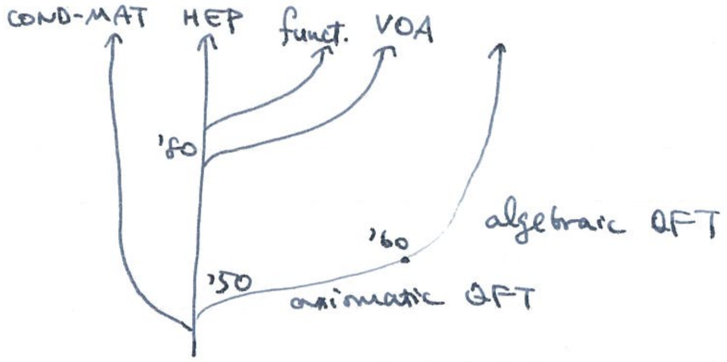
QFT topics that AQFT people might also want to consider

or: QFT topics about which I want some opinions from AQFT people.

0. Random chat
1. Entanglement entropy
2. Familes of QFTs

0. History of various subcommunities of QFTs:



- mostly independent development, with some small interactions
- more and more cross-talks since 2005 or something
- I'm from the HEP community

  there should be many things I should learn from you, and you might find some of the things I would say not totally uninteresting.

- In addition to the subareas given above, $\exists$ "maths inspired by QFTs" such as mirror symmetry, Seiberg-Witten invariants of 4-mfds

A: sympl. mfd $\rightsquigarrow$ Q(A) : qft constructed using A

B : cpx. mfd $\rightsquigarrow$ Q(B) B

"indications" Q(A) = Q(B)

relations between A $\leftrightarrow$ B

X : 4-manifold

$Z_Q(X)$: a certain qft evaluated on X Q

UV IR

= integral over moduli of SU(2) instantons on X $\longleftrightarrow$ = integral over moduli of SW monopole equation on X.

should be the same.

What are the QFTs that underlie these math works?
axiomatic framework of

NOT vanilla AQFT. It's some unwritten, consensus version of updated set of axioms of Segal type...

Roughly, a d-dim QFT Q is a functor that assigns:

 : (d-1) d Riemannian mfd $Y \longmapsto \mathcal{H}(Y)$: a Hilb. sp

Y'

X : d-d Riemannian mfd $X \longmapsto$ a linear map $Z(X) : \mathcal{H}(Y) \to \mathcal{H}(Y')$

Y

<u>but with</u> various "operators" supported on submanifolds of various (co)dimensions.

e.g. given an X without boundary

$X \longmapsto Z(X) \in \mathbb{C}$ "partition function"

you can decorate it with, say

$\longmapsto Z(X; \phi_1(p_1)\, \phi_2(p_2)\, a(\ell)\, n(S)) \in \mathbb{C}$

labels of

ϕ_i : point operators

a : line ops

n : surface ops

⋮

s.t. when $X = \mathbb{R}^d$ with just point operators, we get Wightman / Osterwalder-Schrader functions back.

- Point ops satisfy an algebra structure that generalizes VOA.
- line ops form (something like) a tensor category.
- surface ops 2-category ??

⋮

In particular, by smearing the point operators, in a region $\mathcal{O}$, we have $\mathcal{A}(\mathcal{O})$. meaning that this consensus unwritten axiom system includes the data that enter in an AQFT net. so:

FUNDAMENTAL QUESTION (for me)

given a net $\mathcal{A}(\mathcal{O})$, is there a QFT (in this unwritten, consensus sense) that gives rise to $\mathcal{A}(\mathcal{O})$?

Can't be answered without a definition of the QFT in the above sense! for 2d CFT, there essentially is, and you'll learn alot in this conf.

A smaller question

given a net $\mathcal{A}(\mathcal{O})$ on $\mathbb{R}^d$, how do we find $\mathcal{H}(T^{d-1})$

and $Z($  $t\ ) : \mathcal{H}(T^{d-1}) \to \mathcal{H}(T^{d-1})$?

T^{d-1}, T^{d-1}

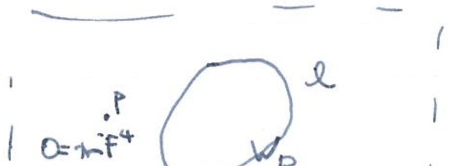
$\shortparallel$

e^{itH}

is it something you can do ?

