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Idiosyncratic review of basic 4d $\mathcal{N}=1$ supersymmetric dynamics

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Contains plagiarisms from many of my old lecture notes and papers. Many coefficients are wrong.

Contents

1 Non-susy preliminaries

1.1 One-loop running

Recall the one-loop renormalization of the gauge coupling in a general Lagrangian field theory:

$$E\frac{d}{dE}g = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C(\text{adj}) - \frac{2}{3}C(R_f) - \frac{1}{3}C(R_s)\right]. \tag{1.1.1}$$

Here, E is the energy scale at which g is measured, and we use the convention that all fermions are written in terms of left-handed Weyl fermions. Then R_f and R_s are the representations of the gauge group to which the Weyl fermions and the complex scalars belong, respectively. The quantity $C(\rho)$ is defined so that

$$\operatorname{tr}\rho(T^a)\rho(T^b) = C(\rho)\delta^{ab} \tag{1.1.2}$$

where T^a are the generators of the gauge algebra and $\rho(T^a)$ is the matrix in the representation ρ, normalized so that $C(\text{adj})$ is equal to the dual Coxeter number. For $\mathrm{SU}(N)$, we have

$$C(\text{adj}) = N, \qquad C(\text{fund}) = \frac{1}{2}. \tag{1.1.3}$$

For $\mathrm{SO}(N)$, we have

$$C(\text{adj}) = N-2, \qquad C(\text{vec}) = 1. \tag{1.1.4}$$

1.2 Flavor anomaly

Non-abelian gauge theories have an important source of non-perturbative effects, called instantons. This is a nontrivial classical field configuration in the Euclidean $\mathbb{R}^4$ with nonzero integral of

$$16\pi^2 k := \int_{\mathbb{R}^4} \operatorname{tr} F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{1.2.1}$$

In the standard normalization of the trace for $\mathrm{SU}(N)$, k is automatically an integer, and is called the instanton number. The theta term in the Euclidean path integral appears as

$$\exp\left[i\frac{\theta}{16\pi^2}\operatorname{tr} F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{1.2.2}$$

Therefore, a configuration with the instanton number k has a nontrivial phase $e^{i\theta k}$. Note that a shift of θ by 2π does not change this phase at all. Therefore, even in a quantum theory, the shift $\theta \to \theta + 2\pi$ is a symmetry.

Using

$$\operatorname{tr} F_{\mu\nu}F_{\mu\nu} = \frac{1}{2}\operatorname{tr}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2 \mp \operatorname{tr} F_{\mu\nu}\tilde{F}_{\mu\nu} \geq \mp \operatorname{tr} F_{\mu\nu}\tilde{F}_{\mu\nu}, \tag{1.2.3}$$

we find that

$$\int d^4x \operatorname{tr} F_{\mu\nu}F_{\mu\nu} \geq 16\pi^2|k| \tag{1.2.4}$$

which is saturated only when

$$F_{\mu\nu} + \tilde{F}_{\mu\nu} \propto F_{\alpha\beta} = 0 \quad \text{or} \quad F_{\mu\nu} - \tilde{F}_{\mu\nu} \propto F_{\dot\alpha\dot\beta} = 0 \tag{1.2.5}$$

depending on the sign of k. Therefore, within configurations of fixed k, those satisfying relations (1.2.5) give the dominant contributions to the path integral. The solutions to (1.2.5) are called instantons or anti-instantons, depending on the sign of k.

In an instanton background, the weight in the path integral coming from the gauge kinetic term is

$$\exp\left[-\frac{1}{2g^2}\int \operatorname{tr} F_{\mu\nu}F^{\mu\nu} + i\frac{\theta}{16\pi^2}\int \operatorname{tr} F_{\mu\nu}\tilde{F}^{\mu\nu}\right] = e^{2\pi i\tau k}. \tag{1.2.6}$$

We similarly have the contribution $e^{2\pi i\bar\tau|k|}$ in an anti-instanton background. The fact that we have just τ or $\bar\tau$, instead of more complicated combinations, is related to the fact that in the instanton background in a supersymmetric theory, $\delta\lambda_{\dot\alpha} = F_{\dot\alpha\dot\beta}\epsilon^{\dot\beta} = 0$ assuming the D-term is also zero, and thus the dotted supertranslation is preserved. Similarly, the undotted supersymmetry is unbroken in the anti-instanton background.

Now, consider charged Weyl fermions ψ_α coupled to the gauge field, with the kinetic term

$$\bar\psi_{\dot\alpha} D_\mu \sigma^{\mu\dot\alpha\alpha}\psi_\alpha. \tag{1.2.7}$$

Let us say ψ_α is in the representation R of the gauge group. It is known that the number of zero modes in ψ_α minus the number of zero modes in $\bar\psi_{\dot\alpha}$ is $2C(R)k$. In particular, the path integral restricted to the k-instanton configuration with positive k is vanishing unless we insert k additional ψ's in the integrand. More explicitly,

$$\langle O_1 O_2 \cdots\rangle = \int [D\psi][D\bar\psi] O_1 O_2 \cdots e^{-S} = 0 \tag{1.2.8}$$

unless the product of the operators $O_1 O_2 \cdots$ contains $2C(R)k$ more ψ's than $\bar\psi$'s. This is interpreted as follows: the path integral measures $[D\psi]$ and $[D\bar\psi]$ contain both infinite number of integrations. However, there is a finite number, $2C(R)k$, of difference in the number of integration variables. Equivalently, under the constant rotation

$$\psi \to e^{i\varphi}\psi, \quad \bar\psi \to e^{-i\varphi}\bar\psi, \tag{1.2.9}$$

the fermionic path integration measure rotates as

$$\begin{aligned} [D\psi] &\to [D\psi] e^{+\infty i\varphi + 2C(R)ki\varphi}, \\ [D\bar\psi] &\to [D\bar\psi] e^{-\infty i\varphi}. \end{aligned} \tag{1.2.10}$$

When combined, we have

$$[D\psi][D\bar\psi] \to [D\psi][D\bar\psi]e^{2C(R)ki\varphi} = [D\psi][D\bar\psi]\exp\left[2C(R)\varphi\frac{i}{16\pi^2}\int \operatorname{tr} F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{1.2.11}$$

This can be compensated by a shift of the θ angle, $\theta \to \theta + 2C(R)\varphi$. As we recalled before, the shift $\theta \to \theta + 2\pi$ is a symmetry. Therefore, the rotation of the field ψ by $\exp(\frac{2\pi i}{2C(R)})$ is a genuine, unbroken symmetry.

1.3 Gauge anomaly

When we perform the path integral of a gauge theory, we first need to consider

$$Z[A_\mu] = \int [D\psi][D\bar\psi] e^{-\int \bar\psi D_\mu \sigma^\mu \psi} \tag{1.3.1}$$

where the chiral fermion ψ is in a representation R of G. To perform a further integration over A_μ consistently, we need

$$Z[A_\mu] = Z[A_\mu^g], \quad A_\mu^g = g^{-1} A_\mu g + g^{-1}\partial_\mu g. \tag{1.3.2}$$

for any gauge transformation $g : \mathbb{R}^4 \to G$.

However, due to the reason similar to the gauge-gauge-flavor anomaly above,

$$Z[A_\mu^g] = (\text{something}) Z[A_\mu] \tag{1.3.3}$$

where the prefactor is not 1 in general. For g continuously connected to a trivial transformation $g \equiv 1$, the prefactor is characterized by an object called *the anomaly polynomial*, which is a cubic polynomial in the formal variables F_a where a is the adjoint index of G. The prefactor is identically 1 if and only if the anomaly polynomial vanishes. Concretely, this anomaly polynomial for a chiral fermion in representation R of G is given by the formula

$$A = \frac{1}{6} \operatorname{tr}_R F^3 \tag{1.3.4}$$

where $F = F_a T^a$ where T^a is the matrix in the representation R.

Since $\operatorname{tr}_{\bar R} F^3/6 = -\operatorname{tr}_R F^3/6$, if a chiral fermion appears in a real representation, the anomaly automatically cancels. In this lectures we only treat this case but the Standard Model is more subtle. The chiral fermions in a generation is given as follows

$$\begin{array}{c|ccccc} & Q_L & \bar u_R & \bar d_R & \ell_L & \bar e_R \\ \mathrm{SU}(3) & 3 & \bar 3 & \bar 3 & 1 & 1 \\ \mathrm{SU}(2) & 2 & 1 & 1 & 2 & 1 \\ \mathrm{U}(1) & 1/6 & -2/3 & 1/3 & -1/2 & 1 \end{array} \tag{1.3.5}$$

Let us check the $\mathrm{U}(1)$ part of the cancellation. Denoting the formal variable for the $\mathrm{U}(1)$ generator T by c, cT evaluates to $c/6$ in the Q_L, etc, so

$$A = \frac{c^3}{6}(3 \cdot 2 \cdot \left(\frac{1}{6}\right)^3 + 3 \cdot \left(-\frac{2}{3}\right)^3 + 3 \cdot \left(\frac{1}{3}\right)^3 + 2 \cdot \left(-\frac{1}{2}\right)^3 + 1 \cdot 1^3) = 0. \tag{1.3.6}$$

Exercise. Check the cancellation of the full anomaly polynomial of the Standard Model.

One also needs to be careful about Witten's global anomaly [1]. Take a chiral fermion in the doublet of gauge $\mathrm{SU}(2)$. There's no perturbative anomaly, since C can be conjugated to $\operatorname{diag}(c, -c)$.

Since continuous change of g doesn't change the phase of $[D\psi][D\bar{\psi}]$, what matters is maps $g : \mathbb{R}^4 \to \mathrm{SU}(2)$ up to continuous change. They are characterized by $\pi_4(\mathrm{SU}(2))$. It is known that

$$\pi_4(\mathrm{SU}(2)) = \pi_4(S^3) = \mathbb{Z}_2. \tag{1.3.7}$$

Let $g_0 : \mathbb{R}^4 \to \mathrm{SU}(2)$ be the one corresponding to the nontrivial element in this $\mathbb{Z}_2$. It is known that under this $[D\psi][D\bar{\psi}]$ gets a minus sign. So, one cannot have an odd number of Weyl fermions in the doublet representation of gauge $\mathrm{SU}(2)$.

Exercise. Check that there are even number of $\mathrm{SU}(2)$ doublets in the Standard Model.

2 Pure super Yang-Mills

2.1 Construction of the Lagrangian

An $\mathcal{N}$=1 vector multiplet consists of a Weyl fermion λ_α and a vector field A_μ, both in the adjoint representation of the gauge group G. We combine them into the superfield W_α with the expansion

$$W_\alpha = \lambda_\alpha + F_{(\alpha\beta)}\theta^\beta + D\theta_\alpha + \cdots \tag{2.1.1}$$

where D is an auxiliary field, again in the adjoint of the gauge group. $F_{\alpha\beta} = \frac{\mathrm{i}}{2}\sigma^{\mu}{}_{\dot\gamma}^{\beta}\bar\sigma^{\nu}{}_{\alpha}^{\dot\gamma}F_{\mu\nu}$ is the anti-self-dual part of the field strength $F_{\mu\nu}$.

The kinetic term for a vector multiplet is given by

$$\int d^2\theta \frac{-\mathrm{i}}{8\pi}\tau \,\mathrm{tr}\, W_\alpha W^\alpha + cc. \tag{2.1.2}$$

where

$$\tau = \frac{4\pi\mathrm{i}}{g^2} + \frac{\theta}{2\pi} \tag{2.1.3}$$

is a complex number combining the inverse of the coupling constant and the theta angle. We call it the complexified coupling of the gauge multiplet. Expanding in components, we have

$$\frac{1}{2g^2}\,\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{16\pi^2}\,\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{g^2}\,\mathrm{tr}\, D^2 - \frac{2\mathrm{i}}{g^2}\,\mathrm{tr}\, \bar\lambda\partial\!\!\!/\lambda. \tag{2.1.4}$$

We use the convention that $\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$ for the standard generators of gauge algebras, which explain why we have the factors $1/(2g^2)$ in front of the gauge kinetic term. The θ term is a total derivative of a gauge-dependent term. Therefore, it does not affect to perturbative computations. It does affect non-perturbative computations.

Here we constructed the Lagrangian using superfields, but it's important that it's just a gauge field and a chiral massless fermion in the adjoint representation. The only tuning is to make the fermion massless, which is technically natural because of the R-symmetry, discussed soon below.

Exercise. Check the supersymmetry of this Lagrangian from various viewpoints.

2.2 One-loop exactness of the perturbative running

There is a renormalization scheme where the superpotential remains a holomorphic function of the chiral superfields, including background fields whose vevs are the gauge and superpotential couplings. This is the core of Seiberg's holomorphy argument.[1]

In a supersymmetric theory, the coupling g is combined with the theta angle θ and enters in the Lagrangian as

$$\int d^2\theta \frac{-i}{8\pi}\tau \operatorname{tr} W_\alpha W^\alpha + cc. \tag{2.2.1}$$

where τ is given by

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}. \tag{2.2.2}$$

We call this τ the complexified gauge coupling. We can consider τ to be an expectation value of a background chiral superfield.

