, $P_{\mu\neq0}$: non-conserved ⇒ bos zero modes

$Q_\alpha^i \curvearrowright \Phi_0(\vec{x})$ $\quad Q_\alpha^i + (Q_\beta^j)^\dagger \epsilon^{ji} \epsilon_{\alpha\beta}$: conserved ←

↳ cpx four components $\quad -$: non-conserved → ferm. zeros

2015 / 1 / 21

Semi-classical quant. of monopoles ②

Add N_f hypers in the doublet of SU(2) gauge

$$Q^i_a \qquad\qquad a=1.2 : \text{gauge}$$
$$\tilde{Q}^b_j \qquad\qquad i=1\cdots N_f : \text{flavor}$$

$$W = Q^i_a \Phi^a_b \tilde{Q}^b_j \qquad\qquad \text{for } SU(2),\ \mathbf{2}=\bar{\mathbf{2}}$$
$$= q_{Ia}\Phi_{ab}q_{Ib} \qquad q_{Ia},\ I=1\cdots 2N_f \quad \leftarrow O(2N_f) \text{ symmetry classically}$$

On the 't Hooft Polyakov monopole background, the ferm. zero m. have the action $L \supset b_I \partial_t b_I$

"quantize" ↳ $\{b_I, b_J\} = g_{IJ}$

⇒ monopole states are { massive hypermultiplets in the Dirac spinor rep. of $O(2N_f)$ $\to$ + chiral sp. / $\to$ − chiral sp.

the 'parity' of $O(2N_f)$, say $q_{I=1} \to -q_{I=1}$, $q_{I\neq 1} \to +q_I$ is <u>anomalous</u>.

In an instanton background, each q_I has one zero mode ⇒ −1 in the 1-inst bkg.

⇒ shifts θ angle by π.

How about $\mathcal{N}=4$ SU(2) theory?

$\Phi \neq 0$, $A=B=0$ $\quad (\Phi \overset{\shortparallel}{=} a)$ ← can set Im $a=0$ without loss of generality.

$$\underbrace{\phi_1}_{\substack{\neq\\0}}\ \underbrace{\phi_2\ \cdots\ \phi_5}_{\substack{\shortparallel\\0}} \qquad \begin{matrix} SO(6)_R & SO(4 \\ \downarrow & \downarrow \\ SO(5)_R & Sp(2) \end{matrix}$$

fermions: $\lambda^{i=1.2.3.4}_{\alpha=1.2}$

ferm. zero modes: $b^{i=1.2.3.4}_{\alpha=1.2}$ $\quad ,\ (b^i_\alpha)^\dagger = J_{ij}\varepsilon_{\alpha\beta}b^j_\beta$

↳ quantize

$$\{b^i_\alpha, b^j_\beta\} = J^{ij}\varepsilon_{\alpha\beta}$$

$i\alpha$: SO(8)'s vector

↳ SO(5) × SO(3)'s chiral spinor.

→ acts on SO(8) Dirac spinor = SO(3) vect + SO(5) vect + SO(3) doublet × SO(5) spinor

massive $\mathcal{N}=4$ vector multiplet! ← $(\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{5}) + (\mathbf{2},\mathbf{4})$

again, zeromodes are gen. by half of susy

2015/1/22

~~Preliminaries on~~ Lagrangian $\mathcal{N}=2$ SCFTs ①

$\mathcal{N}=2$ gauge th with hypers in $R_i \oplus \bar{R}_i$

one-loop beta $\propto 2T(\text{adj}) - \sum_i T(R_i) \oplus T(\bar{R}_i)$

$\boxed{T(R) \text{ is defined as } \operatorname{tr}_R T^a T^b = T(R)\delta^{ab}}$

e.g. $\mathcal{N}=4$ SYM : $\mathcal{N}=2$ with hyper in adj

$\longrightarrow$ one-loop beta $= 2T(\text{adj}) - T(\text{adj}) - T(\text{adj}) = 0$

$\mathcal{N}=2$ $SU(N)$ with N_f $\overline{\text{fund} \oplus \text{fund}}$

$\longrightarrow$ one-loop beta $= 2N - \sum \frac{1}{2} + \frac{1}{2}$

$= 2N - N_f$: 0 if $N_f = 2N$

● FACT if $\mathcal{N}=2$ gauge th. has zero one-loop beta,

it's an SCFT (beta func. vanishes to all order)

and the gauge coupling τ is an exactly marginal par.

note $\mathcal{N}=1$ gauge th. with zero one-loop beta

can be trivial

e.g. $\mathcal{N}=1$ $SU(N)$ with $N_f = 3N$ $\overline{\text{fund} \oplus \text{fund}}$ with zero superp.

$$\int d^2\theta \, \underset{\substack{\uparrow \\ \text{one-loop} \\ \text{exact}}}{\underset{\sim}{\tau}} W^\alpha W_\alpha + \sum_{i=1}^{N_f} \underset{\substack{Z \,:\, \text{wave funct. renorm.} \\ \uparrow \\ \text{nontrivial} \,:\, \text{ruins conformality}}}{\wedge} \int d^4\theta \left(Q_i^\dagger e^V Q_i + \tilde{Q}_i e^V \tilde{Q}_i^\dagger \right)$$

● rough proof of the FACT

the Lagrangian in $\mathcal{N}=1$ language is $\swarrow$ not renormalized

paired by $\mathcal{N}=2$

$$\int d^2\theta \, \underset{\substack{\sim \\ \tau \text{ one-loop exact}}}{\tau} \operatorname{tr} W^\alpha W_\alpha + \int d^2\theta \, 1 \, \tilde{Q} \Phi Q$$

$$+ \frac{1}{g^2} \underset{\substack{\wedge \\ Z_\Phi}}{} \int d^4\theta \operatorname{tr} \Phi^\dagger \Phi + \underset{\substack{\wedge \\ Z_Q}}{} \int d^4\theta \left(Q^\dagger e^V Q + \tilde{Q} e^V \tilde{Q}^\dagger \right)$$

locked by $\mathcal{N}=2$

$Z_\Phi \longleftrightarrow Z_Q$

can be nontrivial if just $\mathcal{N}=1$,

but locked by the $\mathcal{N}=2$ SUSY to the superpotential term

$\Longrightarrow$ no renormalization

non-perturbatively τ_{IR} can be a non-trivial func. of τ_{UV}.

With $\mathcal{N}=4$, this is forbidden because Z_Q is 1 even non-pert. and the possible finite renormalization of τ & Z_Φ are locked by $\mathcal{N}=4$.

Lagrangian 2015 / 1 / 22

~~Prelim. m~~ $\mathcal{N}=2$ SCFTs ②

So you can gradually 'turn on' $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$ (note the weak coupling is $\mathrm{Im}\,\tau \to +\infty$)

What happens when $\mathrm{Im}\,\tau$ is very small ?

$\mathcal{N}=4$ SU(2) SYM

τ : weak coupling. $\phi = \begin{pmatrix} a & \\ & -a \end{pmatrix}$

W-boson : mass $= |2a|$

monopole : mass $= |2\tau a|$ } both $\mathcal{N}=4$ massive vect. mult.

$S \updownarrow$ $-\frac{1}{\tau} = \tau'$

W-boson mass $= |2a'|$

monopole mass $= |2\tau' a'|$

$T: \tau \to \tau+1$ is also a symmetry. S & T generate $SL(2,\mathbb{Z})$.

$\mathcal{N}=2$ SU(2) SYM with $N_f=4$. q_{Ii}, $I=1\cdots 8$, $i=1,2$ $\leftarrow SO(8)$

τ : weak coupling $\phi = \begin{pmatrix} a & \\ & -a \end{pmatrix}$

$\tilde{\tau} = 2\tau$

quark : mass $= |a|$, vector of SO(8)

monopole : mass $= |2\tau a|$, positive chirality sp. of SO(8)

dyon : mass $= |(2\tau+1)a|$ negative — SO(8)

both massive hypers.

$S: \updownarrow$

$-\frac{1}{\tilde{\tau}} = \tilde{\tau}' = 2\tau'$ τ' (weak coupling)

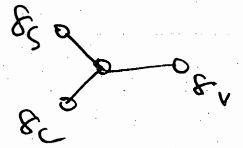
quark mass $= |a'|$ 8_s of SO(8)

monopole mass $= |2\tau' a'|$ 8_v of SO(8)

dyon mass $= |(2\tau'+1)a'|$

$T: \tau \to \tau + \frac{1}{2}$: exchanges $8_s \leftrightarrow 8_c$.

$\tilde{\tau} \to \tilde{\tau} + 1$



{ S & T generate $SL(2,\mathbb{Z})$
triality of SO(8) flavor is involved.

$\mathcal{N}=4$ SU(N) SYM : $SL(2,\mathbb{Z})$, dual $= \mathcal{N}=4$ SU(N) SYM.

$\mathcal{N}=2$ SU(N) + $N_f = 2N$ fund ⊕ fund

: not clean at this stage. triality can't be easily generalized.

⇝ Argyres-Seiberg ; Gaiotto ; Chacaltana-Distler

we'll come back later.

2015 / 1 / 23

n_v, n_H and SCI ①

These S-dualities are at least plausible ; any way to check more ?

Let's define a few quantities that can be assoc. to any 4d $\mathcal{N}=2$ SCFT.

$\boxed{\begin{matrix} n_v \\ n_H \end{matrix}}$ $U(1)_R \times SU(2)_R$ sym $\rightsquigarrow$ anomaly polynomial $\underset{I_6}{\wedge}$ encoding 't Hooft anom.

e.g. hypermult

$$\begin{array}{cccc} & \tilde\psi^\dagger_{\dot\alpha} & \begin{matrix} q \\ \tilde q^\dagger \end{matrix} & \psi_\alpha \\ U(1)_R & 1 & 0 & -1 \end{array} \rightsquigarrow I_6 = 2\times\left(-\frac{c_1(F_{U(1)R})^3}{6} + \frac{c_1(F_{U(1)R})}{24} p_1(TX)\right)$$

vect mult.

$\swarrow$ $SU(2)_R$ doublet

$$\begin{array}{ccc} \phi & \boxed{\begin{matrix} \psi \\ \lambda \end{matrix}} & A \\ 2 & 1 & 0 \end{array} \rightsquigarrow I_6 = 2\left(\frac{c_1(F_{U(1)R})^3}{6} - \frac{c_1(F_{U(1)R})}{24} p_1(TX)\right)$$

$$\underline{\#}\ c_1(F_{U(1)R})\, c_2(F_{SU(2)R}).$$

In general, I_6 is known to be a lin. com. of them (without assuming $\exists$ Lagrangian.)

$$I_6 = (n_v - n_H)\left(\frac{c_1^{\ 3}}{3} - \frac{c_1}{12} p_1\right) \underline{\#}\, n_v c_1 c_2 .$$

<u>defines</u> n_v & n_H.

They are indep. of exact. mar. param. $\Rightarrow$ should be cnst across S-duality ...

$\boxed{\text{SCI}}$ Put the SCFT on $S^3 \times \mathbb{R}$ $\rightsquigarrow$. Pick a supercharge Q

and consider

$$SCI = \mathrm{Tr}_{\mathcal{H}(S^3)} (-1)^F e^{-\beta\{Q, Q^\dagger\}} \ ▨$$

$\uparrow$ charges commuting with Q.

SCFT has $SU(2,2|2)$

Q's commutant : $SU(1,1|2)$: rank 3 $\rightsquigarrow$ 3 params p, q, t + flavor sym.

$$\{Q, Q^\dagger\} = \Delta - 2j_2 - 2I_3 + \frac{r}{2}$$

I_3 : cartan of $SU(2)_R$

r : $U(1)_R$

s.t. Q has $I_3 = \pm\frac{1}{2}$, $r = \pm 1$

$$SCI = \mathrm{Tr}_{\substack{\mathcal{H}(S^3), \\ \Delta - 2j_2 - 2I_3 + \frac{r}{2} = 0}} (-1)^F p^{j_2+j_1} q^{j_2-j_1} t^{I_3} \left(\frac{t}{pq}\right)^{r/2} \prod x_i^{F_i}$$

cf. Gadde-Rastelli-Razamat-Yan 1110.3740. note $I_3 = R$ there, $r_{\text{here}} = 2r_{\text{there}}$

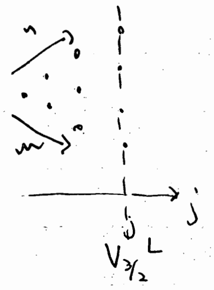
2015/1/23

n_v, n_H and SCI ②

For a free hyper, $\mathcal{H}(S^3)$ is just a Fock space and SCI can be explicitly computed.

$$\text{scalar wave f. on } S^3: \quad \underset{j}{\oplus}\ \overset{\cancel{j\geq 0}}{\boxed{V_j}^L} \otimes \overset{\phi}{\enclose{circle}{V_j}^R} \xleftarrow{\text{top cmp.}} \quad \oplus \quad \underset{j}{\oplus}\ \overset{\bar\phi}{V_j \otimes V_j} \qquad \Delta = j+1$$

$$\text{Weyl ferm. wavef. on } S^3: \quad \underset{j}{\oplus}\ \underset{\psi_\alpha}{V_{j+1/2} \otimes V_j} \quad \oplus \quad \underset{j}{\oplus}\ \underset{\cancel{j\geq 0}\ \ \bar\psi_{\dot\alpha}}{\boxed{V_j} \otimes \enclose{circle}{V_{j+1/2}}} \xleftarrow{\text{top cmp.}} \qquad \Delta = j+\frac{3}{2}$$



- ϕ ψ $\tilde\phi$ $\tilde\psi$ U(1)-charge 1

$$SCI_{p,q,t}(x) := \mathrm{Tr}\,(-1)^F p^{\cdot} q^{\cdot} t^{\cdot} x^F$$

$$= \prod_{n,m\geq 0} \frac{\overset{\bar\psi}{\boxed{1-t^{-1/2}x^{-1}p^{m+1}q^{n+1}}}}{\underset{\phi}{\boxed{1-t^{1/2}x\,p^m q^n}}} \cdot \frac{\overset{\bar{\tilde\psi}}{\boxed{1-t^{-1/2}x\,p^{m+1}q^{n+1}}}}{\underset{\tilde\phi}{\boxed{1-t^{1/2}x^{-1}p^m q^n}}}$$

Let $\Gamma_{p,q}(z) := \prod_{n,m\geq 0} \frac{1-z^{-1}p^{m+1}q^{n+1}}{1-zp^m q^n}$

$$SCI^{hyper}(x) = \Gamma_{p,q}(t^{1/2}x)\,\Gamma_{p,q}(t^{1/2}x^{-1})$$

- For a free vector of SU(N), let $\mathrm{diag}(z_1 \cdots z_N) \in SU(N)$

then $SCI^{vec}(z_1 \cdots z_N) = \left(\frac{1}{\Gamma_{p,q}(t)\,\Gamma_{p,q}\cancel{(t)}}\right)^{N-1} \prod_{i\neq j} \frac{1}{\Gamma_{p,q}(t\frac{z_i}{z_j})\,\Gamma_{p,q}(\frac{z_i}{z_j})}$

when $\Gamma_{p,q} = \lim_{z\to 1}(1-z)\Gamma_{p,q}(z)$

in an weakly interacting system, only gauge-invariant states are kept.