An even smaller question

in a gauge theory, not only the point ops but also line ops are very important, even on $\mathbb{R}^d$.



$$Z = \int e^{-\int \mathrm{tr} F\wedge * F}[dA]$$

$$Z(\mathrm{tr}F^4(p),\ W_R(\ell)) = \int \mathrm{tr}F^4(p)\cdot \mathrm{tr}_R \mathrm{Hol}(\ell)\cdot e^{-\int \mathrm{tr}F\wedge *F}[dA]$$

whether the gauge th in question is confined or not is measured by whether

$$\langle W_R(\ell)\rangle \sim \begin{cases} e^{-(\text{area within } \ell)} & \leftarrow \text{confined} \\ e^{-(\text{length of } \ell)} & \leftarrow \text{not} \end{cases}$$

- do you include smeared version of $W_R(\ell)$ in $\mathcal{A}(\mathcal{O})$, or not.
- I think Ω in $\mathcal{H}(T^{d-1})$ won't be cyclic unless you include $W_R(\ell)$ wrapping homologically nontrivial cycles...

that was the random chat.

1. Entanglement entropy

- One very popular topic in HEP-TH & COND-MAT these days (~10yrs) is the EE of QFTs. I'd like to know how you would think.

— In general, let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, take a state $|\psi\rangle \in \mathcal{H}$.
it gives a state of $B(\mathcal{H}_A)$ via $a \in B(\mathcal{H}_A) \mapsto \langle\psi|a|\psi\rangle$.
can be represented by a density matrix ρ_A on $\mathcal{H}_A$ s.t.

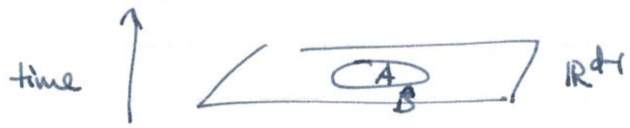
$$\mathrm{tr}_{\mathcal{H}_A} \rho_A a = \langle\psi|a|\psi\rangle .$$

then EE of $|\psi\rangle$ w.r.t. $\mathcal{H}_A \overset{\text{def}}{=}$ vN entropy of ρ_A

$$= -\mathrm{tr}_{\mathcal{H}_A} \rho_A \log \rho_A .$$

OK e.g. when $\mathcal{H}_A$, $\mathcal{H}_B$ both fin. dim!
and people in HEP-TH / COND-MAT often "assume" that, somehow.

— Now, given a d-dim QFT on $\mathbb{R}^d$, split $\mathbb{R}^{d-1} = A \cup B$
sharing a bndry.

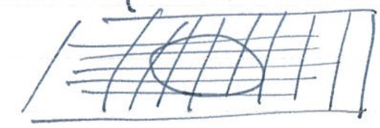


people often say this: $\mathcal{H}(\mathbb{R}^{d-1}) = \mathcal{H}(A) \otimes \mathcal{H}(B)$
and define EE of the region A to be
the EE of $|\Omega\rangle$ w.r.t. $\mathcal{H}(A)$.

But what do we mean by $\mathcal{H}(A)$??

— A usual way out: assume $\exists$ a lattice model Λ_ε with lattice spacing ε s.t.

$$\mathcal{H} = \bigotimes_{\text{vertices } v} \mathcal{H}_v , \quad \mathcal{H}_v : \text{fin. dim}$$



that converges to the continuum QFT Q that you want to study in the limit $\varepsilon \to 0$.

let $\mathcal{H}(A) = \bigotimes_{v \in A} \mathcal{H}_v$. compute EE as a function of ε.
study how it behaves as $\varepsilon \to 0$. Doesn't converge.