In this scheme, the one-loop running coupling at the energy scale E can be expressed as

$$\tau(E) = \tau_{UV} - \frac{b}{2\pi i}\log\frac{E}{\Lambda_{UV}} + \cdots \tag{2.2.3}$$

where b is the rational number appearing on the right hand side of (3.3.4). Note that the coupling τ starts from $1/g^2$, and therefore the n loop diagram would have the dependence $g^{2(n-1)}$. The constant shift as in the imaginary part in (2.2.3) is then a one-loop effect.

Perturbation theory is independent of the θ angle, since $F_{\mu\nu}\tilde{F}_{\mu\nu}$ is a total derivative, although of a gauge-dependent quantity. Therefore the n loop effect is a function of $(\operatorname{Im}\tau)^{1-n}$, which is not holomorphic unless $n = 1$. We conclude that the running (2.2.3) is one-loop exact in the holomorphic scheme. We find that the combination

$$\Lambda^b = E^b e^{2\pi i\tau(E)} \tag{2.2.4}$$

is invariant to all orders in perturbation theory. We call this Λ the complexified dynamical scale of the theory.[2] Note that Λ is a complex quantity, and can be considered as a vev of a background chiral superfield.

In the case of $\mathrm{SU}(N)$ pure Yang-Mills, the one-loop running of the coupling is given by

$$E\frac{\partial}{\partial E}\tau(E) = 3N, \tag{2.2.5}$$

[1]Recently an obstruction to this philosophy was found for 2d $\mathcal{N}=(2,2)$ theories [2]; but the analysis there confirms that there's no problem in 4d $\mathcal{N}=1$.

[2]A redefinition of the form $\Lambda \to c\Lambda$ by a real constant c corresponds to a redefinition of the coupling of the form $1/g^2 \to 1/g^2 - c'$ where c' is another constant, or equivalently $g^2 \to g^2 + c'g^4 + \cdots$. Therefore this is a redefinition starting at the one-loop order, keeping the leading order definition of g^2 fixed. In this lecture note, we do not track such finite renormalization of the coupling very carefully.

and therefore we define the dynamical scale Λ by the relation

$$\Lambda^{3N} = e^{2\pi i \tau_{UV}} \Lambda_{UV}^{3N}. \tag{2.2.6}$$

This is *not* the end of the story: in the infrared the field W_α doesn't make sense. We need to make sense of the dynamics.

2.3 R-symmetry

We assign R-charge zero to the gauge field, and R-charge 1 to the gaugino λ_α. The phase rotation $\lambda_\alpha \to e^{i\varphi}\lambda_\alpha$ is anomalous, and needs to be compensated by $\theta \to \theta + 2N\varphi$. The shift of θ by 2π is still a symmetry, therefore the discrete rotation

$$\lambda_\alpha \to e^{\pi i/N}\lambda_\alpha, \qquad \theta \to \theta + 2\pi \tag{2.3.1}$$

is a symmetry generating $\mathbb{Z}_{2N}$. Note that under the R-symmetry, Λ defined above has the transformation

$$\Lambda \to e^{2\pi i/(3N)}\Lambda. \tag{2.3.2}$$

This theory is believed to confine, with nonzero gaugino condensate $\langle \lambda_\alpha \lambda^\alpha \rangle$. What would be the value of this condensate? This should be of mass dimension 3 and of R-charge 2. The only candidate is

$$\langle \lambda_\alpha \lambda^\alpha \rangle = c\Lambda^3 \tag{2.3.3}$$

for some constant c. The symmetry (2.3.2) acts in the same way on both sides by the multiplication by $e^{2\pi i/N}$. Assuming that the numerical constant c is non-zero, this $\mathbb{Z}_{2N}$ is further spontaneously broken to $\mathbb{Z}_2$, generating N distinct solutions

$$\langle \lambda_\alpha \lambda^\alpha \rangle = c e^{2\pi i \ell/N}\Lambda^3 \tag{2.3.4}$$

where $\ell = 0, 1, \ldots, N-1$. Unbroken $\mathbb{Z}_2$ acts on the fermions by $\lambda_\alpha \to -\lambda_\alpha$, which is a 360° rotation. This $\mathbb{Z}_2$ symmetry is hard to break.

It is now generally believed that this theory has these N supersymmetric vacua and not more. For other gauge groups, the analysis proceeds in the same manner, by replacing N by the dual Coxeter number $C(\text{adj})$ of the gauge group under consideration. For example, we have $N-2$ vacua for the pure $\mathcal{N}$=1 $\mathrm{SO}(N)$ gauge theory.

2.4 The theory in a box

2.4.1 $\mathrm{SU}(N)$

It is instructive to recall another way to compute the number of vacua in the $\mathcal{N}$=1 pure Yang-Mills theory with gauge group G, originally discussed in [3]. We put the system in a spatial box of size $L \times L \times L$ with the periodic boundary condition in each direction. We keep the time direction as $\mathbb{R}$.

By performing the Kaluza-Klein reduction along the three spatial directions, the system becomes supersymmetric quantum mechanics with infinite number of degrees of freedom.

The box still preserves the translation generators P^μ and the supertranslations Q_α unbroken. We just use a linear combination $\mathcal{Q}$ of Q_α and $Q_\alpha^\dagger$, satisfying

$$H = P^0 = \{\mathcal{Q}, \mathcal{Q}^\dagger\}. \tag{2.4.1}$$

We also have the fermion number operator $(-1)^F$ such that

$$\{(-1)^F, \mathcal{Q}\} = 0. \tag{2.4.2}$$

Consider eigenstates of the Hamiltonian H, given by

$$H|E\rangle = E|E\rangle. \tag{2.4.3}$$

In general, the multiplet structure under the algebra of $\mathcal{Q}$, $\mathcal{Q}^\dagger$, H and $(-1)^F$ is of the form

$$\begin{array}{ccccc} & \nearrow\!\!\!\!\swarrow & \mathcal{Q}^\dagger|E\rangle & \leftrightarrow & (\mathcal{Q}^\dagger\mathcal{Q} - \mathcal{Q}\mathcal{Q}^\dagger)|E\rangle \\ |E\rangle & \leftrightarrow & \mathcal{Q}|E\rangle & \nearrow\!\!\!\!\swarrow & \end{array} \tag{2.4.4}$$

involving four states. When $\mathcal{Q}|E\rangle = 0$ or $\mathcal{Q}^\dagger|E\rangle = 0$, the multiplet only has two states. If $\mathcal{Q}|E\rangle = \mathcal{Q}^\dagger|E\rangle = 0$, the multiplet has only one state, and E is automatically zero due to the equality

$$E\langle EE\rangle = \langle E|H|E\rangle = \langle E|(\mathcal{Q}\mathcal{Q}^\dagger + \mathcal{Q}^\dagger\mathcal{Q})|E\rangle = |\mathcal{Q}|E\rangle|^2 + |\mathcal{Q}^\dagger|E\rangle|^2. \tag{2.4.5}$$

We see that a bosonic state is always paired with a fermionic state unless $E = 0$.

This guarantees that the Witten index

$$\operatorname{tr} e^{-\beta H}(-1)^F = \operatorname{tr}\big|_{E=0}(-1)^F \tag{2.4.6}$$

is a robust quantity independent of the change in the size L of the box: when a perturbation makes a number of zero-energy states to non-zero energy $E \neq 0$, the states involved are necessarily composed of pairs of a fermionic state and a bosonic state. Thus it cannot change $\operatorname{tr}(-1)^F$.

Therefore, we can compute the Witten index in the limit where the box size L is far smaller than the scale Λ^{-1} set by the dynamics. Then the system is weakly coupled, and we can use perturbative analysis. To have almost zero energy, we need to have $F_{\mu\nu} = 0$ in the spatial directions, since magnetic fields contribute to the energy. Then the only low-energy degrees of freedom in the system are the holonomies

$$U_x, U_y, U_z \in \mathrm{SU}(N), \tag{2.4.7}$$

which commute with each other. *Assuming that they can be simultaneously diagonalized*, we have

$$U_x = \operatorname{diag}(e^{i\theta_1^x}, \ldots, e^{i\theta_N^x}), \tag{2.4.8}$$

$$U_y = \operatorname{diag}(e^{i\theta_1^y}, \ldots, e^{i\theta_N^y}), \tag{2.4.9}$$

$$U_z = \operatorname{diag}(e^{i\theta_1^z}, \ldots, e^{i\theta_N^z}). \tag{2.4.10}$$

together with gaugino zero modes

$$\lambda_1^{\alpha=1}, \ldots, \lambda_N^{\alpha=1}, \quad \lambda_1^{\alpha=2}, \ldots, \lambda_N^{\alpha=2} \tag{2.4.11}$$

with the condition that

$$\sum_i \theta_i^x = \sum_i \theta_i^y = \sum_i \theta_i^z = 0, \quad \sum_i \lambda_i^{\alpha=1} = \sum_i \lambda_i^{\alpha=2} = 0. \tag{2.4.12}$$

The wavefunction of this truncated quantum system is given by a linear combination of states of the form

$$\lambda_{i_1}^{\alpha_1} \lambda_{i_2}^{\alpha_2} \cdots \lambda_{i_\ell}^{\alpha_\ell} \psi(\theta_i^x; \theta_i^y; \theta_i^z) \tag{2.4.13}$$

which is invariant under the permutation acting on the index $i = 1, \ldots N$. To have zero energy, the wavefunction cannot have dependence on $\theta_i^{x,y,z}$ anyway, since the derivatives with respect to them are the components of the electric field, and they contribute to the energy. Thus the only possible zero energy states are just invariant polynomials of λs. We find N states with the wavefunctions given by

$$1,\ S,\ S^2,\ \ldots,\ S^{N-1} \tag{2.4.14}$$

where $S = \sum_i \lambda_i^{\alpha=1} \lambda_i^{\alpha=2}$. They all have the same Grassmann parity, and contribute to the Witten index with the same sign. Thus we found N states in the limit of small box, too.

2.4.2 SO(3)

It is known that three commuting holonomies $U_{x,y,z} \in G$ for general G can *not* be diagonalized in general; $\mathrm{SU}(N)$ is very special. The simplest case is in fact $\mathrm{SO}(3)$, where we take

$$U_x = \mathrm{diag}(+--), \quad U_y = \mathrm{diag}(-+-), \quad U_x = \mathrm{diag}(--+). \tag{2.4.15}$$

You might exclaim "they are simultaneously diagonalized" but the point is that they are not in the same Cartan subgroup.

What happens is the following [4]. Lifting from $\mathrm{SO}(3)$ to $\mathrm{SU}(2)$, we find that the holonomies $g_{1,2,3}$ lift to Pauli matrices $\sigma_{1,2,3}$. Note that $g_1 g_2 = g_2 g_1$ but $\sigma_1\sigma_2 = -\sigma_2\sigma_1$. This means that the Stiefel-Whitney class[3] w_2 of the $\mathrm{SO}(3)$ bundle, evaluated on the face C_{12} of the T^3, gives -1. Here and in the following, C_{ij} is the T^2 formed by the edges in the i-th and the j-th directions of T^3. We can similarly compute $w_2(C_{23})$ and $w_2(C_{31})$; we have $(w_2(C_{23}), w_2(C_{31}), w_2(C_{12})) = (-1,-1,-1)$.

In general, the possible choices of w_2 are $(\pm1,\pm1,\pm1)$. The commuting triples in the class $(+1,+1,+1)$ are the ones that can be simultaneously conjugated to the Cartan torus $T \subset \mathrm{SO}(3)$ discussed above, and they give 2 states. For each of the other seven choices of w_2, there is one isolated commuting triple, that gives one zero-energy state.[4] In total, we find

$$|Z_{\mathrm{SO}(3)}(L)| = 2 + 7 = 9 \qquad (L\Lambda \ll 1). \tag{2.4.16}$$

[3]This is the w_2 of the gauge bundle. In this note we only consider tori with trivial spin structure.

[4]The fermion number $(-1)^{F_1}$ of these seven states is the same as the fermion number $(-1)^{F_0}$ of the two states we found earlier. To see this, let us consider the partition function on a small T^4 with fixed w_2. When w_2 is trivial

Therefore, we should find the same when L is very, very big. But how? There are still two vacua, with $\langle \mathrm{tr}\,\lambda\lambda \rangle = \pm\Lambda^3$. But one vacuum has magnetic $\mathbb{Z}_2$ gauge symmetry while the other does not [5].

Assuming this, we can see how a very big T^3 gives 9 states: on a very big T^3, the first vacuum gives 2^3 states due to the choice of the holonomies on T^3, and the second vacuum gives just 1. In total[5], we find

$$|Z_{\mathrm{SO}(3)}(L)| = 2^3 + 1 = 9 \qquad (L\Lambda \gg 1). \tag{2.4.17}$$

This is again consistent with the computation in the opposite regime (2.4.16).