E.G.

SCI of SU(N) with $N_f=2N$ flavors, $(x_1 \cdots x_{N_f}) \in SU(N_f)$

$$= \frac{1}{N!}\prod_{i=1}^{N-1}\oint \frac{dz_i}{2\pi i z_i} \cdot SCI^{ves}_{SU(N)}(z_1 \cdots z_N) \cdot \prod_{a,i} \Gamma_{p,q}(t^{1/2}x_a/z_i)\,\Gamma_{p,q}(t^{1/2}z_i/x_a)$$

Horribly messy but readily computable as a series in p, q, t.

2015/1/24

n_v, n_H and SCI ③

Now let's apply this to the duality

$SU(2)$ with $N_f=4$ in 8_V of $SO(8)_F$, τ

$\updownarrow$

$SU(2)$ with $N_f=4$ in 8_S of $SO(8)_F$. τ' $2\tau' = -\frac{1}{2\tau}$.

Let

$$J_{p.q.t}(x_1,x_2,x_3,x_4) = \frac{1}{2}\oint\frac{dz}{2\pi i z}\frac{1}{\Gamma_{p.q}(t)\Gamma_{p.q}}\frac{1}{\prod_\pm\Gamma_{p.q}(tz^{\pm2})\Gamma_{p.q}(z^{\pm2})}\prod_{\pm\pm i}\Gamma_{p.q}(t^{1/2}\cancel{z^{\pm1}}x_i^{\pm1})$$

where $\mathrm{diag}(x_i, x_i^{-1}) \in SO(8)$.

8_S has weights $x_1^{\pm\frac12}x_2^{\pm\frac12}x_3^{\pm\frac12}x_4^{\pm\frac12}$ with even number of pluses;

so we should have

$$J_{p.q.t}(x_1,x_2,x_3,x_4) = J_{p.q.t}\left(\sqrt{x_1x_2x_3x_4}, \sqrt{\frac{x_1x_2}{x_3x_4}}, \sqrt{\frac{x_1x_3}{x_2x_4}}, \sqrt{\frac{x_1x_4}{x_2x_3}}\right). \quad ※$$

Rastelli noticed this should be true in 2009 summer; phoned an expert, who replied his grad student just had a paper proving exactly this equality! (van de Bult)

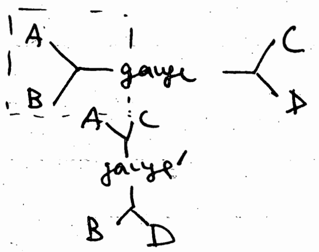
A rewriting: $\overset{SO(8)}{N_f=4} = \overset{SO(4)}{N_f=2} + \overset{SO(4)}{N_f=2}$

$SO(4) \times SO(4)$

$= SU(2)_A\times SU(2)_B \times SU(2)_C\times SU(2)_D$

$8_V = 2_A\otimes 2_B \oplus 2_C\otimes 2_D$

$8_S = 2_A\otimes 2_C \oplus 2_B\otimes 2_D$



Correspondingly, let $x_1=ab$, $x_2=a/b$, $x_3=cd$, $x_4=c/d$

and set $I_{p.q.t}(a.b.c.d) = J_{p.q.t}(x_1\ x_2\ x_3\ x_4)$

then $I_{p.q.t}(a.b.c.d) = I_{p.q.t}(a,c,b,d)$ … equivalent to ※.

$SU(2)_{gauge}\times SU(2)_A\otimes SU(2)_B$ 'trifundamental' Q_{aiu} $\begin{matrix}a=1.2\\ i=1.2\\ u=1.2\end{matrix}$. $n_H=4$.

$$SCI = \prod_{\pm\pm\pm}\Gamma_{p.q}(t^{1/2}z^\pm a^\pm b^\pm).$$

Having p.q.t is too cumbersome!

Set $q=t$. then $\Gamma_{p.q}(t^{1/2}x)\cdot\Gamma_{p.q}(t^{1/2}x^{-1}) = \prod_{n\geq0}\frac{1}{1-q^{1/2+n}x}\cdot\frac{1}{1-q^{1/2+n}x^{-1}}$

p dep drops out. A general property of SCI.

$$\text{SCI of a trifund} = \prod_{\pm\pm\pm}\prod_{n\geq0}\frac{1}{1-q^{1/2+n}z^\pm a^\pm b^\pm}.$$

2015/1/24

n_v, n_H and SCI ④

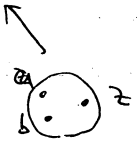
The vector contribution becomes

$$\frac{1}{2}\oint \frac{dz}{2\pi i z}(1-z^2)(1-z^{-2})\ K(z)^{-2}\ (\text{matter contribution})$$

where $K(z)^{-1} = \prod_{n\geq 0}(1-q^{n+1}) \cdot \prod_{\pm}\prod_{n\geq 0}(1-q^{n+1}z^{\pm 2})$.

In this special case $q=t$, the equality ✣ can be shown as follows.
First, the trifund contrib satisfies

$$\prod_{\pm\pm\pm}\prod_{n\geq 0}\frac{1}{1-q^{1/2+n}z^{\pm}a^{\pm}b^{\pm}} = \frac{K(z)K(a)(K(b))}{K_0}\sum_{n\geq 1}\frac{\chi_n(z)\chi_n(a)\chi_n(b)}{\chi_n(q^{1/2})}$$



where $K_0^{-1} = \prod_{n\geq 0}(1-q^{2+n})$, $\chi_n(z) = z^{n-1}+z^{n-3}+\cdots+z^{-n+1}$

$\uparrow$ irrep character. dim $= n$.

Then

$$T_{q=t}(a,b,c,d) = \boxed{\frac{1}{2}\oint\frac{dz}{2\pi i z}(1-z^2)(1-z^{-2})}\ \cancel{K(z)}^{-2}$$

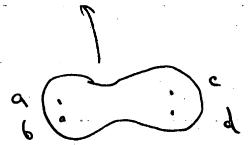
integration on the SU(2) mfd

$$\times\frac{\cancel{K(z)}K(a)K(b)}{K_0}\sum_{n\geq 1}\frac{\chi_n(z)\chi_n(a)\chi_n(b)}{\chi_n(q^{1/2})}$$

$$\times\frac{\cancel{K(z)}K(c)K(d)}{K_0}\sum_{n\geq p}\frac{\chi_n(z)\chi_n(c)\chi_n(d)}{\chi_n(q^{1/2})}$$

$$=\frac{K(a)K(b)K(c)K(d)}{K_0^{\ 2}}\sum_{n\geq 1}\frac{\chi_n(a)\chi_n(b)\chi_n(c)\chi_n(d)}{\chi_n(q^{1/2})^2}$$

: manifestly symmetric in a, b, c, d !



It's been long known that the part. func. of 2d YM on genus g surface with n punctures is given by

$$\frac{\prod_i N(a_i)}{N_0^{2g+n-2}}\sum_{n\geq 1}\frac{\prod_i \chi_n(a_i)}{\chi_n(1)^{2g+n-2}}$$

where $N(a)$ is the 'wavefunction renormalization' of a puncture and N_0 is the 'renormaliz. of Newton constant'

coming from $\lambda\underbrace{\int_C\sqrt{g}R}$

integrates to $2-2g-n$.

2d q-YM : changes the gauge group to be the quantum group.
in the lattice ~~formulation~~, link vars live in the quant. gr.
$\rightsquigarrow$ ends up just changing $\chi_n(1)$ in the den. $\longrightarrow \chi_n(q^{1/2})$.

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6d $\mathcal{N}=(2,0)$ theory ①

① S-duality of $\mathcal{N}=4$ SYM
② relation of the SCI with 2d q-YM } can both be understood by assuming the following

FACT: $\exists$ 6d $\mathcal{N}=(2,0)$ SCFT, labeled by A_n, D_n or $E_{n=6,7,8}$, s.t. its S^1 cpt'tion, in the IR, is given by 5d $\mathcal{N}=2$ SYM with gauge gp A_n, D_n or E_n, respectively.

rough argument for the existence

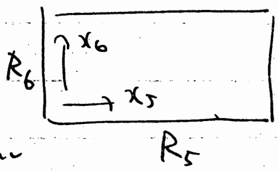
A_n: $(n+1)$ M5 in Mth. S^1 cptf $\rightsquigarrow$ $(n+1)$ D4 branes.

D_n: n M5 in M th with Mth orientifold $\rightsquigarrow$ n D4 on top of O4$^-$

A, D, E: Type IIB on $\mathbb{C}^2/\Gamma$. S^1 cptf & T-dual $\rightsquigarrow$ $\mathbb{C}^2/\Gamma$ Type IIA on S^1.

we don't use string theory any more.

① $N=(2,0)$ in 6d $\xrightarrow[R_6]{}$ $\mathcal{N}=2$ in 5d $\xrightarrow[R_5]{}$ $\mathcal{N}=4$ in 4d



$\Uparrow$ SCFT, no scale.

$$\int d^5x \frac{1}{\underset{\sim\sim}{g^2_{5d}}} \operatorname{tr} F_{\mu\nu}F_{\mu\nu}$$

$\uparrow$ dimensionful.

$g^2 \propto R_6$.

$$\underbrace{\frac{1}{g^2_{5d}} \int dx_5}_{\frac{1}{g^2_{4d}}} \int d^4x \operatorname{tr} F_{\mu\nu}F_{\mu\nu}$$

$\frac{1}{g^2_{4d}} \propto \frac{R_5}{R_6}$.

Which to call $R_{5,6}$ is our choice.

so $d=4$ $\mathcal{N}=4$ SYM with gauge group $G=A, D, E$, $\tau = \frac{R_5}{R_6}$

should be equivalent to the same th. with $\tau' = \frac{R_6}{R_5}$] S-dual

4d S-duality became 6d Lorentz (rotational) inv.

$\uparrow$ exchanges W-boson : quanta

$\updownarrow$ monopoles : semi-classical obj.

$\rightsquigarrow$ writing down a Lagrangian for the 6d theory is clearly hard.
No useful Lag. found yet.

② SCI of 4d $\mathcal{N}=2$ SCFT $= \operatorname{tr}_{\mathcal{H}(S^3)} (-1)^F \cdots$

$= \underset{\text{SUSY}}{\wedge}$ part. func. on $S^3 \times S^1$

6d theory on C: a Riem. surface can preserve 4d $\mathcal{N}=2$.

Its SCI $=$ SUSY part. func. on $S^3 \times S^1 \times C$ of the 6d $\mathcal{N}=(2,0)$ th.

$=$ SUSY part. func. on $S^3 \times C$ of the 5d $\mathcal{N}=2$ SYM

$\underset{\text{localize}}{=}$ 2d YM on C $\underline{\text{dressed with modes on } S^3}$

gives q deformation.

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6d $\mathcal{N}=(2.0)$ theory ②

We want to make this a bit more precise.
As a first step, let's determine the anom. poly. of 6d $\mathcal{N}=(2.0)$ theory.

Free $\mathcal{N}=(2.0)$ tensor mult in 6d :

$$\underset{\substack{\uparrow \\ SO(5)_R \\ \cup \\ SO(4) \\ \| \\ SU(2)_L \times SU(2)_R}}{\phi_{1.2.3.4.5}} \;;\; \underset{\substack{\uparrow \\ \text{6d weyl sp.}}}{\psi_{\alpha, i}} \;;\; B_{\mu\nu}^{\dagger} \qquad \leftarrow \; H=dB, \; H=*H.$$

i: 1.2.3.4 $\curvearrowleft Sp(2)_R \supset Sp(1)_L \times Sp(1)_R$.

$$\text{anomaly poly} = -\frac{1}{48}\left(-p_2(N)+p_2(T)-\frac{1}{4}\left(p_1(N)-p_1(T)\right)^2\right) =: I_8^{\text{free}}$$

T: tangent bundle of the spacetime.
N: $SO(5)_R$ bundle.