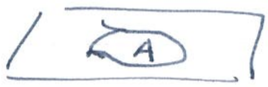
— THE CLAIM — ⑤

In full 2d CFT,

the central charge.

$$\text{EE of } A \sim \frac{c}{3}\log\frac{L}{\varepsilon} + \text{less singular terms}$$

In gapped 3d QFT (i.e. the eigenvalue zero of $|\Omega\rangle$ of the Hamiltonian is isolated)

EE of  $\sim \gamma \dfrac{\text{length of } \partial A}{\varepsilon} + F$

where γ is a constant depending on which $\Lambda_\varepsilon \to \mathbb{Q}$, Λ'_ε is taken

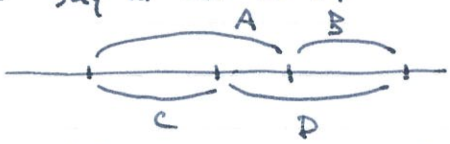
while $F = -\log \mathcal{D} = -\log\sqrt{\sum_i d_i^{\,2}}$.

where $\mathcal{D}$ is the total quantum dim. of the 3d TQFT in the IR lim.

etc. etc.

Is there a way to show this in AQFT? An obstacle: ε.

Possible way out: say in 2d CFT,



cross ratio

$$\begin{aligned}&\text{EE of } A + \text{EE of } B\\ &\quad - \text{EE of } C - \text{EE of } D\end{aligned} = \frac{c}{3}\log\frac{L_A L_B}{L_C L_D}$$

no epsilon

maybe directly definable in AQFT?

In the rest of today's talk, let me explain why $\text{EE} \sim \frac{c}{3}\log\frac{L}{\varepsilon}$ in 2d CFT.

— an explicit, perfectly well-defined example: Ising model. ($c = \frac{1}{2}$)

Prepare $\gamma_1 \cdots \gamma_{2N}$ s.t. $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$ $\curvearrowright$ $\mathbb{C}^{2^N}$

$$H = i\sum\left(\gamma_{2i}\gamma_{2i+1} + \lambda\gamma_{2i+1}\gamma_{2i+2}\right)$$

critical when $\lambda = 1$

$$= i\sum \gamma_i\gamma_{i+1}.$$

in the $N \to \infty$ limit, $\langle \gamma_i \gamma_j \rangle = \dfrac{-2i}{\pi}\dfrac{1}{i-j}$

and higher pt func. is given by Wick contraction e.g.

$$\langle \gamma_i\gamma_j\gamma_k\gamma_l\rangle = \langle\gamma_i\gamma_j\rangle\langle\gamma_k\gamma_l\rangle - \langle\gamma_i\gamma_k\rangle\langle\gamma_j\gamma_l\rangle.$$

Now you restrict to a length 2L subsystem

$$\underbrace{\gamma_1\,\gamma_2}_{\mathbb{C}^2}\otimes\underbrace{\gamma_3\,\gamma_4}_{\mathbb{C}^2}\otimes\underbrace{\quad}_{\mathbb{C}^2}\cdots\quad\underbrace{\gamma\ \cdots\cdots\ \gamma}_{2L\ \gamma\text{'s}}\quad\cdots\cdots\quad\gamma_{2N}$$

$$\mathcal{H}_L := \bigotimes_L \mathbb{C}^2$$

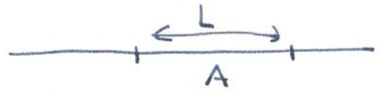
the density matrix ρ_L on $\mathcal{H}_L$ needs to reproduce $\langle \gamma_{i_1}\cdots\gamma_{i_n}\rangle$ within $i_k \in 2L\ \gamma$'s. Since they are free, $\rho_L = e^{\sum c_{ij}\gamma_i\gamma_j}$ where the summation is over $2L\ \gamma$'s.

- c_{ij} can be determined, and depends only on $i-j$.

$\rightsquigarrow$ C_{ij} is a Toeplitz matrix, whose eigenvalues are well studied.

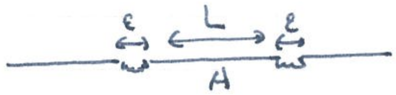
$\rightsquigarrow$ $-\mathrm{tr}_{\mathcal{H}_L}\, \rho_L \log \rho_L$'s $L\to\infty$ behavior can be explicitly determined, and show $\sim \frac{1}{6}\log L$. behavior.