2.5 Discrete gauge field in the confined phase

In the last section we saw that for pure $\mathrm{SU}(2)$ SYM, two vacua are equivalent and trivial, but for pure $\mathrm{SO}(3)$ SYM, one vacuum is trivial but the other has $\mathbb{Z}_2$ gauge field. Is there a way to understand this difference heuristically?

2.5.1 Spectrum of line operators

For $\mathrm{SU}(2)$, Here, the spectrum of operators includes Wilson lines in the fundamental representation $(\lambda_e, \lambda_m) = (1, 0)$, or $(z_e, z_m) = (1, 0)$. Locality determines the remaining lines to be (λ_e, λ_m) with $\lambda_e \in \mathbb{Z}$, $\lambda_m \in 2\mathbb{Z}$, so no other nontrivial representations of the center are allowed.

For $\mathrm{SO}(3)$, the purely electric lines are now $(\lambda_e, \lambda_m = 0)$ with $\lambda_e \in 2\mathbb{Z}$. This set of lines can be completed in two different ways, leading to two distinct theories [6]:

- $\mathrm{SO}(3)_+$: Here the line operators are (λ_e, λ_m) with $\lambda_e \in 2\mathbb{Z}$, $\lambda_m \in \mathbb{Z}$. In other words, they have $(z_e, z_m) = (0, 0)$ or $(z_e, z_m) = (0, 1)$, including the 't Hooft line operator in the fundamental representation of the dual gauge group.

- $\mathrm{SO}(3)_-$: Here the line operators are (λ_e, λ_m) with $\lambda_e, \lambda_m \in \mathbb{Z}$ such that $\lambda_e + \lambda_m \in 2\mathbb{Z}$; they have $(z_e, z_m) = (0, 0)$ or $(z_e, z_m) = (1, 1)$. In particular, the purely electric line $(1, 0)$ and the purely magnetic line $(0, 1)$ are not present, but the dyonic line $(1, 1)$ is present.

along the spatial T^3, the T^4 partition function has the phase $(-1)^{F_0}$, independent of w_2 along the temporal-spatial directions. When w_2 is nontrivial along the spatial T^3 but trivial along the temporal-spatial direction, the partition function has the phase $(-1)^{F_1}$. These two configurations can be mapped to each other by exchanging the time and the space directions. Therefore, we should have $(-1)^{F_0} = (-1)^{F_1}$.

[5]Again, all the states have the same $(-1)^F$. Note that the $\mathbb{Z}_2$ gauge theory on T^3 has a global symmetry $\mathcal{G} := H^1(T^3, \mathbb{Z}_2)$, given by tensoring the gauge bundle by another $\mathbb{Z}_2$ bundle. The charge under $\mathcal{G}$ is $\mathcal{G}^\vee = H^2(T^3, \mathbb{Z}_2)$. Now, the 2^3 states coming from the first vacuum are permuted by $\mathcal{G}$; let us say they have $(-1)^F = (-1)^{F_a}$. The additional state from the second vacuum is invariant under $\mathcal{G}$, with $(-1)^F = (-1)^{F_b}$. Stated differently, there are one state with $(-1)^F = (-1)^{F_a}$ for each charge in $\mathcal{G}^\vee$, and another state with $(-1)^F = (-1)^{F_b}$ with zero charge in $\mathcal{G}^\vee$. Now, the two states with zero charge in $\mathcal{G}^\vee$ are the same two states in the $\mathrm{SU}(2)$ theory, and therefore have the same $(-1)^F$. Therefore we see that $(-1)^{F_a} = (-1)^{F_b}$.

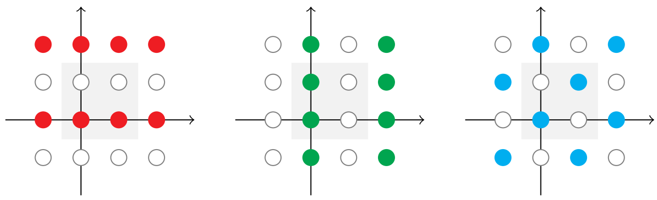

Figure 1: The weights of line operators of gauge theories with the Lie algebra $\mathfrak{su}(2)$. There, the horizontal axis is for λ_e and the vertical axis is for λ_m. The shaded regions in the figure give the $\mathbb{Z}_2$ charges.

We presented the two $\mathrm{SO}(3)$ theories through their different line operators. Alternatively, they can be described by shifting θ by 2π

$$\mathrm{SO}(3)_+^{\theta} = \mathrm{SO}(3)_-^{\theta+2\pi}. \tag{2.5.1}$$

Indeed, the Witten effect shows that under $\theta \to \theta + 2\pi$, $(\lambda_e, \lambda_m) \to (\lambda_e + \lambda_m, \lambda_m)$, which leads to the equation above. This means that when $G = \mathrm{SO}(3)$ the periodicity of θ is 4π. This is due to the fact that on spin manifolds[6], the instanton number of $\mathrm{SO}(3)$ gauge theories is a multiple of $\frac{1}{2}$. Naively, the shift of θ by 2π does not change the local physics. But since the insertion of the line operators in $\mathbb{R}^4$ creates a nontrivial topology, it allows us to distinguish θ from $\theta + 2\pi$ locally on $\mathbb{R}^4$. Note that the insertion of lines in $\mathbb{R}^4$ keeps it a spin manifold, and therefore shifting θ by 4π maps the theory to itself, relabeling the line operators.

2.5.2 Structure of the confined vacua

When $G = \mathrm{SU}(2)$, nothing strange happens. Indeed, if this theory is obtained via a mass deformation of the $\mathcal{N}=2$ pure SYM theory, these two vacua arise from the condensation of a magnetic monopole or a dyon [7], and hence the Wilson line in the fundamental representation (and all lines with $(z_e, z_m) = (1, 0)$) exhibits confinement in both vacua.

How does this story change in the theory with $G = \mathrm{SO}(3)$? First, in this theory we no longer have the $\mathbb{Z}_4$ global symmetry, which is associated with the shift $\theta \to \theta + 2\pi$. Instead, the anomaly free R-symmetry is a $\mathbb{Z}_2$ symmetry, associated with shifting $\theta \to \theta + 4\pi$. This symmetry (taking $\lambda \to -\lambda$) is part of the Lorentz group, given by a 2π rotation in spacetime. Second, it is clear that the theory still has the same two vacua as the $\mathrm{SU}(2)$ theory, but since these two vacua are related by $\theta \to \theta + 2\pi$, they are now inequivalent. The difference between the two vacua can be seen by probing the behavior of the line operators. The $\mathrm{SO}(3)_+$ theory has purely magnetic line operators with charge $(\lambda_e, \lambda_m) = (0, 1)$. In one of the two vacua dyons condense. Since they have both electric and magnetic charges, these 't Hooft lines have an area law – they are confined. In the other vacuum, the condensed particles are purely magnetic. Hence the same line operators have a perimeter law. In fact, the charge of the condensed monopole is twice that of the loop operator,

[6]On non-spin manifolds there can be "quarter instantons" and the periodicity of θ is 8π.

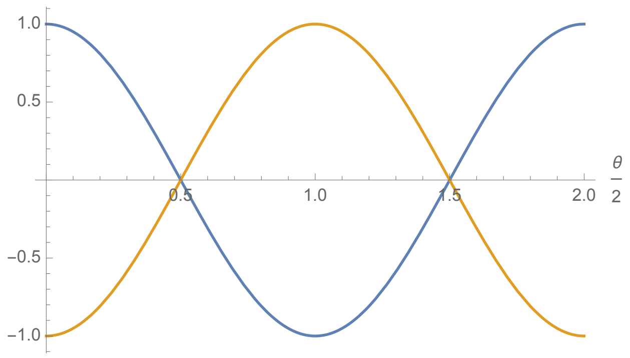

Figure 2: Energy of two vacua for softly broken $\mathcal{N}=1$ $\mathrm{SU}(2)$ SYM.

and is given by $(\lambda_e, \lambda_m) = (0, 2)$. Therefore, at low energy we find in this vacuum an unbroken $\mathbb{Z}_2$ gauge theory, acting by ± 1 on the magnetic line with charge $(\lambda_e, \lambda_m) = (0, 1)$ (or, more generally, on all lines with $(z_e, z_m) = (0, 1)$). This is an explicit example of our comment above about an unbroken discrete gauge symmetry which appears out of the magnetic degrees of freedom. The situation is similar in the $\mathrm{SO}(3)_-$ theory, except that the two vacua are exchanged.

2.6 Non-supersymmetric deformation

We can perform an analogous analysis also for the non-supersymmetric pure gauge theory with gauge groups $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. One way to obtain this theory is by adding a gluino mass m_g to the $\mathcal{N} = 1$ supersymmetric theory discussed above, which generically splits the two vacua, and taking the limit of large $|m_g|$.

Consider first the case $|m_g| \ll |\Lambda|$. Since we have a mass gap, the dynamics in each vacuum is essentially the same as above, Indeed, the soft mass term for the gaugino is just

$$\delta\mathcal{L} = m_g \lambda\lambda + c.c. \tag{2.6.1}$$

and we have the condensate $\langle \lambda\lambda \rangle \simeq \pm\Lambda^3$. So, their vacuum energy is [8, 9].

$$\sim \pm\,\mathrm{Re}(m_g \Lambda^3). \tag{2.6.2}$$

Thus, in the (unique) vacuum of the resulting theory, the $\mathrm{SU}(2)$ theory confines (exhibits an area law for its nontrivial line operator), while (depending on the phase of m_g and on the value of the θ-angle) one of the $\mathrm{SO}(3)$ theories has a perimeter law for its nontrivial line operator, with an unbroken $\mathbb{Z}_2$ gauge symmetry, while the other $\mathrm{SO}(3)$ has an area law. In the $\mathrm{SO}(3)$ gauge theory the θ-angle still has periodicity 4π, such that the two theories (and the two low-energy behaviors) are exchanged by $\theta \to \theta + 2\pi$.

In particular, something happens at $\theta = \pi$. When $|m_g| \ll |\Lambda|$, the CP is spontaneously broken there and there is a 1st order phase transition, realizing Dashen's idea [10] explicitly. The change in the behavior of the line operator at $\theta = \pi$ almost forces something nontrivial there, even when

$|m_g| \gg |\Lambda|$. To study it in detail, we need to study the CP anomaly of the system more carefully [11].

2.7 $\mathrm{Spin}(N)$ vs $\mathrm{SO}(N)$

2.7.1 $\mathrm{Spin}(N)$

Let us first recall the situation when $G = \mathrm{Spin}(N)$, first studied in the Appendix I of [12].

The dual Coxeter number is $N-2$, and therefore, there are $N-2$ vacua in the far infrared, distinguished by the gaugino condensate

$$\langle \mathrm{tr}\,\lambda\lambda \rangle = \Lambda^3, \quad \omega\Lambda^3, \quad \ldots, \quad \omega^{N-3}\Lambda^3 \tag{2.7.1}$$

where $\omega = \exp(2\pi i/(N-2))$. Therefore when the size L of T^3 is very big, we find

$$|Z_{\mathrm{Spin}(N)}(L)| = N-2, \qquad (L\Lambda \gg 1). \tag{2.7.2}$$

The commuting holonomies (g_1, g_2, g_3) can be put into either of the following standard forms:

$$g_a \in T \subset \mathrm{Spin}(N) \tag{2.7.3}$$

where T is the Cartan torus of $\mathrm{Spin}(N)$, or

$$g_a = g_a^{(7)} s_a \tag{2.7.4}$$

where $g^{(7)}_{1,2,3}$ is a lift to $\mathrm{Spin}(7)$ of the following $\mathrm{SO}(7)$ matrices

$$\begin{array}{l} \mathrm{diag}(+1,+1,+1,-1,-1,-1,-1), \\ \mathrm{diag}(+1,-1,-1,+1,+1,-1,-1), \\ \mathrm{diag}(-1,+1,-1,+1,-1,+1,-1), \end{array} \tag{2.7.5}$$

and $s_a \in T'$ where T' is the Cartan torus of $\mathrm{Spin}(N-7) \subset \mathrm{Spin}(N)$ commuting with $g^{(7)}_{1,2,3}$.