<u>FACT</u>. 6d $\mathcal{N}=(2.0)$ th. of type $SU(Q) = A_{Q-1}$ has the anomaly poly

$$I_8^{A_{Q-1}} = \frac{Q^3-Q}{24}p_2(N) + (Q-1)\, I_8^{\text{free } \mathcal{N}=(2,0) \text{ tensor}}.$$

$\uparrow$ the famous N^3 behavior of M5.
cf. N^2 for D-branes ; $N^{3/2}$ for M2.

derivation 1. use M-th.
derivation 2. just use the fact S^1 cptftion is 5d $\mathcal{N}=2$ SYM.

$$\begin{tikzcd}
\text{6d } \mathcal{N}=(2.0) \text{ of type } \cancel{S}U(Q) \arrow[r, "\text{go to tensor br.}"'] \arrow[d, "S^1"] & \text{6d } \mathcal{N}=(2.0) \text{ free tensor} \times (Q\cancel{-1}) \arrow[d] \\
\text{5d } \mathcal{N}=2 \text{ SYM, gauge } \cancel{S}U(Q) \arrow[r, "\text{go to Coulomb br.}"'] & \text{5d } \mathcal{N}=2 \text{ SYM, } U(1)^{Q\cancel{-1}}
\end{tikzcd}$$

$\Phi_{I=1.2.3.4.5}$ $\qquad$ $\Phi_{I=1}=\mathrm{diag}(\phi_1 \cdots \phi_Q)$
$\Phi_{I\neq 1}=0$

$$SO(5)_R \longrightarrow SO(4)_R = SU(2)_L \times SU(2)_R$$

't Hooft anom. is conserved : the 6d 'free' tensor th. should carry all the anomaly. Contributions are :

- one-loop. $\quad\cdots\; Q\, I_8^{\text{free } \mathcal{N}=(2,0) \text{ tensor}}$.
- Green-Schwarz-Sagnotti.

'$dB=H$', $H=*H$, $\quad dH = I_4 \quad \rightsquigarrow$ ~~δ(anom. poly)~~ $I_8^{GSS} = \frac{1}{2}I_4^2$.

roughly : ~~if not~~ $\exists \int B\wedge I_4$ $\quad \rightarrow \delta S = I_2^{(1)} I_4 \rightarrow I_8 = I_4^2$ $\quad$ self duality

at the same time
$H = dB + I_3^{(0)} \rightsquigarrow \delta B = I_2^{(1)}$

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6d $\mathcal{N}=(2.0)$ theory

③

6d $\mathcal{N}=(2.0)$ tensor $B_1, B_2 \cdots B_Q$ $\qquad dH_i = I_{4\,i}$

$\downarrow \qquad \phi_1 < \phi_2 < \cdots < \phi_Q \qquad\qquad \wr$

5d $U(Q) \longrightarrow$ 5d $\mathcal{N}=2$ $U(1)^Q$ $A_1, A_2 \cdots A_Q$ $\qquad \int A_i I_{4\,i}$

$\exists$ massive $\mathcal{N}=2$ vect. mult with charge $e_i - e_j$.

Fermion mass term ~~$\psi \phi_i^I \phi_j \Gamma^I \psi$~~ $\psi(\phi_i-\phi_j)^I \Gamma^I \psi$.

$\rightsquigarrow$ $SU(2)_R$ charged part has mass $\phi_i - \phi_j$

$SU(2)_L$ $\qquad \phi_j - \phi_i$

Now, $\overset{U(1)^Q}{}$ $SU(2)$ charged fermion with mass m, when integrated out, gives the 5d CS $A(c_2 + \frac{1}{12}p_2(T)) \times (\text{sign } m)$.

So, for each i, we have

$$A_i \Big[\sum_{j<i} \Big[\big(c_2(R) + \frac{p_2(T)}{12} \big) - \big(c_2(L) + \frac{p_2(T)}{12} \big) \Big]$$

$$+ \sum_{j>i} \Big[\big(c_2(L) + \frac{p_2(T)}{12} \big) - \big(c_2(R) + \frac{p_2(T)}{12} \big) \Big] \Big]$$

$$= A_i \big(c_2(R) - c_2(L) \big) \frac{1}{2}(Q+1-2i)$$

in 6d, $\qquad dH_i = \frac{1}{2}(Q+1-2i)(c_2(R) - c_2(L))$

$$\rightsquigarrow I_8^{\text{tot}} = Q\, I_8^{\text{free}} + \frac{1}{2} \sum_{i=1}^{Q} \Big[\frac{1}{2}(Q+1-2i)^{\cancel{2}} \Big]^2 \big(c_2(R) - c_2(L)\big)^2.$$

$$= Q\, I_8^{\text{free}} + \frac{Q^3 - Q}{24} p_2(N).$$

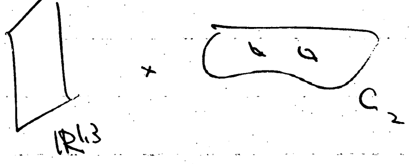
can be easily generalized to D_n & E_n

$$I_8 = \frac{h^\vee d}{24} p_2(N) + r\, I_8^{\text{free}}$$

	$SU(N)$	$SO(2N)$	E_6	7	8
r	$N-1$	N	6	7	8
h	N	$2N-2$	12	18	30
d	N^2-1	$N(2N-1)$	78	133	248

cf. Ohmori-Shimizu-YT-Yonekura 1408.5572

Now, let's put the 6d theory on a Riemann surface



the curvature of C, if not taken care of, destroys the susy.

$\rightsquigarrow$ add appropriate $SO(5)_R$ curvature on C_2 to cancel it!

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Cpt f. of 6d $\mathcal{N}=(2,0)$ theory ①

$TX_4 \times TC$

$\pm\lambda_1, \pm\lambda_2$ $\pm t$: 'Chern roots' : assume tang. bndl. is a direct sum of line bundles. λ_i, t are curvatures of these l. b.

N : $SO(5)_R$-bundle

$\pm n_1\ \pm n_2\ 0$: its Chern roots

supercharge

$+\frac{1}{2}\lambda_1 + \frac{\lambda_2}{2}$ $\boxed{+\frac{t}{2} \pm \frac{n_1}{2} \pm \frac{n_2}{2}}$

take $\begin{cases} n_1 = 2c_1 - t \\ n_2 = 2\alpha \end{cases}$

broken due to 't' $\dashrightarrow t - c_1 \pm \alpha$

unbroken. $\dashrightarrow c_1 \pm \alpha$

$U(1)_R$ charge $+1$, $SU(2)_R$ doublet.

$SO(2)_C \longleftrightarrow SO(2)_R \subset SO(5)_R$

cancel $SO(2)_C$ curvature by $SO(2)_R$ curvature.

$SO(2)_R \times SO(3)_R$ commutes.

$\pm 2c_1$ $\pm 2\alpha, 0$

$U(1)_R \times SU(2)_R$

$\pm c_1$ $\pm\alpha$

cf. Alday-Benini-YT 0909.4776

Anomaly poly of 4d th. $= \int_C \underbrace{\text{Anomaly poly of 6d th.}}_{\frac{h^\vee d}{24} p_2(N) + r I_8^{\text{free}}}$

use $p_1(TY_{2n}) = \sum \lambda_i^2$

$p_2(TY_{2n}) = \sum_{i<j} \lambda_i^2 \lambda_j^2$

$\rightarrow \int_C$ poly of $\lambda_{1,2}$, t, c_1, α

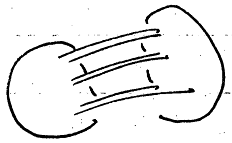
use $\int_C t = 2-2g$

$$= (2-2g)\left[\frac{c_1}{24}(-4c_1^2 + \overset{p_1(TX_4)}{\boxed{\lambda_1^2+\lambda_2^2}})r - \frac{c_1}{6}\overset{-c_2(SU(2)_R)}{\boxed{\alpha^2}}(4h^\vee d + 3r) \right]$$

$$= (g-1)r\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) - (g-1)\left(\frac{4}{3}h^\vee d + r\right)c_1 c_2$$

compare it with $I_4 = (n_v - n_H)\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) - n_v c_1 c_2$

$\rightsquigarrow$ $\underline{n_v = (g-1)\left(\frac{4}{3}h^\vee d + r\right), \quad n_h = (g-1)\frac{4}{3}h^\vee d.}$



$\leftarrow$ $2(g-1)$ three punctured spheres $\Leftarrow$ G^3-symmetric $\boxed{T_G}$ $\mathcal{N}=2$ SCFT

$+$

$3(g-1)$ tubes $\Leftarrow$ $\mathcal{N}=2$ G vect. mult.

$n_v = d$, $n_h = 0$.

T_G theory has

$\rightsquigarrow$ $\underline{n_v = \frac{2}{3}h^\vee d + \frac{r}{2} - \frac{3}{2}d}$, $\underline{n_h = \frac{2}{3}h^\vee d.}$

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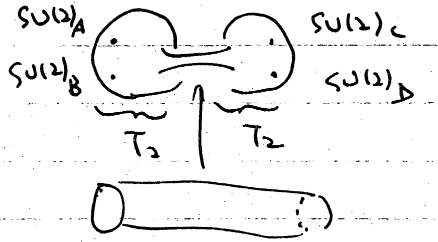
Cpt P. of 6d $\mathcal{N}=(2,0)$ theory ②

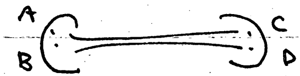
Let's take $G=SU(2)$. $h^\vee=2$, $d=3$, $r=1$

$T_2 := T_{SU(2)}$ theory has $n_v=0$, $n_h=4$. $SU(2)^3$ symmetry.

$\Leftarrow$ trifundamental hyper Q_{aiu}.



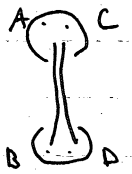
SU(2) vector multiplet (the hyper part is killed by B.C.)



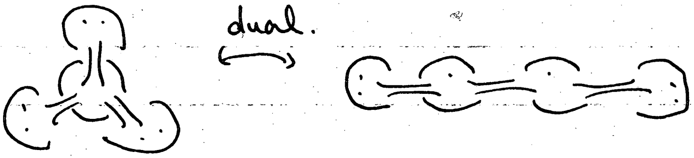
weak coupling.

$4\,c_1 c_2(\text{gauge})$ $\uparrow$ $4c_1 c_2(\text{gauge})$

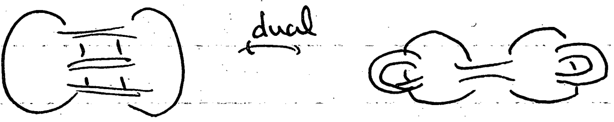
free vector gives $-8\,c_1 c_2(\text{gauge})$ } cancels.



strong coupling
||
weak coupling in another frame



$g=0$, $n=6$



$g=2$, $n=0$

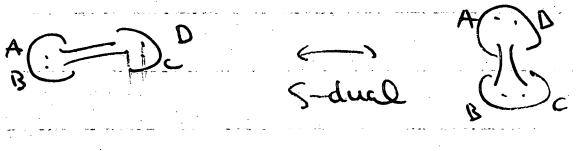
$$\text{anom. poly} = \left[(g-1)r + \frac{n}{2}(r-d)\right]\left(\frac{c_1{}^3}{3} - \frac{c_1}{12}p_1\right)$$

$$-\left[(g-1)\left(\frac{4}{3}h^\vee d + r\right) + \frac{n}{2}\left(\frac{4}{3}h^\vee d + r - d\right)\right] c_1 c_2$$

$$+\sum_i 2h^\vee c_1 c_2(\text{flavor sym assoc. to } i)$$

$\swarrow$ c_2 for general G is def'd so that it's the # inst and for $G=SU(2)$ it's just c_2.

$T_3 := T_{SU(3)}$ has $n_v=5$, $n_h=16$. with $SU(3)^3$ symmetry.



just as for $G=SU(2)$.

But what's this? It's some non-Lagrangian theory.

To get Lagrangian gauge theory, we need to introduce another operation...

No good single reference for this section.

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Partial Closure of punctures

~~CFT. of 6d $\mathcal{N}=(2,0)$ theory~~

~~②~~ ①

. In general, any 4d $\mathcal{N}=2$ SCFT with flavor symmetry F has the F current multiplet

$$\underset{SU(2)_R\ \text{triplet}}{\mu^a_{(ij)}} \quad \underset{SU(2)_R\ \text{doublet}}{\lambda^a_{\alpha i}} \quad J^a_\mu \qquad : \text{adjoint of } F.$$

e.g. Q^i_a $\tilde{Q}^a_i$: bifundamental of ~~S~~$U(N_1)\times U(N_2)$ $i=1\cdots N_1$ $a=1\cdots N_2$

$$\left.\begin{array}{ll} \mu^+ & Q^i_a \tilde{Q}^b_i \\ \mu^0 & (Q^\dagger)^b_i Q^i_a - \tilde{Q}^b_i (\tilde{Q}^\dagger)^i_a \\ \mu^- & (Q^\dagger)^a_i (\tilde{Q}^\dagger)^i_b \end{array}\right\} \quad SU(2)_R \text{ triplet.}$$

chiral

Kähler

antichiral

in $\mathcal{N}=1$ Language.