— general argument for any 2d CFT



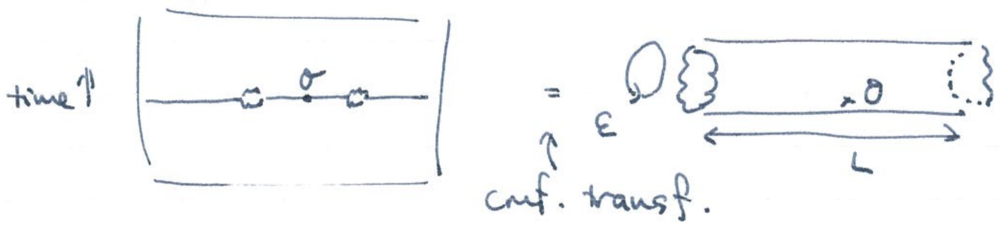
what do we mean by $\mathcal{H}_A$?

and by ε in the continuum theory



insert physical boundary of width ε!

For op's supported on A, $\langle \Omega | \mathcal{O} | \Omega \rangle$ is



i.e. ρ on $\mathcal{H}_A = e^{-\frac{\varepsilon}{L} H_{open}}$

where H_{open} is the Hamiltonian of the CFT on a finite segment of length 1.

$\rightsquigarrow$ $-\mathrm{tr}_{\mathcal{H}_A}\, \rho\log\rho$ can be evaluated via Cardy formula

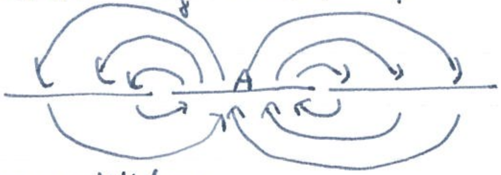
$\frac{L}{\varepsilon}\to 0$ behavior of

$\rightsquigarrow$ gives $\frac{c}{3}\log\frac{L}{\varepsilon}$.

This argument points to something probably important:

ρ^{it} generates the time evolution on the cylinder.

cmf. transforming back, ρ^{it} generates



via Bisognano–Wichmann; Hislop–Longo, this is the modular automorphism of $\mathcal{A}(A)$.

In general, for even non-CFT, non-2d QFTs, assuming $\exists$ lattice reg,

A

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{rest}$, ρ on $\mathcal{H}_A$ s.t. $\mathrm{tr}_{\mathcal{H}_A} \rho \mathcal{O} = \langle \Omega | \mathcal{O} | \Omega \rangle$ defines $\rho^{it} \mathcal{O} \rho^{-it} =: \alpha_t(\mathcal{O})$

that satisfies the KMS condition. ⇝ should converge to the modular automorphism when $\varepsilon \to 0$

EE is $\mathrm{tr}_{\mathcal{H}_A} -\rho \log \rho$. Its $\varepsilon \to 0$ behavior should correspond to (?) some spectral behavior of Δ ...

but Δ acts on $\mathcal{H}$, not $\mathcal{H}_A$ ...

splitness ?? nuclearity ??

— this concludes Sec 1. —

2. Families of QFTs

Interesting math. objects often come in continuous families. say, Riemann surfaces ... QFTs often do so too.

- The Standard Model of particle physics is one such example: parameters in it (e.g. Higgs mass, fine str const etc) determine a point in the family. Experimenters are working hard to measure them!
- So, given a QFT, what's an infinitesimal (1st order) deformation?

  what are obstructions integrating them?

  is there a universal family?

  etc, etc.

- Of course this depends on the axiom system you choose.
- What would be a valid deformation in a net wouldn't be a valid deform in the 'consensus QFT'.

e.g. d-dim free massless scalar field $\langle \phi(x)\phi(y)\rangle = \frac{1}{|x-y|^{d-2}}$

higher pt func is given by Wick thm

this is a valid QFT under any definition!

d-dim generalized free scalar field

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{|x-y|^{2\Delta}}$$

where $2\Delta > d-2$. with higher pt func. given by Wick

I think this gives a valid set of Wightman func.

$\rightsquigarrow$ smearing would give a valid net.

They will form a family ... but <u>not</u> under the consensus axiom.

(e.g. Experimenters don't include this parameter Δ for the Higgs boson in the SM.)

The problem: it doesn't have a nice conserved energy-momentum tensor $T_{\mu\nu}$.