The former component gives $1 + \mathrm{rank}\,T$ zero-energy states, and the latter component gives $1 + \mathrm{rank}\,T'$ zero-energy states. In total, we find

$$|Z_{\mathrm{Spin}(N)}(L)| = (\lfloor \frac{N}{2} \rfloor + 1) + (\lfloor \frac{N-7}{2} \rfloor + 1) = N-2, \qquad (L\Lambda \ll 1). \tag{2.7.6}$$

2.7.2 $\mathrm{SO}(N)$

Now, we move on to the case $G = \mathrm{SO}(N)$. In this case, there are two choices of the discrete theta angle, so there are two theories $\mathrm{SO}(N)_\pm$, see [5]. As argued there, in the $\mathrm{SO}(N)_+$ theory all vacua have $\mathbb{Z}_2$ gauge symmetry for $\mathrm{SO}(N)_+$, but in the $\mathrm{SO}(N)_-$ theory all vacua have just $\mathbb{Z}_1$ gauge symmetry. Therefore, in the infrared, we find

$$|Z_{\mathrm{SO}(N)_+}| = 8(N-2), \quad (L\Lambda \gg 1) \tag{2.7.7}$$

and

$$|Z_{\mathrm{SO}(N)_-}| = (N-2), \quad (L\Lambda \gg 1). \tag{2.7.8}$$

Let us confirm this result in a computation in the ultraviolet, $L\Lambda \ll 1$. The topological type of the bundle is given by the Stiefel-Whitney class evaluated on the faces, $(m_{23}, m_{31}, m_{12}) \in \{\pm 1\}^3$.

When $(m_{23}, m_{31}, m_{12}) = (+1, +1, +1)$, all the commuting holonomies are obtained by projecting the $\mathrm{Spin}(N)$ commuting holonomies down to $\mathrm{SO}(N)$. Then, these give $(1 + \operatorname{rank} T) + (1 + \operatorname{rank} T') = N - 2$ zero-energy states as before.

For seven other choices $(m_{23}, m_{31}, m_{12}) \neq (+1, +1, +1)$, we can always apply $\mathrm{SL}(3, \mathbb{Z})$ to have $(m_{23}, m_{31}, m_{12}) = (-1, +1, +1)$. In [13] it was proved that the commuting holonomies are either of the form

$$g_a = g_a^{(3)} s_a \tag{2.7.9}$$

where $g_{1,2,3}^{(7)}$ is the following $\mathrm{SO}(3)$ matrices

$$\mathrm{diag}(+1, +1, +1), \quad \mathrm{diag}(-1, -1, +1), \quad \mathrm{diag}(-1, +1, -1), \tag{2.7.10}$$

and $s_a \in T''$ where T'' is the Cartan torus of $\mathrm{SO}(N-3) \subset \mathrm{SO}(N)$ commuting with $g_{1,2,3}^{(3)}$, or of the form

$$g_a = g_a^{(4)} s_a \tag{2.7.11}$$

where $g_{1,2,3}^{(4)}$ is the following $\mathrm{SO}(4)$ matrices

$$\mathrm{diag}(-1, -1, -1, -1), \quad \mathrm{diag}(-1, -1, +1, +1), \quad \mathrm{diag}(-1, +1, -1, +1), \tag{2.7.12}$$

and $s_a \in T'''$ where T''' is the Cartan torus of $\mathrm{SO}(N-4) \subset \mathrm{SO}(N)$ commuting with $g_{1,2,3}^{(4)}$.

Quantization of the zero modes then give

$$(1 + \operatorname{rank} T'') + (1 + \operatorname{rank} T''') = N - 2 \tag{2.7.13}$$

states for each of the seven choices $(m_{23}, m_{31}, m_{12}) \neq (+1, +1, +1)$. In the $\mathrm{SO}(N)_+$ theory they are all kept, but in the $\mathrm{SO}(N)_-$ theory, they have a nontrivial induced discrete electric charge $e = (m_{23}, m_{31}, m_{12})$ due to the non-zero theta angle. This causes these states to be projected out.

In total, we find

$$|Z_{\mathrm{SO}(N)_+}| = 8(N-2), \quad (L\Lambda \ll 1) \tag{2.7.14}$$

and

$$|Z_{\mathrm{SO}(N)_-}| = (N-2), \quad (L\Lambda \ll 1) \tag{2.7.15}$$

in the ultraviolet computation, agreeing with the infrared computations.

What is the distinction between $\mathrm{SO}(N)_\pm$ from the point of the view of the line operators and the point of view of the Lagrangian? As lines in N even are complicated, consider the case N odd. Then the spectrum of the line operators look similar to $\mathrm{SO}(3)$. But the smallest instanton in $\mathrm{SO}(5)$ is in fact smaller by a factor of 2 compared to that of $\mathrm{SO}(3)$. This makes the 2π shift of θ preserves $\mathrm{SO}(N)_\pm$, instead of exchanging $\pm$.

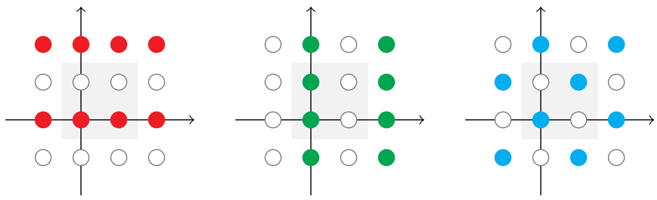

Figure 3: The weights of line operators of gauge theories with the Lie algebra $\mathfrak{so}(N)$ for N odd.

In terms of the Lagrangian, what is going on is that we can have nontrivial Stiefel-Whitney classes w_2. Then we can include a term $\mathcal{P}(w_2)/2$ where $\mathcal{P}$ is the Pointrjagin square. This is roughly of the form $w_2 \wedge w_2$ in the Lagrangian. This is defined mod 2, and therefore can have the coefficient 0 or π. $\mathcal{P}(w_2)$ is defined mod 4 on a spin manifold. We have the relation

$$p_1 = \mathcal{P}(w_2) + w_4 \mod 2 \tag{2.7.16}$$

for any SO bundle. The difference between small N and large enough N is that $p_1 = 4k$ for $N = 3$ but $p_1 = 2k$ for $N \geq 4$. This means that k can be half-integral for $N = 3$ but not for $N \geq 4$, etc.

3 Preliminaries on theories with Matters

3.1 Chiral multiplets

An $\mathcal{N}$=1 chiral multiplet Q consists of a complex scalar Q and a Weyl fermion ψ_α, both in the same representation of the gauge group. It is represented by a chiral superfield satisfying $\bar{D}_{\dot\alpha} Q = 0$, and has the expansion

$$Q(y) = Q\big|_{\theta=0} + 2\psi_\alpha(y)\theta^\alpha + F(y)\theta_\alpha\theta^\alpha \tag{3.1.1}$$

where F is auxiliary. The coefficient 2 in front of the middle component is unconventional, but this choice allows us to remove various annoying factors of $\sqrt{2}$ appearing in the formulas later. Here $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$ is the combination $\bar{D}_{\dot\alpha} y^\mu = 0$.

The complex conjugate is antichiral, satisfying $D_\alpha Q^\dagger = 0$. The product of two chiral superfields is chiral, etc.

The (effective) action has the general form

$$\int d^4\theta K(Q^\dagger, Q) + \int d^2\theta W(Q) + cc. \tag{3.1.2}$$

Expanding in components, we find that the kinetic term contains $\propto F^i \bar{F}^{\bar{j}} g_{i\bar{j}}$ where $g_{i\bar{j}} = \partial_i \bar\partial_{\bar{j}} K$, and the superpotential term contains $\propto F^i \partial_i W$. Eliminating F, we have $\bar{F}^{\bar{j}} g_{i\bar{j}} = \partial_i W$, and the potential is

$$V \sim g^{i\bar{j}} \partial_i W \bar\partial_{\bar{j}} \bar{W}. \tag{3.1.3}$$

The potential is automatically positive. The zero means it's supersymmetric, and then

$$\partial_i W = 0. \tag{3.1.4}$$

3.2 RG of the chiral multiplets

As the UV Lagrangian let's take

$$\int d^4\theta \Phi^\dagger \Phi + \int d^2\theta g \Phi^3 + cc. \tag{3.2.1}$$

How does it renormalize? Promote g to a background chiral superfield Y:

$$\int d^4\theta \Phi^\dagger \Phi + \int d^2\theta Y \Phi^3 + cc. \tag{3.2.2}$$

Assign R-charge $+2$ to Y, zero to Φ. Due to the R-charge conservation and the choice of holomorphic gauge, all-loop computations can only give

$$\int d^4\theta K(\Phi^\dagger, \Phi) + \int d^2\theta Y f(\Phi) + cc. \tag{3.2.3}$$

When Y is small, the perturbation theory is applicable, and this means that there's only the tree term in the superpotential. So we conclude $f(\Phi) = \Phi^3$. This is called the perturbative non-renormalization theorem.

What happens to K? To see this, it is useful to note that $\Phi^\dagger\Phi$ is not only the kinetic term but also the superfield version of the $\mathrm{U}(1)$ current associated to $\Phi \to e^{i?}\Phi$. Indeed, the $\theta\sigma^\mu\bar\theta$ component of $\Phi^\dagger\Phi$ contains $j^\mu = \phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi$. In our Lagrangian, the term Φ^3 breaks $\mathrm{U}(1)$ to $\mathbb{Z}_3$. Accordingly, there's a source term in the (non)conservation equation:

$$\bar D^2(\Phi^\dagger\Phi) = 3g\Phi^3 \tag{3.2.4}$$

This is obtained by taking the variation of $\Phi \to e^X \Phi$ where X is an arbitrary chiral superfield. This is the super version of the Noether procedure. Equivalently, this can be written as the OPE

$$(\Phi^\dagger\Phi)(x)\Phi^3(0) \sim 3 \cdot \frac{1}{|x|^2}\Phi^3(0) + \cdots \tag{3.2.5}$$

Equivalently, this can be written as the three-point function

$$\langle \Phi^3(x)\Phi^{\dagger 3}(y)(\Phi^\dagger\Phi)(z)\rangle = \frac{3}{|x-y|^4|x-z|^2|y-z|^2} \tag{3.2.6}$$

This can be re-written as the OPE of Φ^3 and $\bar\Phi^3$:

$$\Phi^3(x)\Phi^{\dagger 3}(0) \sim \frac{1}{|x|^6} + 3\frac{1}{|x|^4}(\Phi^\dagger\Phi)(0) + \cdots \tag{3.2.7}$$

This is exactly what is needed to compute the leading perturbation in $gg^\dagger$:

$$\int d^4x d^2\theta g \Phi^3(x) \int d^2\bar\theta g^\dagger \Phi^{\dagger 3}(0) \sim \int d^4\theta g g^\dagger 3(2\pi^2)(\log\mu)\Phi^\dagger\Phi(0) \tag{3.2.8}$$

So, if we write the renormalized kinetic term as $\int d^4\theta Z\Phi^\dagger\Phi$, we see

$$\mu\frac{\partial}{\partial\mu}Z = (2\pi^2)\cdot 3\cdot gg^\dagger. \tag{3.2.9}$$

Or equivalently, if you slightly lower the cutoff from μ' to μ,

$$\delta K = ((2\pi^2)\cdot 3\cdot gg^\dagger)\cdot\log(\mu'/\mu)\Phi^\dagger\Phi. \tag{3.2.10}$$

This corresponds to the scaling dimension $1+\gamma$ of the operator Φ given by

$$\gamma = \frac{1}{2}(2\pi^2)\cdot 3\cdot gg^\dagger. \tag{3.2.11}$$

So far we used the holomorphic scheme. To keep K fixed instead to have canonical kinetic term, we use the (non)conservation equation (3.2.4) again:

$$\begin{aligned}\int d^4\theta\delta K &= -\frac{1}{2}\int d^2\theta\bar{D}^2\delta K + c.c.\\ &= -\frac{1}{2}3g[(2\pi^2)\cdot 3\cdot gg^\dagger)\dot{\log}(\mu'/\mu)]\int d^2\theta\Phi^3 + cc.\end{aligned} \tag{3.2.12}$$

Therefore we see

$$\mu\frac{\partial}{\partial\mu}g = \frac{1}{2}3g[(2\pi^2)\cdot 3\cdot gg^\dagger)] \tag{3.2.13}$$

to the leading order; g renormalizes to zero.

Of course this is doable in the ordinary perturbation theory, but this way of doing things manifest every numerical factor rather transparently. For more details, see [14].

Exercise. Confirm this in the standard perturbation theory. This is a two-loop effect. Where in the above computation was the two-loop computation carried out?

3.3 With gauge multiplets

When there are N chiral multiplets Q_a, $(Q_a)^\dagger Q_b$ is the $\mathrm{U}(N)$ current, as we saw. So, to the leading order

$$\int d^4V^{\bar{a}b}(Q_a)^\dagger Q_b \tag{3.3.1}$$

is the coupling to the vector multiplet; the all-order version is

$$\int d^4(Q_a)^\dagger(e^V)^{\bar{a}b}Q_b. \tag{3.3.2}$$

Here we want to gauge with $\mathrm{SU}(N)$, so V is assumed to be traceless. This is anomalous (due to the perturbative triangle anomaly for $N\geq 3$, and due to the global anomaly for $N=2$.) So we add $\tilde{Q}^b$ to cancel the anomaly.