. So you can give a vev to μ^+ : $\langle\mu^+\rangle \neq 0$.

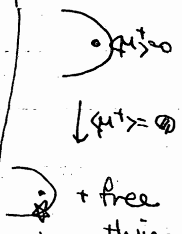
Here we only consider nilpotent vevs.

When $F=SU(N)$, upto (complexified) flavor rotation,

$$\langle\mu^+\rangle = \left(\begin{array}{ccc|cc|c} 0&1&&&& \\ &0&1&&& \\ &&0&&& \\ \hline &&&0&1& \\ &&&&0& \\ \hline &&&&&0 \end{array}\right) \quad \text{etc, sum of Jordan blocks.}$$
$$\underbrace{}_{n_1}\underbrace{}_{n_2}\underbrace{}_{n_3}$$

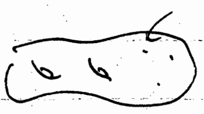
characterized by a partition of N
$N=n_1+n_2+\cdots n_k$

2015.2.3

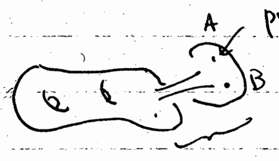


Let's analyse the simplest case. Take 6d th of type SU(2).

pick a puncture. Has SU(2) flavor sym. and the assoc μ^+.



let $\langle\mu^+\rangle = \begin{pmatrix} 0&1\\0&0 \end{pmatrix}$. what happens?



$+ \ \underset{SU(2)\ \text{gauge}}{} \ Q_{aiu} \leftarrow A$: trifund.
gauge B

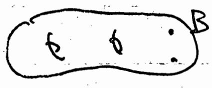
$\mu^+{}_{uv} = Q_{aiu} Q_{bjv} \varepsilon^{ab} \varepsilon^{ij}$

$\langle Q_{aiu} \rangle = \varepsilon_{ai} \delta_{u1}$ does the job.

$SU(2)_{gauge}$ broken completley $\Rightarrow$ eats 3 hypers.

~~one uneaten hyper~~ : $\delta Q_{aiu} = \varepsilon_{ai} \xi_u$.

$SU(2)_{gauge}$ index and $SU(2)_B$ identified via ε_{ai}.



+ one free hyper.

This can be done more abstractly:

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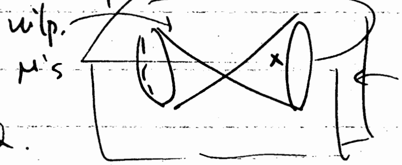
Partial closure of punctures ②

$\mu^+ = \begin{pmatrix} a & b \\ -c & -a \end{pmatrix}$ $\langle \mu^+ \rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ nilpotent $\Leftrightarrow a^2 + bc = 0 . \Leftrightarrow \mathbb{C}^2/\mathbb{Z}_2 .$

$SU(2)_R \times SU(2)_A$ act on (z, w) as $SO(4)$ $\longrightarrow$ cpx 2-dim'l.



$\begin{cases} a = zw \\ b = z^2 \\ c = w^2 \end{cases}$

In addition, ⇝ clearly ~~there'ae two~~ there's one free hyper.

$\langle \mu^+ \rangle$ is ~~the top component of $SU(2)_R$ triplet and the bottom component of $SU(2)_A$ triplet~~ (invariant under the diagonal combination.

$SU(2)_{R\,new} = SU(2)_R + SU(2)_A$

Recall one puncture has a contrib. to the anom. poly by

$$-1\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) - 3c_1 c_2(R) + 4c_1 c_2(A)$$

R_{new} is a diagonal combination. ⇝ $c_2(R) = c_2(A) = c_2(R_{new})$

$$\rightsquigarrow \; -1\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) + c_1 c_2(R_{new}).$$

$$\begin{array}{cccccc} & \phi & \psi & & \phi & \psi \\ & SU(2)_R \times SU(2)_A & SU(2)_A & \longrightarrow & SU(2)_{R\,new} & SU(2)_{R\,new} \\ & 2 \times 2 & 2 & & 1+3 & 2 \\ U(1)_R & 0 & -1 & U(1)_R & 0 & -1 \end{array}$$

gives this.

let's consider $G = SU(3)$. one full puncture has the contribution

$$-3\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) - \overset{13}{\cancel{12}}\, c_1 c_2 + \overset{6}{\cancel{4}}\, c_1 c_2(A)$$

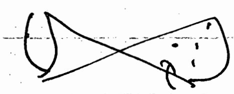
three choices of $\langle \mu^+ \rangle$:

① $\langle \mu^+ \rangle = 0$ ⇝ do nothing.

② $\langle \mu^+ \rangle = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$: invariant under $SU(2)_{R\,new} \overset{\mathrm{id}}{\subset} SU(2)_{R\,old} \times SU(3)_A$

embed via 3-dim irrep

$c_2(R) = c_2(R_{new})$
$c_2(A) = 4\,c_2(R_{new})$



$$\begin{array}{cc} \phi & \psi \\ & SU(2)_{R\,new} \\ & 2 \oplus 4 \\ U(1)_R & -1 \end{array}$$

$$-3\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) + (1+1+3^2)\, c_1 c_2$$

$$-3\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) + \overset{11}{\cancel{9}}\, c_1 c_2$$

agree !

⇝ nothing remains except free hypers.

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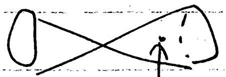
Partial Closure of punctures ③

③ $\mu^+ = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$ invariant under $SU(2)_{R_{new}} \subset SU(2)_{R_{old}} \times SU(3)_A$

id; embed via 2-dim map.



cpx 4 dim

$C_2(R) = C_2(R_{new})$

$C_2(A) = C_2(R_{new})$

ϕ $\quad\Big\{ \psi$

$SU(2)_{R_{new}}$

$2 \oplus 1 \oplus 1$

$\Big\{$

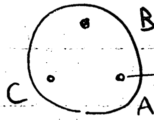
$$-2\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) + c_1 c_2 .$$

$$-3\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) - 7\, c_1 c_2(R_{new})$$

$\longleftrightarrow$ mismatch by

$$-\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) - 8\, c_1 c_2 .$$

Apply this analysis to $T_{SU(3)}$



$\rightarrow \langle \mu^+ \rangle = \begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}$

originally, $n_v = 5$, $n_h = 16$: $-11\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) \overset{2\times + \times}{-5} c_1 c_2 + 6 c_1 c_2(A)$

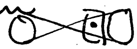
$$\downarrow$$

$$-11\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) + c_1 c_2(R_{new})$$

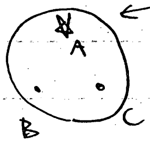
$$\shortparallel$$

$$-9\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) + \left[-2\left(\frac{c_1^3}{3} - \frac{c_1}{12}p_1\right) + c_1 c_2\right]$$

contrib. from



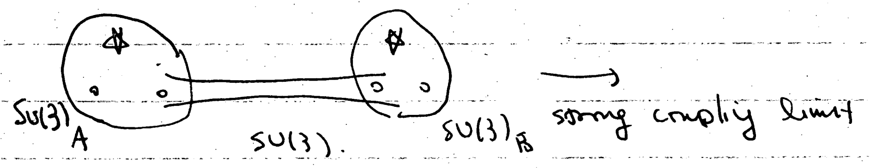
$SU(3) \times SU(3)$ bifundamental. $\Longleftarrow$ what remains is $n_v = 0$, $n_h = 9$, with $SU(3)_B \times SU(3)_C$ symmetry

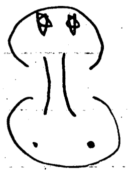


$\leftarrow$ set $\langle \mu_+ \rangle = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$, remove free hypers associated to the flavor rotation of $\langle \mu_+ \rangle$

Now we know the 6d realization of SU(3) with 6 flavors.

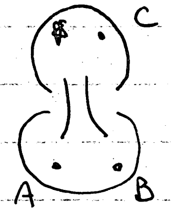
Take 6d th. of type SU(3).



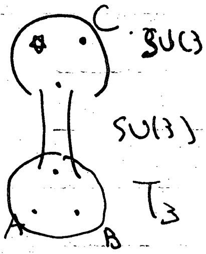


$\leftarrow$ what's this ?

We know



: this is

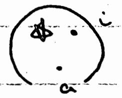


$SU(3)^2$ bifundamental.

we know need to partially close the puncture C. $\langle \mu^+ \rangle = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$.

2015/1/25

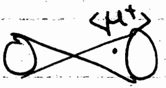
Partial Closure of punctures ④



$Q^a_i \; \tilde{Q}^i_a \quad a, i = 1.2.3 \quad ; \quad Q^a_i \tilde{Q}^j_a = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$

We can take the result for SU(2) trifundamental Higgsing ;

We see $SU(3)_g \xrightarrow[\text{breaks to}]{} SU(2)_{\text{gauge}}$.



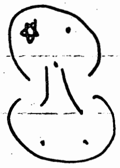
eats $8-3=5$ hyper.

← cpx 4 dim'l ⇝ 2 hyper

} 7 hypers to be removed.

originally $3^2=9$ hypers ⇒ 2 hypers remain. doublet of SU(2) gauge.

So

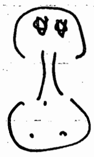


$SU(3)^2$ bifund

SU(3) gauge

T_3

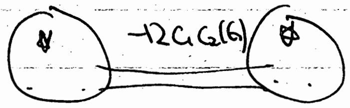
$\xrightarrow[\text{closure}]{\text{partial}}$



← one SU(2) doublet hyper.

← $SU(2)_{\text{gauge}} \subset SU(3)$

T_3



n_v	0	8	0	8
n_h	9	0	9	18

↑ SU(3) with 3+3 flavors

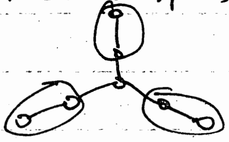
↑ not just $SU(3)^2$ sym ; SU(6) sym.

8	18	
0	2	$2C_1C_2(G)$
3	0	$-8C_1C_2(G)$
5	16	$6C_1C_2(G)$
n_v	n_h	

Then

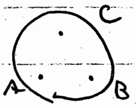
$SU(3)^2 \hookrightarrow SU(6)$

T_3 should have E_6 flavor symmetry.



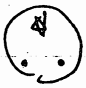
⟸ the same should be true for AB, BG & GA

<u>Generalization</u> $T_{SU(N)} = T_N$ theory. Setting $\langle \mu^+_c \rangle = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}$ 'erases' the puncture.

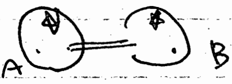


: has $SU(N)^3$ sym.

↓ set $\langle \mu^+_c \rangle = \left(\begin{array}{ccc|c} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \hline & & & 0 \end{array}\right) = [N-1, 1]$.

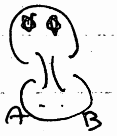


: $SU(N)^2$ bifund.

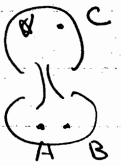


SU(N) with 2N flavors

$\xrightarrow{\text{dual}}$

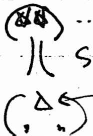


$\xleftarrow[C]{\text{Higgs}}$



$Q \; \tilde{Q}$

T_N



… a doublet

SU(2)

← [N−2, 1, 1]

2015 / 1 / 26

Cpt. of 6d $\mathcal{N}=(2,0)$ th and SCI ①

Recall the SCI of an 4d $\mathcal{N}=2$ SCFT:

$$\mathrm{SCI}_{p,q,t}(x_i) := \mathrm{tr}_{\mathcal{H}(S^3),\ \Delta=2j_2+2I_3+\frac{r}{2}} (-1)^F \quad p^{j_2+j_1} q^{j_2-j_1} t^{I_3} \left(\frac{t}{pq}\right)^{r/2} \prod x_i^{J_i} .$$

Essentially a supersymmetric $S^1\times S^3$ partition function.

For simplicity set $q=t$.

$$\mathrm{tr}(-1)^F e^{-2\beta(\frac{\Delta}{2}-j_2-I_3+\frac{r}{4})} \; p^{\frac{\Delta}{2}+j_1-I_3-\frac{r}{4}} \; q^{j_2-j_1+I_3}$$

$$//$$

$$\mathrm{SCI} = \mathrm{tr}_{\Delta=2j_2+2I_3+r/2} (-1)^F \; p^{j_2+j_1-r/2} \; q^{j_2-j_1+I_3} \; \prod x_i^{J_i}$$

① non negative

~~Note $\Delta \overset{(\geq)}{=} 2j_2+2I_3+\frac{r}{2} = (j_2-j_1+I_3)+I_3-(j_1+j_2-\frac{r}{2})$~~

② equal

~~$\Delta \geq -2j_1+2I_3-\frac{r}{2} = (j_2-j_1+I_3)+I_3+(j_1+j_2-\frac{r}{2})$~~

$\Downarrow$

~~$j_1+j_2-\frac{r}{2}=0 \rightsquigarrow p$ dependence drops out.~~

* Annihilated by 4 supercharges!

$$= \mathrm{tr}_{\mathcal{H}_{\mathrm{full}}} (-1)^F q^{\Delta-I_3} .$$

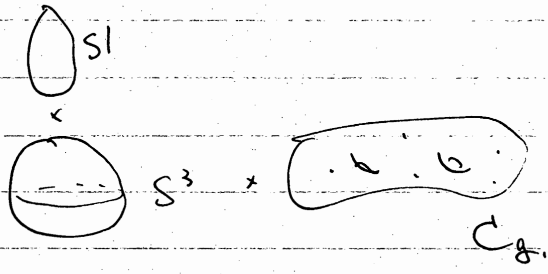
note $e^{-2\beta}$ and p play the same role after $j_2 \leftrightarrow -j_1$, $r \leftrightarrow -r$.

set $e^{-2\beta}=p=q$ $\Rightarrow$ both drop out.