Its existence required to describe the QFT's coupling to gravity

via $Z_Q[M, g+\delta g] = Z_Q[M, g] \langle \int_M \delta g_{\mu\nu} T^{\mu\nu}\sqrt{g}\, dx \rangle$

part of the 'axiom'

- But at least for well-behaved (?) 2d full CFT, I believe the deformation as a net would be equivalent to the deformation under the 'consensus axioms'.

- Under 'consensus axioms', any 1st order deformation is of the form

$$\langle \phi_1(x_1) \cdots \phi_n(x_n)\rangle_\varepsilon = \langle \phi_1(x_1)\cdots\phi_n(x_n)\, e^{\int_M \varepsilon \mathcal{O}(x)\, d\mathrm{vol}_M}\rangle_0$$

where $\mathcal{O}(x)$ is a scalar point operator.

. when a path integral description is available,

$$\langle \cdots \rangle_0 = \int \cdots \; e^{-\int_M L_0[\phi_1 \cdots \phi_n]\, dx} \; D\phi_1 \cdots D\phi_n$$

then $$\langle \cdots \rangle_\varepsilon = \int \cdots \; e^{-\int_M (L_0[\phi_1 \cdots \phi_n] + \varepsilon \mathcal{O})\, dx} \; D\phi_1 \cdots D\phi_n$$

deformation of the Lagrangian !

In general, the space of point operators V has a filtration

$V_\Delta \subset V_{\Delta'} \subset V$ for $\Delta' \in \mathbb{R}_+$ called scaling dimensions

defined by $\mathcal{O} \in V_\Delta$ iff $\langle \mathcal{O}(x)\, \mathcal{O}(y)^\dagger \rangle \sim O\left(\frac{1}{|x-y|^{2\Delta}}\right)$ when $x \to y$

(For CFTs, this becomes genuine grading)

<u>Expectation</u> For a d-dim QFT,

i) $\mathcal{O} \in V_{\Delta<d}$ (called relevant/ superrenormalizable) is not obstructed at all.

ii) $\mathcal{O} \in V_{\Delta=d}$ tricky (marginal / renormalizable) ~~but~~ sometimes obstructed

iii) $\mathcal{O} \in V_{\Delta>d}$ more tricky (irrelevant / non-renormalizable) extremely often obstructed.

(Results in constructive field theory concerning $P(\phi)_2$ and $(\phi^4)_3$ are specific instances of i) . There should be a general proof. Difficulty of $(\phi^4)_4$ is an instance of ii).)

For a dim-d <u>CFT</u>, deformations of type i) will lead to non CFT. deformations as CFT are of type ii).

Take d=2 for definiteness.

point operators $\longleftrightarrow$ $\mathcal{H}$ on S^1 $\curvearrowleft Vir \times \overline{Vir}$

scalar $V_{\Delta=2}$ $\longleftrightarrow$ those with $L_0 = \overline{L_0} = 1$.

1st order deformations.

product as $VOA \otimes \overline{VOA}$ induces an bilinear map

$\bullet$: scalar $V_{\Delta=2}$ $\otimes$ scalar $V_{\Delta=2}$ $\longrightarrow$ scalar $V_{\Delta=2}$

that comes from Borcherds' $a_{(n)}b$ on both hol / antihol sides.

expectation

$v \in$ scalar $V_{\Delta=2}$ is not obstructed at all

if $v \cdot v = 0$ under this product.

Example

$\boxed{c=1 \text{ full CFTs}}$. hol side: $\partial X(z)\partial X(w) \sim \frac{1}{(z-w)^2}$ ← generates $U(1)$ aff. or Heisenberg alg.

$:e^{ikX}: \quad L_0 = \frac{k^2}{2}$. h.w. vec. generates V_k.

glue the hol side and antihol side:

$e^{ikX(z)} e^{ik'X(\bar z)}$ $\quad (L_0, \bar L_0) = (\frac{k^2}{2}, \frac{k'^2}{2})$.