In fact we can add N_f pairs $(Q^i_a, \tilde{Q}^a_i)$ for $a = 1, \ldots, N$ and $j = 1, \ldots, N_f$. We'd like to know what happens to this theory.

Let's first analyze the one-loop running. The general formula reduces to $\mathcal{N}=1$ case to

$$E\frac{d}{dE}g = -\frac{g^3}{(4\pi)^2}\left[3C(\text{adj}) - C(R)\right] \tag{3.3.3}$$

or equivalently

$$E\frac{d}{dE}\frac{\operatorname{Im}\tau}{8\pi} = \frac{1}{16\pi^2}[3C(\text{adj}) - C(R)],. \tag{3.3.4}$$

Then

$$\eta := \Lambda^{3C(\text{adj})-C(R)} := E^{3C(\text{adj})-C(R)}e^{2\pi i\tau(E)} \tag{3.3.5}$$

is RG-invariant in the holomorphic scheme, and can be regarded as a background chiral superfield. Note that this contains $e^{i\theta}$ in the exponent, and represents the one-instanton contribution.

In our case $3C(\text{adj}) - C(R)$ is $3N - N_f$. So it is IR free when $N_f > 3N$, it is almost conformal when $N_f = 3N$ (but turns out to be IR free as you soon see), and $N_f < 3N$ the coupling starts to grow.

Let's first analyze the region where $(3N - N_f) \ll N_f$. The change in $\int d^2\theta\tau \operatorname{tr} WW$ is one-loop exact in the holomorphic gauge. What's the change in $K = Q^\dagger Q + \tilde{Q}^\dagger\tilde{Q}$? Again the essential relation is the non-conservation

$$\bar{D}^2 Q^\dagger_i Q_i = 2C(R)\frac{1}{16\pi^2}\operatorname{tr} WW, \tag{3.3.6}$$

where the sum is over a but not over i. This is known as the Konishi anomaly [15] and due to this reason the operator $Q^\dagger Q$ is often called the Konishi operator.

The leading order change in K is

$$\delta K = -\frac{\dim G}{\dim R}4C(R)\frac{1}{2\pi\operatorname{Im}\tau}\log(\mu'/\mu)(Q^\dagger Q + \tilde{Q}^\dagger\tilde{Q}) \tag{3.3.7}$$

from the standard perturbation theory. (This can be deduced from the non-conservation equation too. For details, see [14]).

Now, to keep the kinetic term canonically normalized, we use the non-conservation to rewrite this as a change in τ, as always using $\int d^4\theta\delta K = -(1/2)\int d^2\theta\bar{D}^2 K + c.c$:

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\frac{\operatorname{Im}\tau}{8\pi} = \frac{1}{16\pi^2}&(3C(\text{adj}) - 2C(R)N_f) - \\ &2N_f\cdot\frac{1}{2}\cdot(\frac{\dim G}{\dim R}4C(R)\frac{1}{2\pi\operatorname{Im}\tau})(2C(R)\frac{1}{16\pi^2}) + \cdots\end{aligned} \tag{3.3.8}$$

to this order.

We find the zero of (3.3.8) at

$$\frac{g^2}{8\pi^2} = \frac{1}{2\pi\operatorname{Im}\tau} = \frac{3N - 2N_f C(R)}{2N_f}\frac{\dim R}{4C(R)^2\dim G} + \cdots \tag{3.3.9}$$

Note that the δK above means that the dimension $1+\gamma$ of the field Q is given by

$$\gamma = -\frac{\dim G}{\dim R} 2C(R) \frac{1}{2\pi \operatorname{Im} \tau} + \cdots \tag{3.3.10}$$

as in the case of the Wess-Zumino model. This becomes

$$= -\frac{1}{2}\frac{3N - 2N_f C(R)}{2N_f C(R)} + \cdots = -\frac{3N - N_f}{2N_f} + \cdots \tag{3.3.11}$$

at the conformal point. We will soon see that in fact that the value of γ at the conformal point is exact.

4 SQCD and Seiberg duality

4.1 The conformal window

At the zero of the beta function, the system is invariant under the scaling transformation D. It's expected that the theory is not only invariant under scale symmetry, but also under conformal symmetry. The inversion is a particular conformal symmetry disconnected to the identity:

$$\mathcal{I} : x_\mu \mapsto \frac{x_\mu}{|x|^2}. \tag{4.1.1}$$

Conjugating P_μ with $\mathcal{I}$, we get K_μ. In the supersymmetric case, we get $S_{\dot\alpha}$ from conjugating Q_α by $\mathcal{I}$.

One important commutation relation is

$$\{Q_\alpha, S^\dagger{}_\beta\} = \epsilon_{\alpha\beta}(2iD + 3R) + M_{\alpha\beta} \tag{4.1.2}$$

where R here is the superconformal R symmetry. This implies that a chiral scalar operator, annihilated by Q_α, satisfies $\Delta = (3/2)R$. Indeed, in a free theory, Φ has dimension 1 and R-charge $2/3$.

Note that any $\mathrm{U}(1)$ symmetry that rotates Q with charge ± 1 is called *an* R-symmetry. The superconformal symmetry picks a particular R-symmetry, which needs to be conserved. In favorable cases, this fact can be used to fix the scaling dimension of the operators.

So, let us consider $\mathrm{SU}(N)$ with N_f flavors again. We are forced to assign R-charge 1 to the gaugino λ. To make the R-gauge-gauge anomaly vanish, the R-charge r of the fermion components ψ, $\bar\psi$ of Q, $\tilde Q$ should satisfy

$$N + r \cdot 2 \cdot \frac{1}{2} \cdot N_f = 0 \tag{4.1.3}$$

meaning that $r = -N/N_f$, meaning that $R(Q) = 1 - N/N_f$, meaning that

$$\Delta(Q) = \frac{3}{2} - \frac{3N}{2N_f} \tag{4.1.4}$$

meaning that $\gamma = -(3N - N_f)/(2N_f)$. This is *exact*.

When N_f is very close to $3N$, we have a weakly-coupled conformal fixed point. What happens when we gradually lower N_f? Consider the gauge invariant operator $\tilde{Q}Q$, which has dimension $3(1 - N/N_f)$. As we will soon review below, this is ≥ 1. Therefore, $N_f \geq 3N/2$ to have a conformal point.

4.2 Aside: unitarity bound

Let us review how such a unitarity bound is derived. Here we recall a lesser-known version [16].[7] Consider a scalar operator $\mathcal{O}$ of dimension Δ. Then its Euclidean two-point function behaves as

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{C}{|x|^{2\Delta}}. \tag{4.2.1}$$

The constant C is guaranteed to be positive: Consider smearing the operators around x and 0 with compact support. Then it should be the norm of a wave function.

A slightly more detailed use of unitarity leads $\Delta \geq 1$. To see this, we first Fourier-transform the 2pt function and write

$$\frac{C}{|x|^{2\Delta}} = C\frac{(2\pi)^2\Gamma(2-\Delta)}{4^{\Delta-1}\Gamma(\Delta)}\int \frac{d^4k}{(2\pi)^4}e^{ikx}|k|^{2(\Delta-2)}. \tag{4.2.2}$$

Wick rotating back to Minkowski signature, we find the spectral density at 4-momentum k, which is given by

$$\begin{aligned} C\frac{(2\pi)^2\Gamma(2-\Delta)}{4^{\Delta-1}\Gamma(\Delta)}\operatorname{Im}(-k^2 - i\epsilon)^{\Delta-2} &= C\frac{(2\pi)^2\Gamma(2-\Delta)}{4^{\Delta-1}\Gamma(\Delta)}\sin(\pi(\Delta-2))|k|^{2(\Delta-2)} \\ &= C\frac{(2\pi)^2\pi(\Delta-1)}{4^{\Delta-1}\Gamma(\Delta)^2}|k|^{2(\Delta-2)}. \end{aligned} \tag{4.2.3}$$

This requires $\Delta > 1$.

4.3 Seiberg duality

Long before N_f hits this lower bound $3N/2$, the anomalous dimension is of order 1 and we lose perturbative control. Is there any way out? Here comes Seiberg duality to the rescue [17].

So far we considered the $\mathrm{SU}(N)$ with N_f pairs of Q^i and $\tilde{Q}_{\tilde{i}}$, with zero superpotential. Consider another gauge theory, with the gauge group $\mathrm{SU}(N')$ with N_f pairs of q_i and $\tilde{q}^{\tilde{i}}$, a set of gauge invariant scalars $M^i_{\tilde{j}}$, and with $W = q_i\tilde{q}^{\tilde{j}}M^i_{\tilde{j}}$. Seiberg said that these two theories become the same in the infrared limit, when $N + N' = N_f$, $N, N' \geq 2$. (When one of N and N' is 1, we need a small modification.)

[7] I learned this from Yonekura-kun.

There are many checks of this duality. We'll perform an extremely detailed check in Sec. 6. For now let's see the basic ones. Firstly, the list of gauge-invariant chiral operators matches:

$$\begin{array}{rcl} \text{original} & & \text{dual} \\ \hline M^i_{\tilde{j}} := Q^i \tilde{Q}_{\tilde{j}} & \leftrightarrow & M^i_{\tilde{j}} \\ B^{i_1,\ldots,i_N} := \epsilon^{a_1,\ldots,a_N} Q^{i_1}_{a_1} \cdots Q^{i_N}_{a_N} & \leftrightarrow & b_{i_1,\ldots,i_{N'}} := \epsilon_{a_1,\ldots,a_{N'}} q^{a_1}_{i_1} \cdots q^{a_{N'}}_{i_{N'}} \\ \tilde{B}_{\tilde{i}_1,\ldots,\tilde{i}_N} := \epsilon_{a_1,\ldots,a_N} \tilde{Q}^{a_1}_{\tilde{i}_1} \cdots \tilde{Q}^{a_N}_{\tilde{i}_N} & \leftrightarrow & \tilde{b}^{\tilde{i}_1,\ldots,\tilde{i}_{N'}} := \epsilon_{a_1,\ldots,a_{N'}} \tilde{q}^{\tilde{i}_1}_{a_1} \cdots q^{\tilde{i}_{N'}}_{a_{N'}} \end{array} \tag{4.3.1}$$

where $B \leftrightarrow b$ and $\tilde{B} \leftrightarrow \tilde{b}$ are to be related by the ϵ symbol for $\mathrm{SU}(N_f)$. Note also that $m_i^{\tilde{j}} := q_i \tilde{q}^{\tilde{j}}$ is killed by the superpotential:

$$\frac{\partial W}{\partial M^i_{\tilde{j}}} = m_i^{\tilde{j}}. \tag{4.3.2}$$

[NEED TO BE CAREFUL ABOUT THE CONVERSION FACTORS]

The R-charges of these operators do match too. We first need to fix the superconformal R-charge on the dual side. M doesn't contribute to R-gauge-gauge anomaly anyway, so $R(q) = R(\tilde{q}) = 1 - N'/N_f = N/N_f$. Since $R(W) = 2$, we have $R(M) = 2 - 2N/N_f$, as before. We also see $R(b) = NN'/N_f = R(B)$.

The flavor-flavor-flavor anomalies also match. Let's just consider $\mathrm{SU}(N_f)^3_{\text{untilded}}$. We have the formal variable $F = F_s T^s$ where $s = 1, \ldots, N_f^2 - 1$. On the original side, we just have

$$\frac{1}{6} N \operatorname{tr}_{N_f} F^3. \tag{4.3.3}$$

On the dual side, we have

$$\frac{1}{6}(N' \operatorname{tr}_{\bar{N}_f} F^3 + N_f \operatorname{tr}_{N_f} F^3) = \frac{1}{6}(N_f - N') \operatorname{tr}_{N_f} F^3 \tag{4.3.4}$$

where the first comes from q and the second comes from M. They agree.

Just for fun, consider R-$\mathrm{SU}(N_f)_{\text{untilded}}$-$\mathrm{SU}(N_f)_{\text{untilded}}$. Introduce the formal variable $C = cx$ where x is the generator of $\mathrm{U}(1)_R$. On the original side, we have

$$\frac{1}{2} N(-\frac{N}{N_f}) c \operatorname{tr}_{N_f} F^2 \tag{4.3.5}$$

and on the dual side

$$\frac{1}{2}\left[N'(-\frac{N'}{N_f}) c \operatorname{tr}_{\bar{N}_f} F^2 + N_f(1 - \frac{2N}{N_f}) c \operatorname{tr}_{N_f} F^2\right]. \tag{4.3.6}$$

Again they agree, thanks to $N + N' = N_f$.

Exercise. Check the agreement of other anomalies, such as R-R-R.

Let us next check that the duality is compatible with decoupling a flavor. Consider giving a non-zero mass to one flavor, by adding

$$W = mQ^{i=N_f}\tilde{Q}_{i=N_f}. \tag{4.3.7}$$

In the scale far below m, we get ${N_f}^{\text{new}} = N_f - 1$, keeping N fixed.