Note $q = e^{-2\pi\frac{R_{S^1}}{R_{S^3}}}$.

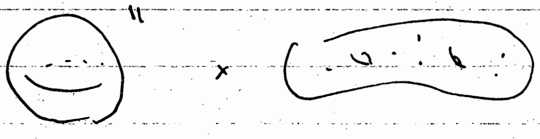
So, what's the SCI with single parameter q, of the 4D theory obtained by putting 6d th of type G on a Riem. surf C ?

6d on



with 4 sup. preserved

5d on



with 4 sup. preserved

with $g_{5d}^2 \propto R_{S^1}$

$\Downarrow$ honest computation. [F-K-M] ← In fact they did a slightly different partial twisting, but the part. F. is guaranteed to agree with this

$$\mathrm{SCI} = \frac{\prod_i N(a_i)}{N_0^{n+2g-2}} \sum_\lambda \frac{\prod_i \chi_\lambda(a_i)}{\chi_\lambda(q^{\frac{N-1}{2}} q^{\frac{N-3}{2}} \cdots)^{n+2g-2}}$$

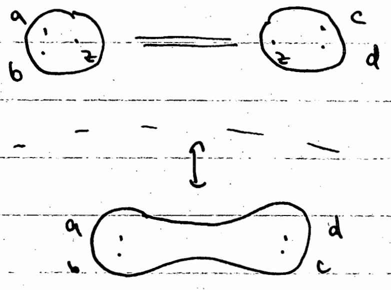
when $q = e^{-2\pi\frac{R_{S^1}}{R_{S^3}}}$.

unfortunately they didn't determine $N(a_i)$ & N_0.

But we can use some tricks to determine them.

Fukuda-Kawano-Mat
1210.2855

Cptf of 6d $\mathcal{N}=(2,0)$ th and SCI ③



$$\frac{1}{N!}\int\frac{dz_i}{2\pi i z_i}\prod\left(1-\frac{z_i}{z_j}\right)\cdot K(z)^{-2}\cdot\frac{N(a)N(b)N(z)}{N_0}\sum_{\lambda}\frac{\chi\,\chi\,\chi}{\chi_\lambda(q^\rho)}\cdot\frac{N(c)N(d)N(z)}{N_0}\sum_{\lambda'}\frac{\lambda\,\chi\,\chi}{\chi_\lambda(q^\rho)}$$

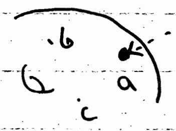
$$K(z)^{-1}=\prod_{n\geq0}\left[(1-q^{n+1})^{N-1}\prod_{i,j}\left(1-q^{n+1}\frac{z_i}{z_j}\right)\right]$$

$$\|$$

$$\frac{N(a)N(b)\,N(c)N(d)}{N_0^2}\sum_{\lambda}\frac{\chi(a)\,\chi(b)\,\chi(c)\,\chi(d)}{\chi_\lambda(q^\rho)^2}$$

$\rightsquigarrow$ $N(z)=K(z)$.

- To determine N_0, recall



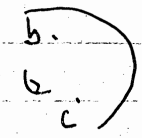
...has SU(N) symmetry. has μ.

Let $\langle\mu^+\rangle=\begin{pmatrix}0&1&&\\&\ddots&\ddots&\\&&&1\\&&&0\end{pmatrix}$.

$\Downarrow$

$SU(2)_{R\,\mathrm{old}}$ $\supset$ $SU(N)$ $\uparrow$ $SU(2)_{R\,\mathrm{new}}$ irrep id

effectively sets $(a_1\cdots a_N)=(q^{\frac{N-1}{2}},\cdots,q^{-\frac{N-1}{2}})=:q^\rho$



no puncture + some free multiplet

$K(a)$

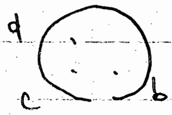
$\Big\downarrow a\to q^\rho$

$K_0^{-1}:=\prod_{d=2}^{N}\prod_{n\geq0}(1-q^{d+n})$ × contribution from this

- Therefore we should have

$$\frac{K(a)K(b)K(c)K(d)}{N_0^2}\sum_{\lambda}\frac{\chi(a)\chi(b)\chi(c)\chi(d)}{\chi(q^\rho)^2}$$

$\downarrow$

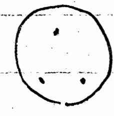


$$\frac{K(a)K(b)K(c)K_0}{N_0^2}\left(\sum_{\lambda}\frac{\chi(\overset{q^\rho}{a})\chi(b)\chi(c)\chi(d)}{\chi(q^\rho)^2}\right)\times\text{contribution}$$

$\|$ should be

$$\frac{K(b)K(c)K(d)}{N_0}\sum_{\lambda}\frac{\chi(b)\chi(c)\chi(d)}{\chi(q^\rho)^2}$$

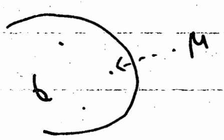
$\therefore$ $N_0=K_0$

$\rightsquigarrow$  's index $=\dfrac{K(a)K(b)K(c)}{K_0}\sum_{\lambda}\dfrac{\chi(a)\chi(b)\chi(c)}{\chi(q^\rho)}$

2015 / 1 / 27

Cpt'f of 6d $\mathcal{N}=(2,0)$ th and SCI ③

Using the same idea, you can determine $\overset{\text{SCI}}{\vee}$ of partial closures



$\langle\mu^+\rangle = \begin{pmatrix} 0 & \cdots & 1 \\ & & 0 \end{pmatrix} \xrightarrow{\qquad} [N-1,1]$

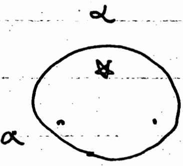
+ some free hypers

$\chi(a)K(a) \overset{\text{replace}}{\rightsquigarrow} \chi(q^{\frac{N-2}{2}}\alpha, \cdots, q^{-\frac{N-2}{2}}\alpha; \alpha^{1-N}) K_{[N-1,1]}(q)$

where $K(q^{\frac{N-2}{2}}\alpha, \cdots, q^{-\frac{N-2}{2}}\alpha; \alpha^{1-N}) = K_{[N-1,1]} \times$ contrib! from this

Concretely, $K_{[N-1,1]}(\alpha)^{-1} = \left[\prod_{d=2}^{N-1} \prod_{n\geq 0} (1-q^{d+n})\right] \prod_{\pm} \prod_{n\geq 0} (1-q^{\frac{N}{2}+n}\alpha^{\pm N})$

Then



's SCI $= \dfrac{K(a)K(b)K_{[N-1,1]}(\alpha)}{K_0} \sum_\lambda \dfrac{\chi_\lambda(a)\chi_\lambda(b^{-1})\chi_\lambda(q^{\frac{N-2}{2}}\alpha \cdots q^{\frac{2-N}{2}}\alpha, \alpha^{1-N})}{\chi_\lambda(q^\rho)}$

$\Updownarrow$ should equal, and indeed.

↑ should be just a bifund. hyper $Q^i_u \tilde{Q}^u_i$. SCI $= \prod_{u,i} \prod_{n\geq 0} \dfrac{1}{1-q^{\frac{1}{2}+n}(a_i/b_u)\alpha} \dfrac{1}{1-q^{\frac{1}{2}+n}(b_u/a_i)\alpha^{-1}}$

It is easy to check to order $q^{1/2}$.

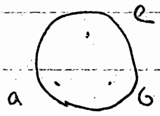
$K_{\cdot\cdot} = 1 + O(q)$.

$\lambda=0 \rightsquigarrow 1$. $\lambda = \square \rightarrow (\Sigma a_i)(\Sigma b_u^{-1}) q^{\frac{2-N}{2}}\alpha / (q^{\frac{1-N}{2}} + \cdots)$

$\bar{\square} \rightarrow (\Sigma a_i^{-1})(\Sigma b_u) q^{\frac{2-N}{2}}\alpha^{-1} / (q^{\frac{1-N}{2}} + \cdots)$

λ bigger : $O(q)$.

Or more simply: T_3



$= \dfrac{K(a)K(b)K(c)}{K_0} \sum_\lambda \dfrac{\chi(a)\chi(b)\chi(c)}{\chi_\lambda(q^1, 1, q^{-1})}$

let's compute it to order q^1.

$K(a) = 1 + \chi_{adj}(a) q + O(q^2)$. $K_0 = 1 + O(q^2)$

$\lambda = 0 \rightsquigarrow 1$.

$\lambda = \square \rightsquigarrow \chi_\square(a)\chi_\square(b)\chi_\square(c) / (q^{-1} + \cdots)$

$\bar{\square} \rightsquigarrow \chi_{\bar\square}(a)\chi_{\bar\square}(b)\chi_{\bar\square}(c) / (q^{-1} + \cdots)$

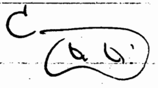
λ : bigger $\rightsquigarrow O(1)/(q^{-2} + \cdots) \sim O(q^2)$.

E_6 adj decomposition under $SU(3)^3$ $\Downarrow$

$\Rightarrow 1 + \left[\chi_{adj}(a) + \chi_{adj}(b) + \chi_{adj}(c) + \chi_\square(a)\chi_\square(b)\chi_\square(c) + \chi_{\bar\square}(a)\chi_{\bar\square}(b)\chi_{\bar\square}(c)\right] q + O(q^2)$.

2015/1/27

Other topics in class S theory ①

So far we considered the cptf. of 6d $\mathcal{N}=(2,0)$ th on  C with punctures. preserving $\mathcal{N}=2$ SUSY in 4d. Now known as <u>class S theories</u>.

<u>Further works</u> and/or other

✱ More detailed properties such as
- Seiberg-Witten curves
- chiral ring relations (SCI just counts operators)

$\mathrm{tr}\,\mu_A{}^k = \mathrm{tr}\,\mu_B{}^k = \mathrm{tr}\,\mu_C{}^k \quad \forall k$ etc.

- S^4 partition function, $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ part func.
  $\rightsquigarrow$ Liouville / Toda on C, conformal blocks on C
- '2d' chiral algebras obtained by restricting on $\mathbb{R}^2 \subset \mathbb{R}^4$;
  has ~~the~~ structure of holo. side of 2d CFT.
  s.t. the ~~cylinder~~ torus part. func = $\mathrm{SCI}_{q=t}$
- $S^3/\mathbb{Z}_k \times S^1$; $S^4/\mathbb{Z}_k$; $\mathbb{RP}^4$ etc.

✱ Cptf. preserving only $\mathcal{N}=1$ in 4d
- ~~class~~ 6d S th has $Sp(2)_R \simeq SO(5)_R$ symmetry.
  to cancel the curvature of C, we used the decomp

$$SO(5)_R \supset \underset{\uparrow}{\underbrace{SO(2)_R}} \times SO(3)_R$$

embed $SO(2)_{\mathrm{rot}}$

If we do $SO(5)_R \supset SO(4)_R \times SO(1)_R$

$$SO(4)_R = \underbrace{SU(2)_R} \times \underline{SU(2)_L} \quad \text{and}$$

embed $SO(2)_{\mathrm{rot}}$ → $SU(2)_R$; ↑ $SU(2)_L$: can add nontrivial flux in the Cartan without breaking $\mathcal{N}=1$

this preserves just 4d $\mathcal{N}=1$.

✱ other punctures.
- we considered just the
  full $\langle\mu\rangle = 0$
  nothing $\langle\mu^+\rangle = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 0 \end{pmatrix}$
  simple $\langle\mu^+\rangle = \left(\begin{array}{c|c} 0 \; 1 \; \ddots \; 0 & \\ \hline & 0 \end{array}\right)$
  intermediate ones ; analogues in type D, E $\Rightarrow$ newly found S-W curves of Lagrangian gauge theories
- irregular punctures

Many things to do! TN quite well understood for BPS purposes.

2015 / 1 / 30

5d $\mathcal{N}=1$ gauge theories

we used the following relation:

$$\left\{\begin{array}{l} \text{6d } \mathcal{N}=(2,0) \text{ th of type } G = A_n, D_n \text{ or } E_n \\ \qquad \downarrow S^1 \\ \text{5d } \mathcal{N}=2 \text{ SYM with gauge group } G. \end{array}\right.$$

$\mathcal{N}=1$ " SYM with with an adjoint hyper.

What does this procedure exactly mean?

When we cptyfy D-d scalar th. massless

$\downarrow S^1$, coordinate x^D, radius R

(D-1)-d scalar th,

$$\phi(\vec{x}', x^D) = \sum \phi_{(n)}(\vec{x}')\, e^{n i x^D / R}$$

$\hookrightarrow$ 4d mass $= \left|\frac{n}{R}\right|$: KK tower.

n is a conserved charge assoc'd to S^1 rotation.