$V_k \otimes \overline{V_{k'}}$

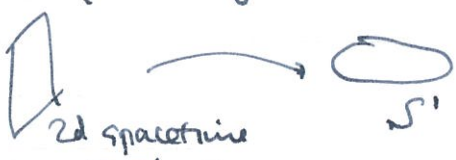
$$\mathcal{H}_{S^1} = \bigoplus_{(k,k') \in ▨} V_k \otimes \overline{V_{k'}}$$

modular inv. on the torus dictates

$$▨ = \mathbb{Z}\left(\frac{1}{R}, \frac{1}{R}\right) + \mathbb{Z}\left(\frac{R}{2}, -\frac{R}{2}\right).$$

called free boson whose radius is R.

obtained by quantizing



of radius R.

call this CFT "S^1_R".

comes in a family param. by R. $\leftrightarrow$ deformation by $v := \partial X \bar\partial X$. $\left(\begin{array}{l}\text{can check}\\ v \cdot v = 0\end{array}\right)$

inv. under $R \leftrightarrow \frac{2}{R} = R'$: T-duality.

what happens at $R = \sqrt{2}$?

$V_{\pm\sqrt{2}} \otimes \overline{V_0}$ is in the spectrum: $\left\{\begin{array}{l} e^{\pm i\sqrt{2}X(z)}. \\ \partial X \end{array}\right.$

generates $SU(2)_1$ aff. alg. call them $J^{1,2,3}$.

similarly on the antihol. side.

$\Rightarrow \quad +\partial X \bar\partial X$ equiv. with $-\partial X \bar\partial X$.

(more generally, $J^i \bar J^j$ for any i, j is $(L_0, \bar L_0) = (1, 1)$.

but they are all equiv. under the $SU(2) \times SU(2)$ action.)

: $c=1$ full CFT with $U(1)$ aff. alg. on both sides.] $\forall$

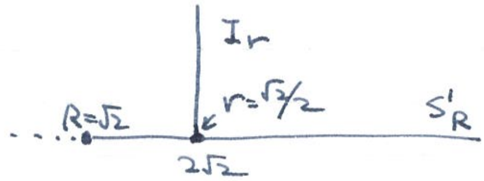
what happens if we drop the condition $\forall$?

$\exists$ symmetry $X(z), X(\bar{z}) \to -X(z), -X(\bar{z})$ at any R.

$I_{R/2} := S^1_R / \mathbb{Z}_2$: orbifold model.

∂X, $\bar{\partial} X$ projected out. $\Rightarrow$ no $U(1)$ aff sym.

$\partial X \bar{\partial} X$ is kept : still deformable. only Virasoro at generic R.



$I_{\sqrt{2}/2} = S^1_{2\sqrt{2}}$. this is because $X(z), X(\bar{z}) \to -X(z), -X(\bar{z})$ at $R=\sqrt{2}$

is equivalent to $X \to X + \frac{2\pi}{\sqrt{2}}$: half shift

due to $SU(2) \times SU(2)$ invariance there.

Therefore $I_{\sqrt{2}/2} = S^1_{\sqrt{2}} / \mathbb{Z}_2$ reflection $= S^1_{\sqrt{2}} / \mathbb{Z}_2$ half shift

$= S^1_{\sqrt{2}/2} = S^1_{2\sqrt{2}}$.

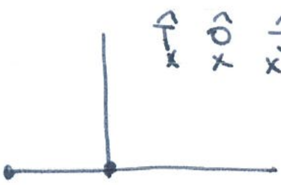
You can try taking finite subgroup $\Gamma_{\text{full}} \subset SU(2) \times SU(2)$ and take the orbifold.

One finds it's consistent only when $\Gamma \times \Gamma \subset SU(2) \times SU(2)$

$\Gamma = \mathbb{Z}_N$: $S^1_{\sqrt{2}}/\Gamma = S^1_{\sqrt{2}n}$ ⎫ $\partial X \bar{\partial} X$ kept

$\Gamma =$ dihedral$_n$: $S^1_{\sqrt{2}}/\Gamma = I_{\sqrt{2}/2\, n}$ ⎭

$\Gamma =$ tetra, octa / cube, dodeca / icosa : $\hat{T}$, $\hat{O}$, $\hat{I}$ } $\partial X \bar{\partial} X$ proj. out



: all presently known $c=1$ models (which are modular inv & only fin# of states below any given scaling dim.)

Are they all ?