What happens on the dual side? The superpotential is now

$$W = qM\tilde{q} + mM_{i=N_f}^{i=N_f}. \tag{4.3.8}$$

Setting $\partial W/\partial M_{i=N_f}^{i=N_f} = 0$, we have

$$q_{i=N_f}^a \tilde{q}_a^{i=N_f} + m = 0. \tag{4.3.9}$$

This gives a vev to $q_{i=N_f}$ and $\tilde{q}^{i=N_f}$, breaking $\mathrm{SU}(N')$ to $\mathrm{SU}(N'-1)$. Seiberg duality is compatible with this, since $N'^{\text{new}} = {N_f}^{\text{new}} - N = N' - 1$.

Finally, let us consider re-dualizing the dual theory. The dual theory has the structure

1. $\mathrm{SU}(N')$ with N_f pairs q_i, $\tilde{q}^i$.
2. Then add N_f^2 singlets M_j^i and add the coupling $\delta W = M_j^i q_i \tilde{q}^j$.

Let us dualize the first part. Then we have

1. $\mathrm{SU}(N)$ with N_f pairs $\mathcal{Q}^i$, $\mathcal{Q}_i$ with N_f^2 singlets $\mathcal{M}_i^j$ with the coupling $W = \mathcal{M}_i^j \mathcal{Q}^i \tilde{\mathcal{Q}}_j$
2. then add N_f^2 singlets M_j^i and add the coupling $\delta W = M_j^i \mathcal{M}_i^j$.

The total coupling is now

$$W_{\text{total}} = \mathcal{M}_i^j \mathcal{Q}^i \underline{\tilde{Q}}_j + M_j^i \mathcal{M}_i^j. \tag{4.3.10}$$

Taking the variation with M_j^i, we see $\mathcal{M}_i^j$ is now massive, and taking the variation with $\mathcal{M}_i^j$, we have

$$M_j^i = \mathcal{Q}^i \underline{\tilde{Q}}_j \tag{4.3.11}$$

and can eliminate M_j^i. So we get back the original theory.

4.4 Behavior of SQCD, $N < N_f < 3N$

Assuming now the validity of Seiberg duality, let's try to understand the behavior of SQCD. When N/N_f is close to $1/3$, it's a weakly-coupled superconformal theory. The dual theory has N'/N_f slightly above $2/3$. Note that the dual is very strongly coupled.

We raise N/N_f gradually, making the original theory more and more strongly coupled. The dual theory's N'/N_f decreases accordingly, making it more and more weakly coupled. When

N/N_f becomes $2/3$, we hit the unitarity bound of M. The dual theory's N'/N_f hits $1/3$, which is very weakly coupled.

Now, we can raise N/N_f even further. We can no longer expect superconformal symmetry in the infrared. But note that the check of Seiberg duality performed above only cared about having a conserved R-symmetry, not that this conserved R-symmetry is in the superconformal symmetry. So we can continue: the dual theory has N'/N_f below $1/3$. This means that the dual theory is infrared free from the start; recall that the one-loop beta function coefficient is $3N' - N_f < 0$. There is a logarithmic running of the coupling toward zero in the infrared, and indeed this is not superconformal.

How far can we go? Of course we can only have $N' \geq 0$. What happens when N' is very low can be understood by being more careful about the decoupling.

4.5 Behavior of SQCD, $N_f = N + 1$

Let us start from $\mathrm{SU}(N)$ with N_f flavors, and decouple $N' - k$ flavors by giving mass

$$W = m^i_j M^j_i \tag{4.5.1}$$

here the indices go over the last $N' - k$ of $1, \ldots, N_f$. On the dual side, we have $\mathrm{SU}(N')$ with N_f flavors, and the superpotential is

$$W = qM\tilde{q} + m^i_j M^j_i. \tag{4.5.2}$$

As before, this gives vevs to q_i and $\tilde{q}^i$ for the last $N' - k$ flavors, breaking $\mathrm{SU}(N')$ with N_f flavors to $\mathrm{SU}(k)$ with $N + k$ flavors. Assume that the resulting theories are in the infrared free region. Then everything is weakly coupled in the infrared, and we can perform the instanton computation.

Recall from (3.3.5) that one-instanton effects come with the factor $\Lambda_D^{3N'-N_f}$, which contains $e^{i\theta}$. Here the subscript D emphasizes that this is the instanton factor in the dual theory. Under the rotation $q_i \to q_i e^{i\varphi}$ fixing other q and $\tilde{q}$, this factor is also of charge 1. An invariant combination can then be written:

$$\Lambda_D^{3N'-N_f} \frac{\det M_{(N+k)\times(N+k)}}{\det m_{(N'-k)\times(N'-k)}}. \tag{4.5.3}$$

This has R-charge

$$\frac{2N'}{N_f}(N + k) - \frac{2N}{N_f}(N - k) = 2k. \tag{4.5.4}$$

Therefore, if and only if $k = 1$, one-instanton configurations can produce this superpotential.

From the point of view of the $\mathrm{SU}(1)$ theory with $N + 1$ flavors, $\Lambda_D^{3N'-N_f}/\det m$ is just a numerical factor. This means that only in this edge case, the superpotential on the dual $\mathrm{SU}(1)$ side is modified to be

$$W = qM\tilde{q} + \det M. \tag{4.5.5}$$

Note also that in this case $q_i \propto b_i = \epsilon_{ii_1\ldots i_N} B^{i_1\ldots i_N}$ and similarly for the tilded variables. Therefore we conclude: the infrared theory of $\mathrm{SU}(N)$ with $N_f = N + 1$ flavors are described by an almost free theory of the mesons $M^i_{\tilde{j}}$ and baryons B_i, $\tilde{B}^{\tilde{j}}$ with the superpotential

$$W = \frac{1}{\Lambda^{3N-(N+1)}}(B_i M^i_{\tilde{j}} B^{\tilde{j}} + \det M). \tag{4.5.6}$$

Here the powers of Λ is introduced to match the mass dimension. Those who know supersymmetric instanton calculus might worry: this looks like a (-1)-instanton effect, while in instanton computations only a positive-instanton contribution generates the supersymmetry. It is fine, since the instanton computation is only applicable in the weakly-coupled theories, whereas this is an extremely strongly coupled situation in the original variables.

The appearance of $\det M$ can be also checked by considering the dual of $\mathrm{SU}(2)$ with $N_f = 3$. In this case, $Q^{i=1,2,3}_a$ and $\tilde{Q}^a_{i=1,2,3}$ both transform in the doublet representation of $\mathrm{SU}(2)$, so can be combined to $\mathsf{Q}^a_{I=1,2,3,4,5,6}$ with $\mathrm{SU}(6)$ symmetry. The baryon B_i and $\tilde{B}^i$ are just quadratic, and can be combined with M^j_i to form

$$\mathsf{M}_{[IJ]} = \mathsf{Q}^a_I \mathsf{Q}^b_J \epsilon_{ab}. \tag{4.5.7}$$

Then the superpotential (4.5.6) can be written as

$$W = \epsilon^{IJKLMN} \mathsf{M}_{IJ} \mathsf{M}_{KL} \mathsf{M}_{MN}. \tag{4.5.8}$$

Without $\det M$, the superpotential would not be $\mathrm{SU}(6)$ invariant as it should have been.

4.6 Behavior of SQCD, $N_f = N$

The behavior with less flavors can be understood by decoupling the flavors further. Before proceeding, it is useful to understand how the instanton factors are related. We compare $\mathrm{SU}(N)$ with N_f flavors and $\mathrm{SU}(N)$ with $N_f^{\text{new}} = N_f - 1$ flavors. Let us add $mQ^{i=N_f}\tilde{Q}_{i=N_f}$ to decouple one flavor to get the latter from the former.

The one-instanton factors are respectively $\eta_{N_f} = \Lambda_{N_f}^{3N-N_f}$ and $\eta_{N_f-1} = \Lambda_{N_f-1}^{3N-(N_f-1)}$. To relate them, we can only write

$$\eta_{N_f-1} = m\eta_{N_f}, \quad \text{equivalently} \quad {\Lambda_{N_f}}^{3N-N_f} = m{\Lambda_{N_f-1}}^{3N-(N_f-1)}. \tag{4.6.1}$$

This is also natural because

$${\Lambda_{N_f}}^{3N-N_f} = E^{3N-N_f} e^{2\pi i \tau_{N_f}(E)}, \tag{4.6.2}$$

$${\Lambda_{N_f-1}}^{3N-(N_f-1)} = E^{3N-(N_f-1)} e^{2\pi i \tau_{N_f-1}(E)}, \tag{4.6.3}$$

and $\tau(E)$ needs to match around $E = m$, see figure.

Now, let us add $mM^{i=N+1}_{i=N+1}$ to (4.5.8). Taking the variation with respect to $M^{i=N+1}_{i=N+1}$, we get

$$\det M_{N\times N} - B_{i=N+1}\tilde{B}^{i=N+1} = m{\Lambda_{N_f=N+1}}^{3N-N+1}. \tag{4.6.4}$$

Translating to the notation for $N_f = N$, we get

$$\det M - B\tilde{B} = \Lambda^{2N}. \tag{4.6.5}$$

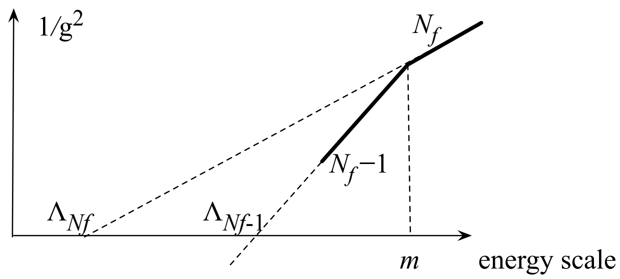

Figure 4: Running coupling.

This is a constraint rather than a superpotential.

Note that classically, $B = \det Q^i_a$, $\tilde{B} = \det \tilde{Q}^a_i$ and $M^i_j = Q^i_a \tilde{Q}^a_j$ and therefore $\det M = \det Q \det \tilde{Q} = B\tilde{B}$. This is deformed by the one-instanton effect. This deformation can be checked directly by an instanton computation [18].

4.7 Behavior of SQCD, $0 < N_f < N$

Let us decouple another flavor. To do this, implement the constraint above by a Lagrange multiplier X and add $mQ^{i=N_f=N}Q_{i=N_f=N}$:

$$W = X(\det M - B\tilde{B} - \Lambda^{2N}) + mM^{i=N}_{i=N}. \tag{4.7.1}$$

Eliminating X and $M^{i=N}_{i=N}$, we get

$$W = \frac{m{\Lambda_{N_f=N}}^{2N}}{\det M_{(N-1)\times(N-1)}}. \tag{4.7.2}$$

In the variables appropriate for $N_f = N - 1$, we have

$$W = \frac{{\Lambda_{N_f=N-1}}^{3N-(N-1)}}{\det M}. \tag{4.7.3}$$

This has the one-instanton form, correctly invariant under the rotation $Q^i \to Q^i e^{i\varphi}$, with the correct R-charge. It is known that this can be reproduced from an honest instanton computation. A generic vev to Q and $\tilde{Q}$ breaks $\mathrm{SU}(N)$ to $\mathrm{SU}(1)$, and therefore it is reliable. This result was originally obtained by Affleck, Dine and Seiberg [19], and therefore this is called the Affleck-Dine-Seiberg superpotential.

Note that the potential computed from this potential is nonzero as long as M is nonzero and finite, and decreases toward infinity. This behavior is called the runaway.

We can equally decouple k flavors from $N_f = N$. Then we have

$$W = k\left[\frac{{\Lambda_{N_f=N-k}}^{3N-(N-k)}}{\det M}\right]^{1/k} \tag{4.7.4}$$

instead. This is also called the ADS superpotential. This might be more puzzling: it looks like a $1/k$-instanton effect. Again this is fine: when we give a generic vev to M, the gauge group $\mathrm{SU}(N)$ is broken to $\mathrm{SU}(k)$ with zero massless flavors, which becomes strongly coupled and standard instanton computation is unreliable.

If we decouple all N flavors, we get

$$W = N(\Lambda^{3N})^{1/N}. \tag{4.7.5}$$

This reproduces N vacua of the pure $\mathrm{SU}(N)$ theory we saw earlier. Indeed, as UV Lagrangian is $\int d^2\theta \tau_{UV} \operatorname{tr} WW$, we have

$$\langle \operatorname{tr} WW \rangle = \frac{\partial}{\partial \tau_{UV}} W_{\text{effective}} \propto \Lambda^3. \tag{4.7.6}$$

with N branches.

5 Behavior of $\mathrm{Sp}(N)$ with N_f flavors

You think you understood the behavior of $\mathrm{SU}(N)$ SQCD? Let's try to check if you really understand, by considering other groups and other matters. The simplest generalization turns out to be to consider $\mathrm{Sp}(N)$. The analysis was originally done in [20].