In this particular case,

$\frac{1}{g^2_{5d}} \propto \frac{1}{R}$; instanton  $\mathbb{R}^4 \times \uparrow \mathbb{R}_t$ has mass $\sim \frac{1}{g^2}\underbrace{\int \mathrm{tr}\, F\wedge F}_{n \times \text{same const}}$

$\sim \frac{n}{R}$: behaves like a KK mode.

Assuming they match, the precise relation is

$$\frac{8\pi^2}{g^2_{5d}} = \frac{1}{R} = \frac{2\pi}{L}.$$

KK charge current = inst number current.

$J_\mu^{KK} = \epsilon F_{\mu\nu} F_{\rho\sigma}$.

Empirical Fact

At least at the BPS level, all the KK modes from 6d $\mathcal{N}=(2,0)$ to 5d is accounted for by the instantons.

One should not add KK gauge/scalar fields; they give overcounting etc.

Comments

1. Status for the non-BPS quantities is not clear.
2. Any 6d action that contains gauge fields that reduce to 5d g. f. has big problems: they need to eliminate BPS parts of the KK tower fields.

How about other $\mathcal{N}=1$ gauge theories in 5d?

2015 / 11 / 30

5d $\mathcal{N}=1$ gauge theories ②

- Consider SU(2) + N_f hypermultiplets in the doublet.
  (classically, $2N_f+1$ half-hypers is also possible,
  but this has $\pi_5(SU(2)) = \mathbb{Z}_2$ anomaly.)
  τ ホント? ⇒ App. 2
- Classically has ~~S~~O($2N_f$) symmetry.
- 'theta angle' associated to $\pi_4(SU(2)) = \mathbb{Z}_2$ $\theta = 0$, π
- parity part of O($2N_f$) shifts the theta angle by π.
  ⇝ when $N_f \geq 1$, just one choice with SO($2N_f$) symmetry.
  when $N_f = 0$, two choices, $\theta = 0$ or π.
- vector multiplet contains $\underset{\substack{\uparrow \\ \text{real scalar}}}{\Phi}$ λ A_μ.

● can give a supersymmetric vev $\Phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$. $a \sim -a$ equivalent. assume $a>0$.
$a \neq 0$ breaks SU(2) → U(1).

. U(1) coupling at one loop $\propto \dfrac{1}{g^2_{YM}} + (8 - N_f)\,a$.
↑ what's the source of this factor 2 ?

cf. in 4d $\dfrac{1}{g^2_{YM}} + (4-N_f)\log\dfrac{a}{\Lambda_0}$

β-func contrib.		e.g. 4d $\mathcal{N}=4$	4d $\mathcal{N}=2$ hyp	vec
				5d $\mathcal{N}=1$ vec
real scalar	1	6	4	2
real spinor	dim spin.	4×4	2×4	2×4
vector+ghost	d−26	4−26		4−26

⇝ d=4 $\mathcal{N}=4$ multiplet gives zero one-loop in any dimension.

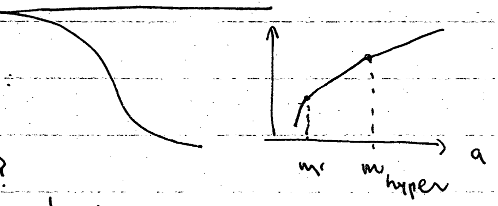
$$\delta\left(\frac{1}{g^2_{YM}}\right) = \underbrace{(\text{spin dep coeff})}_{\substack{\text{doesn't change} \\ \text{upon dim reduction.}}} \times (\text{dim dep coeff}) \times |\text{mass}|^{d-4} \times \text{tr (gauge rep)}^2.$$

SU(2) adj, SU(2) vec } diff by fac 2.

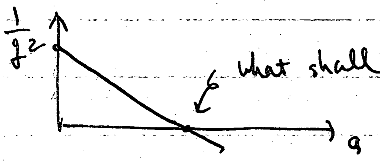
●

. this is one-loop exact : $a\, F_{\mu\nu} F_{\mu\nu} \overset{\text{related}}{\underset{\text{SUSY}}{\longleftrightarrow}} A_\mu F_{\nu\rho} F_{\sigma\tau} \epsilon^{\mu\nu\rho\sigma\tau}$

$f(a) F_{\mu\nu} F_{\mu\nu} \longrightarrow \dfrac{\partial f(a)}{\partial a} A_\mu F_{\nu\rho} F_{\sigma\tau} \epsilon^{\cdots}$: not gauge inv !

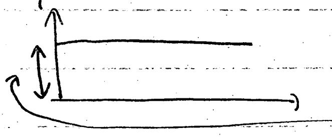


. theory with $N_f > 8$ looks quite sick :

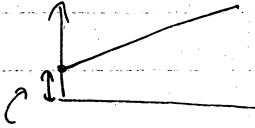


what shall we do ?
you can make it worse, but not better.

. theory with $N_f = 8$

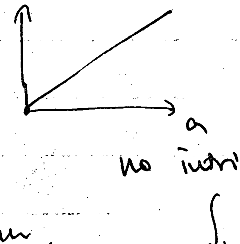


$N_f < 8$



tune

$N_f < 8$



mass param in the Lagrangian.

no intrinsic scale.
↓
believed to be a SCFT

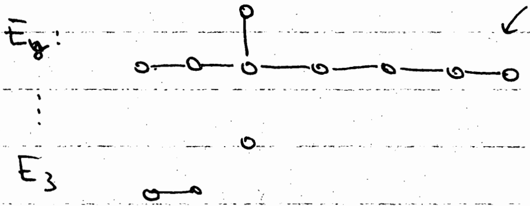
↓
rather similar to $\mathcal{N}=2$.
believed to come from 6d $\mathcal{N}=(1,0)$.

2015.1.30

5d $\mathcal{N}=1$ gauge theories ③

5d SU(2)	$N_f=8$	: SO(16) × U(1)$_{\text{inst}}$	← 6d $\mathcal{N}=(1,0)$,	E_8 flavor sym
	$N_f=7$	: SO(14) × U(1) inst	← 5d $\mathcal{N}=1$	E_8 theory
	6	12		E_7
	5	10		E_6
	4	8		E_5 = SO(10)
	3	6		E_4 = SU(5)
	2	4		E_3 = SU(3) × SU(2)
	1	2		E_2 = U(1) × SU(2)
	0, $\theta=0$			E_1 = SU(2)
	$\theta=\pi$			$\tilde{E}_1$ = U(1).

cf. E_8:



lower E_n is obtained by chopping this branch off.

cf. Seiberg 9608111

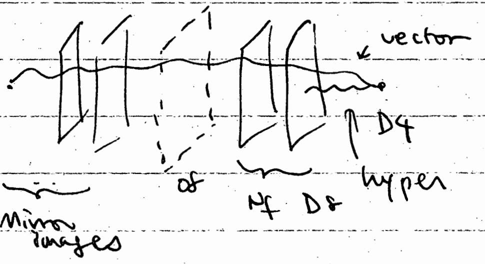
cf. U-duality group of type II on T^d

$$\begin{tikzcd} \text{IIA} \; \mathbb{Z}_2 \arrow[dr] & & & & & & & \\ & 9d \arrow[r] & 8d \arrow[r] & 7d \arrow[r] & 6d \arrow[r] & 5d \arrow[r] & 4d \arrow[r] & 3d \\ \text{IIB} \; SL(2,\mathbb{Z}) \arrow[ur] & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 & E_8 \end{tikzcd}$$

$E_2 = SL(2,\mathbb{Z}) \times \mathbb{Z}_2$, $E_4 = SL(5)$

This comes from combining $\overset{\text{T-dual}}{SO(\overset{10-}{d},\overset{10-}{d})}$ and $SL(2,\mathbb{Z}) \supset \mathbb{Z}$: axion shift by 1. : the rest doesn't commute with T-duality.

- Coming back to the 5d gauge th question. the UV structure was first found by considering string theory: in type IIA, O8 + N_f D8 ∽ probe via 1 D4.



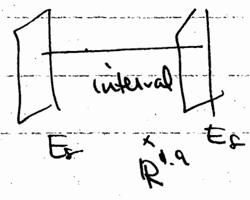
: SU(2) with N_f flavors.

Now,

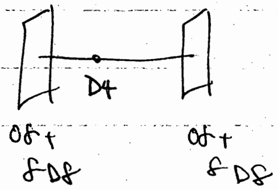
O8+N_fD8 is known to have E_{N_f+1} sym.

How did people know it?

Heterotic M th.



S^1 cpt'f ⇝ IIA



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5d $N=1$ gauge th.

④

Today we'd like to see this symmetry enhancement more field theoretically.

In 4d $N=2$ or 5d $N=1$, every mass deformation comes from a flavor symmetry.

- Hyper $Q, \tilde{Q}$ mass term $\int d^2\theta\, m Q\tilde{Q}$

cf. YT 1501.01031

both in 4d/5d. $\uparrow$ vev of a background U(1) gauge supermultiplet. $\quad$ " $\int d^2\theta \langle\Phi\rangle Q\tilde{Q}$

$Q\tilde{Q}$

$Q^\dagger Q - \tilde{Q}^\dagger\tilde{Q}$ $\quad$ ferm. cmp. $\quad$ J_μ $\qquad$ $X=\int d^4\theta (Q^\dagger Q - \tilde{Q}^\dagger\tilde{Q})$

$\tilde{Q}^\dagger Q^\dagger$

~~scalar~~ scalar — ferm. — vector : spacetime : scalar

~~scalars~~ vector — ferm. — scalar : $SU(2)_R$: scalar

- 5d gauge coupling $\int d^4\theta \frac{1}{g^2} W_\alpha W^\alpha$ is also a mass term.

corresponding current : $\mathrm{tr} F_{\mu\nu} \mathrm{tr} F_{\rho\sigma}\, \varepsilon^{\mu\nu\rho\sigma\tau}$.

$\lambda_\alpha^{(i} \lambda_\beta^{j)} J^{\alpha\beta}$ $\quad$? $\quad$ $J_\mu = \mathrm{tr} F\wedge F$ $\qquad$ $X = \mathrm{tr} F_{\mu\nu} F_{\mu\nu}$

- so, the relation between

E_{N_f+1} symmetric SCFT and $SU(2)$ with N_f flavors

$U(1)_{\text{inst}} \times SO(2N_f)$

is that J_μ^a : E_{N_f+1} current.

pick one component $J_\mu^{a=1}$

take the superpartner X $\qquad$ $\delta L = \mu X$ $\rightsquigarrow$ in the IR becomes a gauge th.

components commuting with $J_\mu^{a=1}$

are $J_\mu^{a=1}$ itself $\leftrightsquigarrow$ $U(1)_{\text{inst}}$

others $\leftrightsquigarrow$ $SO(2N_f)$.

What happens to the broken flavor symmetry currents?

$\partial_\mu J_\mu^b \propto$ variation by the b-th generator of $\delta L = \mu X^{a=1}$

$\propto$ var. by the $a=1$ st gen of μX^b

$\propto \mu X^b$ $\quad$ (if $a=1$ is taken to be a Cartan and b corresponds to a root)

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5d $\mathcal{N}=1$ gauge ⑤

so this is just a rearrangement within the original multiplet.

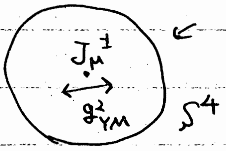
BPS mult. still BPS $\rightsquigarrow$ we should be able to see them in the IR.

$\underline{N_f=0}$ $\overset{\text{IR}}{SU(2)}$ $\underline{\theta=0}$, $U(1)$ inst $\longleftarrow$ $\overset{\text{UV}}{SU(2)}$

J^0_μ : conserved $\qquad J^0_\mu, J^\pm_\mu$

broken current superm. $\left\{\begin{array}{l} J^\pm_\mu : \text{broken. should have inst. number } \pm 1. \\ \psi_{\alpha i} \\ M_{(ij)} \end{array}\right.$

to see them in the IR?



$\leftarrow \int_{S^4} \mathrm{tr}\, F\wedge F = \pm 1.$

Classically this config breaks all the SUSY.

$\rightsquigarrow$ $\lambda_{\alpha i}$: fermion zeromodes from gauginos. $\quad 4\times 2$, $\overset{\text{sympl.}}{\text{maj.}}$ cond. $\rightsquigarrow$ real 8 components

Radial quantization should be done with the condition

$$\{\lambda_{\alpha i}, \lambda_{\beta j}\} = J_{\alpha\beta}\,\epsilon_{ij} \qquad \rightsquigarrow 2^{8/2} = 16 \text{ comp.}$$

$\searrow$ act on the $\underset{\overset{\shortparallel}{1\otimes 5\oplus 3\otimes 1}}{8_s} \oplus \underset{\overset{\shortparallel}{4\otimes 2}}{8_c}$ of $SO(8)_{\text{aux}}$: exactly the correct spect. of a broken current mult!

λ: $\underset{\overset{\shortparallel}{4\otimes 2}}{8_v}$ of $SO(8)$ aux.

$\underline{N_f=0,\ \theta=\pi}$ one-inst is projected out. no enh. (from this sector.)

$N_f>0$: $SU(2)$ gauge doublets $q_{1\cdots 2N_f}$ give

spinless zeromodes $b_{1\cdots 2N_f}$

$$\{b_a, b_b\} = \delta_{ab} \qquad a, b = 1\cdots, 2N_f$$

$\rightsquigarrow$ inst. ops $=$ (broken current mult) $\otimes$ (Dirac spin of $SO(2N_f)$).

note: inst bkg only breaks $SU(2)\to SO(3)$, $\mathbb{Z}_2$ unbroken.

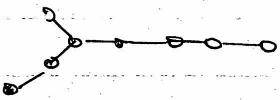
$b_{a,b}$ odd under $\mathbb{Z}_2$.

acts on the flavor spinors as Γ^{2N_f+1}

$SU(2)$ gauge inv $\Rightarrow$ only pos. chirality spinor remain.

$N_f=7$ $\quad SO(2N_f)=SO(14) \quad 91 \quad 64_+$

$\qquad U(1)$ inst. $\quad 1 \quad 64_-$



$\rightsquigarrow$ only possibility is E_8. ($14_{++}, 14_{--}$ automatic.)

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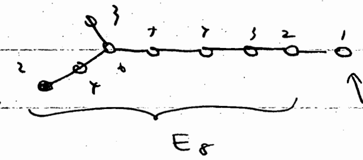
5d $\mathcal{N}=1$ gauge ⑥

$N_f < 7$: similar. get E_{N_f+1}

$N_f = 8$:

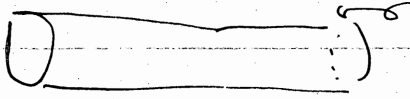


$E_9 = \hat{E}_8$.

Flavor symmetry is an affine symmetry !

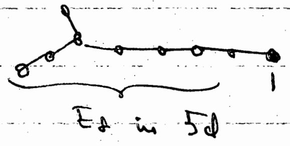
⇝ existence of additional S^1 in the sp. time

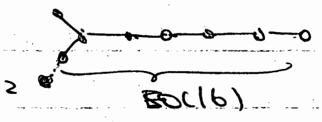
(, $N_f > 8$: E_{N_f+1} is Kac-Moody but hyperbolic ... quite sick.)



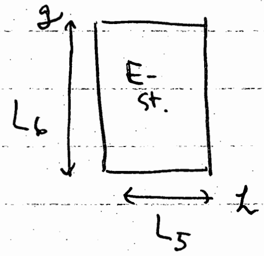
⇝ 6d $\mathcal{N}=(1,0)$ th with E_8 symmetry, called the '6 E-string'

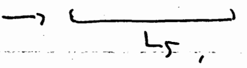
↓ cptf without any hol → cptf with hol, $\mathbb{Z}_2 \subset E_8$ given by $\overset{\shortparallel}{g}$





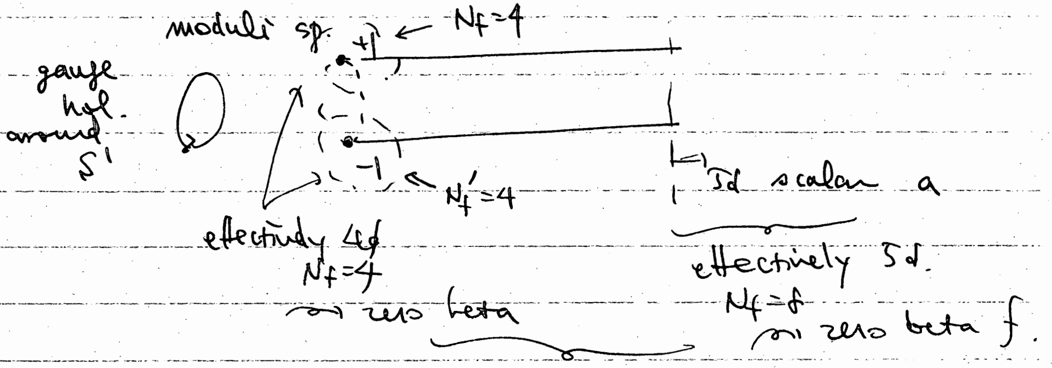
Now, consider $\text{diag}(+^8 -^8) \in SO(16)$ and put the 5d th on S^1 with h.
$\overset{\shortparallel}{h}$



SU(2) with $N_f=8$ →  → 4d SU(2) with $\tau = \frac{L_5}{L_6}$ with $N_f=4$.

$SO(16) \supset SO(8)\times SO(8)$

moduli sp.



⇒ 4d SU(2) with $N_f = 4$ is S-dual with itself.

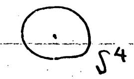
How do we see the triality action on SO(8) ?

$$\begin{tikzcd} & E_8 \arrow[dl, "g"'] \arrow[d, "h"] \arrow[dr] & \\ SO(16) \arrow[dr, "h"'] & SO(16)' \arrow[d, "g"] & SO(16)'' \arrow[dl] \\ & SO(8)\times SO(8) & \end{tikzcd}$$

$$248 = \underbrace{\boxed{\begin{matrix} 28 + 28' \\ + 8_v \times 8_v \end{matrix}}}_{} \; SO(16) \\ + 8_s \times 8_s \\ + 8_c \times 8_c$$

5d $\mathcal{N}=1$ gauge ⑦

<u>$\mathcal{N}=2$ SU(2)</u>



zero modes $\lambda_{\alpha i}$ $\alpha=1\cdots4$ 4 of SO(5) spacetime.
$i=1\cdots4$: 4 of $Sp(2)_R$

$$\{\lambda_{\alpha i}, \lambda_{\beta j}\} = J_{\alpha\beta} J_{ij}$$

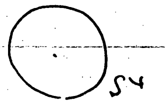
⤳ inst op	SO(5)	$Sp(2)_{IR}$		
	sym. traceless		14	$T_{(\mu\nu)}$
		sym. traceless	14	$\mu_{(ab)}$
	adj	vec	50	
	vec	adj	50	$J_\mu^{[ab]}$
	16	4	64	
	4	16	64	

KK mode of 6d e.m. tensor. →

KK mode of 6d $Sp(2)_{IR}$ current →

$\mathcal{N}=1$ SU(N) ⋯ + adj ⇒ $\mathcal{N}=2$

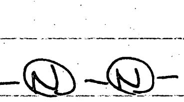
+ fund hypers. CS term $k\, tr(AFF + \cdots)$



one-inst config = essentially SU(2) instanton ⊂ SU(N).

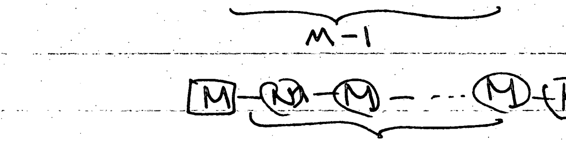
can be analyzed similarly. broken to U(1) × SU(N−2)

SU(N) with 2N flavors : SU(2N) × $U(1)_{inst}$ × $U(1)_{baryon}$.



$U(1)_{inst} \pm U(1)_{baryon}/N$

enhance to $SU(2)_\pm$

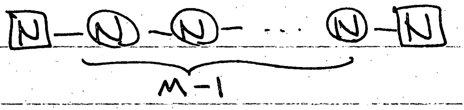


$U(1) \times SU(2N-N') \times \dfrac{U(1)_{inst}}{U(1)_{bar}} \times \dfrac{U(1)_{inst'}}{U(1)_{bar}} \times SU(2N'-N)$

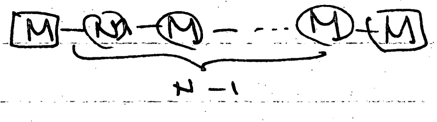
U(1): rotates every hyper by +1.

can be shown to enh. to $SU(3)_\pm$

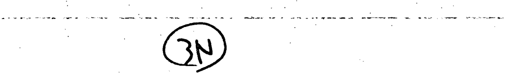
Then:



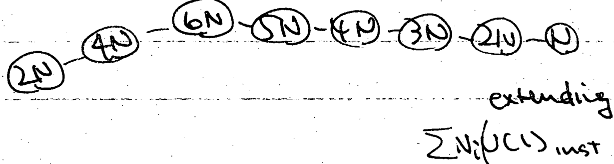
both have $SU(N)^2 \times SU(M)^2$ in the UV.



believed to be the same UV th.



: $SU(N) \times SU(N)^2$: 5d ver. of T_N.



: $\hat{E}_8 \times \hat{E}_8$ symmetry.

extending node of both is the same :

$$\sum_{\text{nodes}} N_i \left(U(1)_{inst} \pm \frac{U(1)_{bar}}{N_i} \right) = \sum_{\text{nodes}} N_i\, U(1)_{inst}$$

6d $\mathcal{N}=(1,0)$ theories

We saw a few (1,0) theories already:

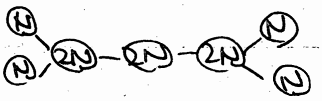
$$\begin{tikzcd}
\text{$\mathcal{N}=(2,0)$ theories of type $G=A_n, D_n, E_n$} \arrow[r] \arrow[d, "S^1"] & \text{tensor branch with rank G tensors} \arrow[d, "S^1"] \\
\text{$\mathcal{N}=2$ SYM with gauge group G} \arrow[r] & U(1)^{\text{rank } G}
\end{tikzcd}$$

$$\begin{tikzcd}
\text{E-string theory} \arrow[r] \arrow[d, "S^1"] & \text{rank 1 tensor br.} \arrow[d, "S^1"] \\
\text{$SU(2)$ with $N_f=8$} \arrow[r] & U(1)
\end{tikzcd}$$

$\boxed{SO(12)} - Sp(2) - SO(12) - \cdots - Sp(2) - \boxed{SO(12)}$

$$\begin{tikzcd}
\text{$\mathcal{N}=(1,0)$ th. with $G\times G$ flavor sym.} \arrow[r] \arrow[d, "S^1"] & \text{? + $2N-1$ tensors} \arrow[d, "S^1"] \\
\text{affine Dynkin quiver theory} \arrow[r] & ?
\end{tikzcd}$$

e.g. $G=D_6$



$10N-7$
$= 4(N-1)+3(2N-1)$
$= (2N-1)+2([illegible]N [illegible])$
$+ 6([illegible]N-[illegible])$

These have 'stringy' realizations.

$\mathcal{N}=(2,0)$ type A_{n-1} ← n M5s
type D ← M orientif + n M5s
$\mathcal{N}=(2,0)$ type $G=A.D.E$ ← IIB $\mathbb{C}^2/\Gamma_G$
E-string ← 1 M5 at the E.O.W. brane
of rank Q ← Q M5 at
$\mathcal{N}=(1,0)$ th with $G\times G$ ← M on $\mathbb{C}^2/\Gamma_G$ + N M5s

Many more: $\left\{\begin{matrix} \text{M} \\ \text{IIB} \\ \text{type I} \end{matrix}\right\}$ on $\mathbb{C}^2/\Gamma_G$ (including Γ = trivial) with N $\left\{\begin{matrix} \text{M5} \\ \text{NS5} \\ \text{D5} \end{matrix}\right\}$ $\left\{\begin{matrix} \text{with} \\ \text{without} \end{matrix}\right\}$ more structu

also: F-theory on singular CY_3
~ IIB with varying dilaton on a singular B_2

I don't have a coherent story to tell you yet.

2015. 2. 4

6d $\mathcal{N}=(1,0)$ theories ②

free multiplets : $\begin{cases} \mathcal{N}=(1,0) \text{ hyper} & \Phi^{4} \quad \psi^{+\,4} \\ \text{tensor} & \phi^{1} \quad \psi^{+\,4} \quad B_{\mu\nu}^{+\,3} \\ \text{gauge} & \psi^{-\,4} \quad A_{\mu}^{4} \end{cases}$

What are the 6d $\mathcal{N}=(1,0)$ SCFTs with rank-1 tensor branch ?

$\mathcal{N}=(2,0)$ type SU(2) $\Big\{$ $\xrightarrow{\text{tensor br.}}$ 1 tensor + 1 hyper

$\mathcal{N}=(1,0)$ E-string rank 1 — 1 tensor

How about 1 tensor + 1 gauge + ? hyper ?

cf. Seiberg 9609161
Bershadsky–Vafa 9703167

- Let's consider the simplest case SU(2).

gauge mult. has the $\overset{\wedge}{}$ gauge anomaly $-\frac{1}{24}\left[(2x)^4+(-2x)^4\right] = -\frac{32}{24}c_2{}^2$.

adjoint hyper : $+\frac{32}{24}c_2{}^2$ $\Rightarrow$ $\mathcal{N}=(1,1)$ gauge th.

$\boxed{\frac{1}{g^2}}\,\mathrm{tr}F^2$ $\leftarrow$ dimensionfull. UV: "little string"
non-gravitational
non-field theory ??

half hyper in the doublet : $\frac{1}{24}\cdot\frac{1}{2}\cdot\left[x^4+(-x)^4\right] = \frac{1}{24}c_2{}^2$

tensor $dH = I_4$, $H=*H$ $\Big\}$ $\delta(\text{anom}) = +\frac{g}{2}I_4{}^2$

g is related to the k.m. tensor of H $T_{\mu\nu} \supset g H_{\mu\cdot\cdot}H_{\nu}{}^{\cdot\cdot}$

Setting $dH=c_2$ (i.e. the SU(2) instanton-string has charge 1)

- the total anomaly is $\frac{g}{2}c_2{}^2 + \frac{2N_f}{24}c_2{}^2 - \frac{32}{24}c_2{}^2$.

$2N_f=32$, $g=0$ $\rightsquigarrow$ anom. free without tensor ; $\frac{1}{g^2}$ dimfull cnst.
I don't know if $\exists$ 'little string'.