5.1 What's the group $\mathrm{Sp}(N)$?

$\mathbb{R}$ and $\mathbb{C}$ have the absolute value function that satisfies $|a||b| = |ab|$. In particular, for $\mathbb{C} = \mathbb{R}^2$, this leads to the formula

$$(a^2 + b^2)(s^2 + t^2) = (as - bt)^2 + (at + bs)^2. \tag{5.1.1}$$

It is natural to wonder if we introduce similarly a bilinear product to $\mathbb{R}^n$ and have a formula

$$\sum {a_i}^2 + \sum {s_i}^2 = \sum_k (\sum_{i,j} c_k^{ij} a_i s_j)^2. \tag{5.1.2}$$

It's possible only for $n = 1, 2, 4, 8$. The product

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \tag{5.1.3}$$

defined by c_k^{ij} loses commutativity for $n = 4$, and associativity for $n = 8$.

The $n = 4$ case is known as the quaternion $\mathbb{H}$ and the $n = 8$ case is known as the octonion $\mathbb{O}$. It is standard to use the basis 1, i, j, k over $\mathbb{R}$ for $\mathbb{H}$. The multiplications are

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ij = k \tag{5.1.4}$$

and cyclic permutations.

A general element is

$$q = a + bi + cj + dk, \tag{5.1.5}$$

The conjugate is

$$\bar{q} = a - bi - cj - dk \tag{5.1.6}$$

and we define

$$|q|^2 = q\bar{q} = a^2 + b^2 + c^2 + d^2. \tag{5.1.7}$$

We can check $|qq'| = |q||q'|$.

Consider $\mathbb{H}^n$, consisting of column vectors with n elements of $\mathbb{H}$. This is a $\mathbb{H}$-linear space, where the scalar multiplication is *from the right*. A $\mathbb{H}$-linear transformation is then the matrix multiplication *from the left*

$$q_i \mapsto m_i^j q_j. \tag{5.1.8}$$

This commutes with the scalar multiplication thanks to the associativity

$$m_i^j(q_j c) = (m_i^j q_j)c. \tag{5.1.9}$$

This fails over $\mathbb{O}$ and that's why it's hard to do things over $\mathbb{O}$. Also, already for $\mathbb{H}$, it is difficult to define the determinant, due to noncommutativity.

Now, $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$ have a natural norm

$$|v|^2 = \sum_{i=1}^{n} |x_i|^2. \tag{5.1.10}$$

$\mathbb{R}$-, $\mathbb{C}$-, $\mathbb{H}$- linear transformations which preserve the norm are respectively called $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$. Note that

- For the first two, we can demand that the determinant is 1, which determine the subgroups $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$.
- When $n = 1$, they respectively become $\mathbb{Z}_2$, $\mathrm{U}(1)$, $\mathrm{Sp}(1) = \mathrm{SU}(2)$.

Another way to represent $\mathrm{Sp}(n)$ is as follows. $\mathbb{H}^n = \mathbb{C}^{2n}$, so $\mathrm{Sp}(n) \subset \mathrm{U}(2n)$. $g \in \mathrm{U}(2n)$ is in $\mathrm{Sp}(n)$ when g preserves J_{ij}, $i, j = 1, \ldots, n$ given by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.1.11}$$

$\mathrm{Sp}(n)$ is particularly simple, since the only invariant tensor is this J; the epsilon symbol can be constructed from this. The baryons can be rewritten as a polynomial of mesons.

5.2 $\mathrm{Sp}(N)$ with N_f flavors

Consider $\mathcal{N}=1$ supersymmetric $\mathrm{Sp}(N)$ gauge theory with chiral multiplets Q_i^a, where $a = 1, \ldots, 2N$. To avoid Witten's global anomaly, we need to have $i = 1, \ldots, 2N_f$. The one-instanton factor is

$$\eta = \Lambda^{3(N+1)-N_f}. \tag{5.2.1}$$

Under the anomaly-free R-symmetry, $R(Q) = 1 - (N+1)/N_f = (N'+1)/N_f$, where $(N+1) + (N'+1) = N_f$. The only gauge invariant chiral scalar operators are mesons

$$M_{ij} = Q_i^a Q_j^b J_{ab} \tag{5.2.2}$$

which is automatically antisymmetric under $i \leftrightarrow j$.

When $N_f > N+2$**:** When N_f is close to the upper bound $3(N+1)$, it's in the weakly-coupled conformal phase. We analyze what happens when we lower N_f by considering the dual, which is given by $\mathrm{Sp}(N')$ with N_f flavors q^i, $(i = 1, \ldots, 2N_f)$ and gauge singlets $M_{[ij]}$ with the superpotential $W = q^i q^j M_{ij}$. The operators match rather obviously, with the correct R-charges. The 't Hooft anomalies also match. Deforming both sides by $mM_{i=2N_f-1,j=2N_f}$, we can check the consistency under the decoupling.

Exercise. Confirm these statements.

When $N_f = N+2$**:** Decoupling $N_f - (N+2)$ flavors, as before, we see that the superpotential $W \propto \mathrm{Pf}\, M$ can be generated on the dual side by an instanton effect. In terms of the variables in the original variables, this means that $\mathrm{Sp}(N)$ with $N+2$ flavors in the infrared becomes almost free theories of mesons $M_{[ij]}$ with the superpotential

$$W = \frac{\mathrm{Pf}\, M}{\Lambda^{3(N+1)-(N+2)}}, \tag{5.2.3}$$

which has the correct R-charge and mass dimension.

When $N_f = N+1$**:** Decoupling another, one finds the constraint

$$\mathrm{Pf}\, M = \Lambda^{3(N+1)-(N+1)}. \tag{5.2.4}$$

When $N_f \leq N$**:** When $N_f = N$, we find

$$W = \frac{\Lambda^{3(N+1)-N}}{\mathrm{Pf}\, M} \tag{5.2.5}$$

which is produced by one instanton. Decoupling further, one finds

$$W = (N+1-N_f)\left[\frac{\Lambda^{3(N+1)-N_f}}{\mathrm{Pf}\, M}\right]^{1/(N+1-N_f)} \tag{5.2.6}$$

and in the extreme $N_f = 0$, one just finds

$$W = (N+1)(\Lambda^{3(N+1)})^{1/(N+1)}. \tag{5.2.7}$$

5.3 $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$ revisited

This has been totally un-surprising so far, but there is a small surprise when one recalls $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$.

The moral of this section will be the following:

- Seiberg duality tells us that there can be *two* different Lagrangians can describe the same theory in the infrared.
- In fact there can be *more than two* different Lagrangians that describe the same theory in the infrared.
- This also suggests that there might be *no* Lagrangian theory that describe a given theory in the infrared.

5.3.1 $N_f = 0, 1, 2, 3$

When $N_f = 0$, it's just the pure theory. When $N_f = 1$, there is only one meson M and

$$W = \frac{\Lambda^{6-1}}{M}. \tag{5.3.1}$$

When $N_f = 2$, the gauge invariant variables are B, M^i_j and $\tilde{B}$ (having $1 + 2^2 + 1 = 6$ in total) can be combined into $\mathsf{M}_{[IJ]}$ (having $4 \cdot 3/2 = 6$ in total) and we have the constraint

$$\det M - B\tilde{B} = \mathrm{Pf}\,\mathsf{M} = \Lambda^{6-2}. \tag{5.3.2}$$

The case $N_f = 3$ we get

$$W = \frac{\det M - B_i M^i_j \tilde{B}^j}{\Lambda^{6-3}} = \frac{\mathrm{Pf}\,\mathsf{M}}{\Lambda^{6-3}}. \tag{5.3.3}$$

5.3.2 $N_f = 4$

Now consider the case $N_f = 4$. Here $i, j = 1, \ldots, N_f$ and $I, J = 1, \ldots, 2N_f$.

As $\mathrm{SU}(2)$ with 4 flavors, the dual $\mathrm{SU}(4-2) = \mathrm{SU}(2)$ theory has the fundamentals q_i , $\tilde{q}^i$ and the 4×4 singlets M^i_j with the superpotential

$$W_1 = q_i \tilde{q}^i M^i_j. \tag{5.3.4}$$

As $\mathrm{Sp}(1)$ with 4 flavors, the dual theory is again $\mathrm{Sp}(1)$, has fundamentals q^I and a gauge-singlet antisymmetric $\mathsf{M}_{[IJ]}$ which has six $8 \cdot 7/2 = 28$ components, with the superpotential

$$W_2 = \mathsf{q}^I \mathsf{q}^J \mathsf{M}_{[IJ]}. \tag{5.3.5}$$

They are clearly different. Only $\mathrm{SU}(4)^2 \times \mathrm{U}(1)_B$ flavor symmetry is manifest in the former, while the full $\mathrm{SU}(8)$ is manifest in the latter.

The important point here is that there can be multiple duals of a single theory. We can produce more, in fact. In the $\mathrm{SU}(2)$ variables, we can rewrite W_2 as

$$W_2 = q_i \tilde{q}^i M^i_j + (q_i q_j) B^{[ij]} + (\tilde{q}^i \tilde{q}^j) \tilde{B}_{[ij]}. \tag{5.3.6}$$

Now, this theory has the structure

1. We have the $\mathrm{SU}(2)$ with four flavors q_i, $\tilde{q}^i$.
2. Add M^i_j, $B^{[ij]}$, $\tilde{B}_{[ij]}$ and add the coupling

$$\delta W = q_i \tilde{q}^i M^i_j + (q_i q_j) B^{[ij]} + (\tilde{q}^i \tilde{q}^j) \tilde{B}_{[ij]}. \tag{5.3.7}$$

Now, let us dualize the first entry as $\mathrm{SU}(2)$ with four flavors. We get

1. We have the $\mathrm{SU}(2)$ with four flavors $\mathcal{Q}^i$, $\tilde{\mathcal{Q}}_i$, and 4×4 singlets $\mathcal{M}^j_i$, and the coupling

$$W = \mathcal{Q}^i \tilde{\mathcal{Q}}_j \mathcal{M}^j_i. \tag{5.3.8}$$

2. Add M^i_j, $B^{[ij]}$, $\tilde{B}_{[ij]}$ and the coupling

$$\delta W = \mathcal{M}^j_i M^i_j + \epsilon_{ijkl} (\mathcal{Q}^k \mathcal{Q}^l) B^{[ij]} + \epsilon^{ijkl} (\tilde{\mathcal{Q}}_k \tilde{\mathcal{Q}}_l) \tilde{B}_{[ij]}. \tag{5.3.9}$$

Adding the superpotential, we can eliminate M^i_j and $\mathcal{M}^j_i$, we obtain the third dual [21]: it is again $\mathrm{SU}(2)$ with four flavors $\mathcal{Q}^i$, $\tilde{\mathcal{Q}}_i$, and gauge singlets $B^{[ij]}$, $\tilde{B}_{[ij]}$, with the superpotential

$$W = \epsilon_{ijkl} (\mathcal{Q}^k \mathcal{Q}^l) B^{[ij]} + \epsilon^{ijkl} (\tilde{\mathcal{Q}}_k \tilde{\mathcal{Q}}_l) \tilde{B}_{[ij]}. \tag{5.3.10}$$

In fact there are many ways to split $\mathsf{Q}_{I=1,\ldots,8}$ to $(Q^{i=1,\ldots,4}, \tilde{Q}_{i=1,\ldots,4})$ and there are many more duals one can consider, the entire web of which is controlled by the Weyl group of E_7, as shown in [22].

5.3.3 $N_f = 5$

The dual as $\mathrm{SU}(2)$ is an $\mathrm{SU}(3)$ gauge theory with five flavors, whereas the dual as $\mathrm{Sp}(1)$ theory is an $\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$ gauge theory with five flavors. So the duals are clearly different.

6 Supersymmetric index on $S^3 \times S^1$

Let us perform more detailed checks of Seiberg duality. I don't have the time to type a handwritten set of notes I used before, so let me just include it here. Unfortunately it's in Japanese. The rest of this page is intentionally left blank...

Super "conformal" index ①

pure SYM を T^3 にしてより深いチェックをしたように、

SQCDのSeiberg dualityも何か出来ないか？

T^3 はなされていない。$S^3 \times \mathbb{R}_t$ を考える。 Römelsberger 0510060

Festuccia-Seiberg 1105.0689

boson sym は $SU(2)_l \times SU(2)_r \times U(1)$ ← time translation.

$\mathcal{N}=1$ SUSYにするには、先のギロンより。（空間のisom が simple だから）

$SU(2|1) \times SU(2)_r$ ~~$U(1)$~~ にするしかない。

~~$U(1)_R$ を含む ← 単純な R-sym が必要~~.