$0 \le 2N_f < 32$ ··· only $2N_f=8$ & 20 allowed.

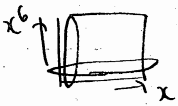
Two ways to see this :

① $\pi_6(SU(2)) = \mathbb{Z}_{12}$ ··· global anomaly ! adjoint : 32 mod 12
half doublet : 1 mod 12

$\rightsquigarrow$ $2N_f = 8$ or 20

I don't know why they give the same constraint ...

② Dirac quantization law in 6d.



$\rightsquigarrow$ an el. particle & a mag. particle in 4d. $\leftrightarrows$ angular momentum carried by EM field $=\frac{\hbar}{2}g$

$\rightsquigarrow$ g needs to be an integer.

$\Rightarrow \begin{cases} g=1, & 2N_f=20 \\ g=2, & 2N_f=8 \end{cases}$.

2015.2.1

6d $\mathcal{N}=(1,0)$ theories ③

$dH = \mathrm{tr} F \wedge F \rightsquigarrow L > \underset{\sim}{\phi}\, \mathrm{tr} F_{\mu\nu} F_{\mu\nu}$
tensor multiplet scalar

$\langle\phi\rangle$ controls i) gauge coupling
ii) and therefore the tension of the instanton-string.

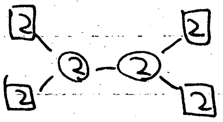
$\langle\phi\rangle = 0$ can be a strongly-coupled SCFT, where strings become tensionless.

SU(2) with $2N_f = 8$, $2N_f = 20$ + one tensor : both known to be realized in M-theory

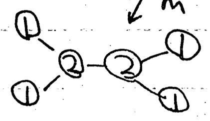
$2N_f = 8 \Leftarrow$ 2 M5s on $\mathbb{C}^2/\mathbb{Z}_2$ ⟶ on S^1

$2N_f = 20 \Leftarrow$ 1 M5 on $\mathbb{C}^2/\Gamma_{D_5}$

↙ on S^1

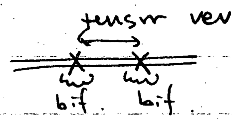


⟵ ⇡

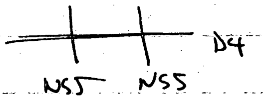


2 flavors for 'SU(1)' gauged
becomes 2 flavors for SU(2) ungauged via stringy effect

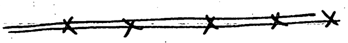
$\underline{2N_f = 8}$ 2 NS5s on 2 D6s



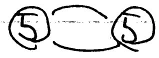
cf. in smaller dimensions the distance between NS5s is not a dynamical field



generalize
N NS5s on 2D6s



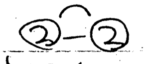
$\boxed{2}$ – ② – ② – ② – ② – $\boxed{2}$ + 4 tensors, say. $\xrightarrow{\text{go to the origin, put on } S^1}$



● when $N>2$, fl. sym $\underset{\text{at least}}{\wedge}$ SU(2) × SU(2)

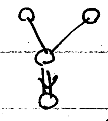
has fl. sym $\underset{\text{at least}}{\wedge}$ SU(2) × SU(2).

when N=2, $\boxed{2}$–②–$\boxed{2}$ has SO(8).



is special

$\widehat{SO(7)} \rightsquigarrow$



has $E_{4+1} = SO(10) = 28^0 + ①^0 + 8_s^+ + 8_s^-$

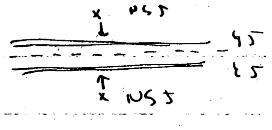
$\cup$

$SO(8) \supset SO(5) \times \underset{\sim}{SO(3)}$ ⑤+3 ⑤+3

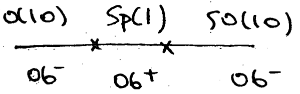
How can I see further enh. to $\widehat{SO(8)}$? SO(7) remain ⟵ where $8_v \to 4 \times 2$. gauged

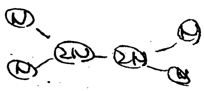
$28 \to$ ⑩ + 3 + 15.

$\underline{2N_f = 20}$ 1 'NS5' on 5 D6 on O6



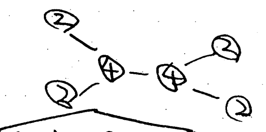
$\rightsquigarrow$





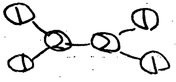
generalize {

$\boxed{SO(10)} - \underbrace{\text{Sp(1)} - \text{SO(10)} - \text{Sp(1)}}_{2N-1} - \boxed{SO(10)}$ $\xrightarrow[\text{put on } S^1]{\text{go to the origin}}$



has $\overbrace{SO(10) \times SO(10)}$.

$\boxed{SO(10)}$–(Sp(1))–$\boxed{SO(10)}$ has SO(20).



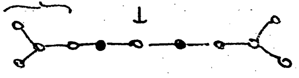
is special

$\underbrace{\boxed{4} - ② - ②}_{E_{6+1}} - \boxed{4}$

$\cup$

$SU(2) \times SO(12)$

$\underbrace{SO(8)}$ of $\boxed{4}$ ② of bifund



$\Leftarrow \widehat{D_{10}}$.

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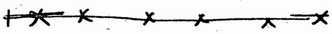
6d $\mathcal{N}=(1,0)$ theories

④

$\mathbb{C}^2/\mathbb{Z}_k$ N M5s 

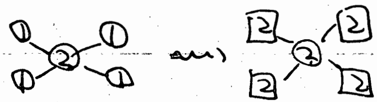
$\boxed{k}-(k)-(k)-(k)-(k)-(k)-\boxed{k}$

$\mathbb{C}^2/\Gamma_{D_n}$ N M5s $\boxed{SO(2k)}-(Sp(k-4))-(SO(2k))-(Sp(k-4))-(SO(2k))-\cdots-(Sp(k-4))-\boxed{SO(2k)}$

| or: $\boxed{SO(2k)}-\bullet-(SO(2k))-\bullet-(SO(2k))-\cdots-\bullet-\boxed{SO(2k)}$

↑ 6d SCFT for 1 M5 on $\mathbb{C}^2/\Gamma_{D_n}$

n=4, for 1 M5 : 'Sp(0) with 8 flavors'

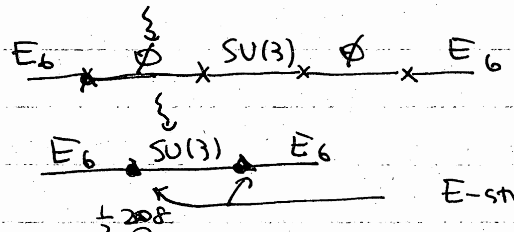
on S^1 ⇝ 

: SU(2) with 8 fl.

⇝ E-string!

● $\mathbb{C}^2/\Gamma_{E_6}$: 1 M5 $\boxed{E_6}-\bullet-\boxed{E_6}$

cf. del Zotto - Heckman - Tomasiello - Vafa 1407.6359



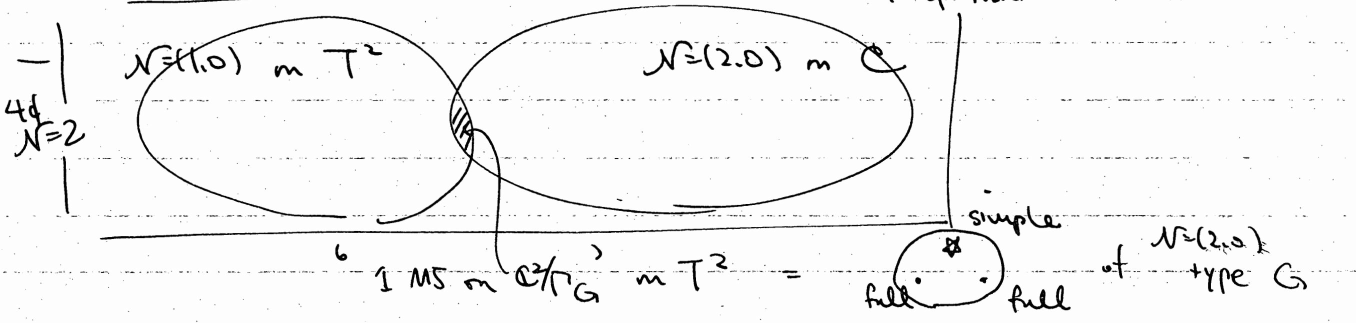
E-string as matter content.

$\mathbb{C}^2/\Gamma_{E_7}$ $E_7 \mid \phi \mid SU(2) \mid SO(7) \mid SU(2) \mid \phi \mid E_7$

$\mathbb{C}^2/\Gamma_{E_8}$ $E_8 \mid \phi \mid \phi \mid SU(2) \mid G_2 \mid \phi \mid F_4 \mid \phi \mid G_2 \mid SU(2) \mid \phi \mid \phi \mid E_8$

$\frac{1}{2}2$ $\frac{1}{2}$207

— these funny sequences can be found either in $\left\{\begin{array}{l}\text{F-theory} \\ \quad\text{or} \\ T^3 \text{ compactification} \to \text{just on } T^3\times\mathbb{R}\end{array}\right.$

● 

— what happens if we consider $\mathcal{N}=(1,0)$ on C_2 ?
can preserve 4d $\mathcal{N}=1$. generalized Gaiotto construction
for each $\mathcal{N}=(1,0)$ 6d theory ???

e.g. just using E-string (of rank ②) gives a zoo of E_8-symmetric $\mathcal{N}=1$ theories in 4d.

Can compute a & c & SCI. But can we identify them as some known 4d theory ???

THE END

2015 2 1

Appendix : $\pi_i(G)$.

Facts
- Let $\tilde{G}$: universal cover of G s.t. $G = \tilde{G}/\Gamma$.
  $\pi_1(G) = \Gamma$, $\pi_i(G) = \pi_i(\tilde{G})$ $i \geq 2$.
- $\pi_i(G \times G') = \pi_i(G) \times \pi_i(G')$.

then $G = U(1)$ $\rightsquigarrow$ $\pi_1(U(1)) = \mathbb{Z}$, $\pi_{i>1}(U(1)) = 0$.

Assume G : simple non-Abelian connected simply connected.

$\pi_2(G) = 0$.

$\pi_3(G) = \mathbb{Z}$.

$\pi_4(Sp(n)) = \mathbb{Z}_2$ (including $Sp(1)$), $= 0$ otherwise.

$$\begin{cases} \pi_5(Sp(n)) = \mathbb{Z}_2 & (\text{incl. } Sp(1)) \\ \pi_5(SU(n)) = \mathbb{Z} & (n \geq 3) \\ \pi_5(G) = 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} \pi_6(SU(2)) = \mathbb{Z}_{12} & \pi_6(SU(3)) = \mathbb{Z}_6 & \pi_6(G_2) = \mathbb{Z}_3 \\ \pi_6(G) = 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} \pi_7(SU(2)) = \mathbb{Z}_2 & \pi_7(SU(3)) = 0 & \pi_7(SO(8)) = \mathbb{Z} \oplus \mathbb{Z} \\ \underset{n \geq 2}{\pi_7(Sp(n)) = \mathbb{Z}} & \underset{n \geq 4}{\pi_7(SU(n)) = \mathbb{Z}} & \underset{n = 7, \geq 9}{\pi_7(SO(n)) = \mathbb{Z}} \\ \pi_7(\text{exceptional}) = 0 \end{cases}$$

$\vdots$

In a d-dim'l gauge theory,

$\pi_{d-1}(G)$ gives the theta angle (that doesn't dep. on the topology of the spacetime)

when d: even

$\pi_d(G)$ gives the (possible) global gauge anomaly

d: odd

$\pi_d(G)$: CS if $\mathbb{Z}$

global anomaly if $\mathbb{Z}_d$

2015. 2. 3

Appendix 2 global anomaly in odd dimensions

Recall the 'parity anomaly' in 3d / 5d :

- a Dirac fermion in the fundamental of $SU(N)$ changes the sign of the partition function under the global gauge transformation generator in $\pi_{\substack{d \\ \shortparallel \\ 3.5}}(SU(N)) = \mathbb{Z}$

- this anomaly can be cured by adding a half-integral CS

- a Dirac fermion in the doublet of $SU(2)$ in 5d does not have this problem ; However, as $\mathbf{2}$ of $SU(2)$ and $\mathbf{4}$ of $SO(4,1)$ are both pseudo-real, we can impose symplectic Majorana condition.

  The fermion determinant of this half-doublet changes sign under the generator of $\pi_5(SU(2)) = \mathbb{Z}_2$
  $\rightarrow$ anomalous.

- To see this more explicitly consider $Sp(n)$ in general, and reduce the spacetime as $\mathbb{R} \times \underbrace{\mathbb{R}^4}$

  introduce one-instanton for $Sp(1)$ in $Sp(1) \otimes O(n) \subset Sp(n)$.

the half ~~doublet~~ fundamental gives n real fermions under $O(n)$ of the 1d theory.

Canonical quantization tells us that the states can only be in a spinor of $Pin(n)$; the ferm. det gives minus one upon a generator of $\pi_1(O(n)) = \mathbb{Z}_2$.

But this path is contractible within $Sp(n)$ $\rightsquigarrow$ inconsistent !

- shows the periodicity $\pi_d(O) = \pi_{d+4}(Sp)$.
- works only for $n \geq 2$ strictly speaking.