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^0_{\alpha\dot\beta} P + \frac{2}{r}\sigma^i_{\alpha\dot\alpha} J^i_l$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

$$[P_0, Q_\alpha] = \frac{1}{r} Q_\alpha$$ ← supercharge は時間方向に回転する。

$S^3 \times S^1$ にして、勝手な S^1 の半径で SUSYを保つには、回転をcompensateしないといけない。

continuous unbroken
⇒ $U(1)_R$ 対称性の存在を必要とする。

$[R, Q_\alpha] = -Q_\alpha$ として $H = P_0 + \frac{1}{r}R$ を考える。

$[H, Q_\alpha] = 0$.

$$Z_{S^3\times S^1} = \mathrm{tr}(-1)^F e^{-\beta(P_0 + \frac{1}{r}R) + \mu_i J_i}$$

J_i: その他の flavor symmetry 及び J_R: 右角運動量

μ_i: chemical potential

寄与する short rep. は上記 $\{Q_\alpha, \bar{Q}_{\dot\beta}\}$ commutator より、

$P_0 = \frac{2}{r} J^z_l$ のもの.

$$= \mathrm{tr}|_{\text{short}} (-1)^F e^{-\frac{\beta}{r}(2J^z_l + R) + \mu_i J_i}.$$

ラグランジアンは書け、SUSYがある。しかし、使わなくてもよい。

（主な変更点… (w.r.t. flat sp)
$\partial_t \to \partial_t + \frac{iq}{r}$. where q is the R-charge of the field.）

計算法：Witten index だから、理論の詳細によらない。

S^3 上なので gap があるから、特に安全.

S^3-index ②

$\Phi \to \phi$, ψ

S^3 上のスカラー場は $SU(2)_\ell \times SU(2)_r$ のもとでは $\phi \to \bigoplus_j V_j \otimes V_j$ と分解.

フェルミオン場

$\phi: q$ $\quad \bar{\phi}: -q$

$\psi: q-1$ $\quad \bar{\psi}: -q+1$

$P_0 = 2j+1$ → $\bigoplus V_{j+\frac{1}{2}} \otimes V_j \oplus V_j \otimes V_{j+\frac{1}{2}}$ ← $P_0 = 2j+1$

top cmp. だけ のこる.

ψ_α $\quad \psi_\beta$

$P_0 = 2j+2$ → $V_j \otimes V_j \quad \oplus \quad V_{j+\frac{1}{2}} \otimes V_{j+\frac{1}{2}}$ ← $P_0 = 2j$

$\bar{\phi}$ $\quad \phi$

top. cmp だけ のこる.

$2(j+\frac{1}{2}) - q + 1$

残る mode:

$j=0 \quad j=\frac{1}{2} \quad j=1$ → m, n

$\bar{\psi}$

ϕ

$j=0 \quad j=\frac{1}{2} \quad j=1$

N.B. ゼロ点 energy.

$$Z = \prod_{m,n \geq 0} \frac{1 - t^{m+n-q+2}\, y^{m-n}/z}{1 - t^{m+n+q}\, z\, y^{m-n}}.$$

但し $t = e^{-\frac{\beta}{r}}$, $y = e^{-ML}$, $z = e^{-M\Phi(1)}$ …

$q = 2/3$ だと $\quad \phi^{2}{}^{?} \quad \psi^\dagger$

$1 + z t^{2/3} + (z^{+2} - z^{-1}) t^{4/3} + (y + y^{-1}) z t^{5/3} + \ldots$

ϕ with angl. mom. … cnst mode of ϕ

elliptic gamma func:

$$\Gamma_{p,q}(z) := \frac{\prod (1 - z^{-1} p^{i+1} q^{j+1})}{\prod (1 - z p^i q^j)}$$

より $\quad Z_{\text{chiral}} = \Gamma_{ty,\, t/y}(t^q z)$ である.

誘導表現の言葉

$P_0 = 2j,\ R = q \quad \phi \quad \boxed{V_j \otimes V_j}$ ココ $\quad \oplus \quad \phi^\dagger \quad \boxed{V_j \otimes V_j} \quad P_0 = 2j+2 \quad R = -q$ ← $D = 2j+…$

$P_0 = 2j+1,\ R = q-1,\ \psi \quad V_{j-1/2} \otimes V_j \qquad \bar{\psi} \quad V_{j+1/2} \otimes V_j \quad P_0 = 2j+1,\ R = -q+1$

ココ が残る.

$SU(2|1)$ の表現. $\qquad D = 2j + \frac{3}{2}$

bc $\quad V_j \otimes V_j$

$\lambda \quad V_{j+1/2} \otimes V_j \oplus V_j \otimes V_{j+1/2}$

$A \quad V_{j+1} \otimes V_j \oplus V_j \otimes V_j \oplus V_j \otimes V_{j+1}$

$A_0 = \text{cnst}.$

$$\boxed{P_0 = D - \frac{3}{2} R_{UV}}$$

$A_0 = \text{diag}(a_1, a_2 \ldots a_N)$

として $z_i = e^{-\beta a_i}$

同様に vect. mult. をやると $SU(N)$ では

$$Z = \prod_{i \neq j} \frac{1 - t^{m+n}\, y^{m-n}\, z_i/z_j}{1 - t^{m+n+2}\, y^{m-n}\, z_j/z_i} = \frac{1}{\prod_{i \neq j} \Gamma_{ty,\, t/y}\left(\frac{z_i}{z_j}\right)} \left(\frac{1}{\Gamma_{ty,\, t/y}(\ldots)} \ldots \right)$$

~~分子は…~~ ghost を ~~…~~ 考える.

但し $m=n=0$, $i=j$ は除く.

S^3 index ③

SU(2) で N_f flavor ある場合. Q の $R = 1-\frac{N_c}{N_f}$ だから

z_1, z_2 ; z, z_1^{-1} $\quad \mu_1 \cdots \mu_{N_f}$ ν

$\tilde\mu_1 \cdots \tilde\mu_{N_f}$ ν として $\prod\mu = \prod\tilde\mu = 1$

$$\underbrace{\frac{1}{\Gamma'(1)\prod_{\pm}\Gamma(z^{\pm2})}}_{\text{vector}} \underbrace{\prod_{\pm,i}\Gamma(t^{1-\frac{2}{N_f}} z^{\pm1}\mu_i \nu)}_{Q} \underbrace{\prod_{\pm,i}\Gamma(t^{1-\frac{2}{N_f}} z^{\pm1}\tilde\mu_i \nu^{-1})}_{\tilde Q}$$

SU(2) gauge inv part に proj. out.

$\|$

$$\begin{pmatrix} z & \\ & z^{-1}\end{pmatrix} = e^{-i\beta\begin{pmatrix}A_0 & \\ & -A_0\end{pmatrix}}$$ と書いたときの A_0 の path integral.

単に $\oint \frac{dz}{2\pi i z}$ すればよい.

(SU(2) を U(1) に gauge fix するときの Vandermonde det. は既に $\frac{1}{\prod\Gamma(z_i/z_j)}$ の $m=n=0$ の項に $\prod_{i\neq j}\left(1-\frac{z_i}{z_j}\right)$ として含まれている.)

$N_f=3$ だと, low energy desc dual は $M = Q_i\tilde Q_j$ $\cancel{\quad}$ $B^k = Q_iQ_j\epsilon^{ijk}$, $\tilde B^k = \tilde Q\tilde Q$, のはず.

$R=2/3$

$$\stackrel{?}{=} \prod\Gamma(t^{2/3}\mu_i^{-1}\nu^2) \prod\Gamma(t^{2/3}\tilde\mu_i^{-1}\nu^{-2})\prod\Gamma(t^{2/3}\mu_i\tilde\mu_j)$$

べき展開して, 下から確認可能.

$$\text{UV} \quad SU(2)\ N_f=3$$
$$\downarrow$$
$$\text{IR} \quad M, B^k, \tilde B^k$$

$N_f=4$

$$\text{UV}_1 \searrow \text{IR} \swarrow \text{UV}_2$$

S^1/S^3 index は IR では同じはず.
$\Rightarrow$ UV でも同じはず.

$$\oint\frac{dz}{2\pi i z}\frac{1}{\Gamma'(1)\prod\Gamma(z^{\pm2})}\prod\Gamma(t^{1/2}z^{\pm1}\mu_i\nu)\prod\Gamma(t^{1/2}z^{\pm1}\tilde\mu_i\nu^{-1})$$

$$\stackrel{?}{=}\left[\oint\frac{dz}{2\pi i z}\frac{1}{\Gamma'(1)\prod\Gamma(z^{\pm2})}\prod\Gamma(t^{1/2}z^{\pm1}\mu_i^{-1}\nu)\prod\Gamma(t^{1/2}z^{\pm1}\tilde\mu_i^{-1}\nu^{-1})\right]\prod\Gamma(t\mu_i\tilde\mu_j)$$

Sheet Date

S^3-index ④

一般に $SU(N_c)$ N_f $\overset{Q}{\underset{\tilde{Q}}{}}$ ↔ $SU(N_f-N_c)$ N_f $\overset{q}{\underset{\tilde{q}}{}} + M$

の S.D. は elliptic Γ の積の(積分の)とんでも無い公式を導く。

Rains, math.QA/0309252 で証明されている
(その当時は S.D. との関係は未知).

その他沢山 Seiberg duality は知られている。
→ 沢山の Γ の等式。証明されているものは少ない

困 この Rains の論文で、$SU(2)$ $N_f=4$ をちゃんと考えると $SU(8) \to E_7$ を示している。

困 5d $\mathcal{N}=1$ on $S^4\times S^1$ と同様のことが出来る。

物理的には

$$\text{free hyper } Z = \frac{1}{(tz)_{ty,t/y}\;(t/z)_{ty,t/y}} \qquad (z)_{p,q} = \prod_{i,j\geq 0}(1-zp^iq^j)$$

X ← $(tz)_{ty,t/y}$, Y ← $(t/z)_{ty,t/y}$

Dimotte, Gaiotto 1209.1404

5d hyper → 4d chiral (X, Y) (R, $\bar{R}$)

boundary superp. $\begin{cases} W = X\mathcal{O} \\ Y|_{\text{bndry}} = \mathcal{O} \end{cases}$ は half-BPS condition.

hemisphere index

$$= \frac{1}{(tz)_{ty,t/y}} \times (S^3\text{-index}).$$

$SU(2)$ 4 flavor Q_i $\tilde{Q}^i$
$q_{i=1\cdots 8}$ と思っておく。fugacity $\mu_1 \cdots \mu_8$ $\prod \mu = 1$

$M_{[ij]} = q_i q_j$ ← 28個 (さっきは $\underset{16}{M}, \underset{6}{B}, \underset{6}{\bar{B}}$ と書いた。)

5d bulk に X^{ij}, Y_{ij} を導入。$\begin{cases} W = XM \\ Y| = M \end{cases}$ とする。

乙 $J(\mu_1 \cdots \mu_8)$ ~~index~~

$$= \frac{1}{\prod_{i<j}(t\mu_i^{-1}\mu_j^{-1})_{ty,t/y}} \int \frac{dz}{2\pi i z} \frac{1}{\Gamma(1)} \frac{1}{\Gamma(z^{\pm 2})} \prod_{i=1}^{8} \Gamma(t^{1/2} z^{\pm}\mu_i)$$

この prefactor をつければよい。

~~構成から SU(8) 対称性を持つ。~~

↓ 次頁につづいた後

~~数学者 ⇒ E_7 sym が(index の段階で)ある (上記 Rains の論文.~~)

Sheet ______ Date ______

S^3-index ⑤

もう少し説明すると. SU(2) $N_f=4$ の index を $I(\mu_1 \cdots \mu_f)$ と書くと.
Seiberg duality は

$$I(\mu_1 v, \mu_2 v, \mu_3 v, \mu_4 v, \tilde{\mu}_1 v^{-1}, \tilde{\mu}_2 v^{-1}, \tilde{\mu}_3 v^{-1}, \tilde{\mu}_4 v^{-1})$$

$$= \frac{(t \mu_i^{-1} \tilde{\mu}_j^{-1})_{ty, t/y}}{(t \mu_i \tilde{\mu}_j)_{ty, t/y}} I(\mu_1^{-1} v, \cdots \mu_4^{-1} v; \tilde{\mu}_1 v^{-1} \cdots \tilde{\mu}_4^{-1} v^{-1})$$

両辺に $\dfrac{1}{(t\mu_i^{-1}\tilde{\mu}_j^{-1})} \dfrac{1}{t\mu_i^{-1}\mu_j^{-1}v^{-2})(t\tilde{\mu}_i^{-1}\tilde{\mu}^{-1}v^{+2})}$ をかけると

i) ☆ $J(\mu_1 v \cdots \mu_4 v, \tilde{\mu}_1 v^{-1} \cdots \tilde{\mu}_4 v^{-1}) = J(\mu_1^{-1} v \cdots \mu_i^{-1} v, \tilde{\mu}_1^{-1} v^{-1} \cdots \tilde{\mu}_4^{-1} v^{-1})$

iii) ☆ $J(\mu_1 \cdots \mu_f)$ は $\mathfrak{S}_8$ で不変.

$SU(2) = Sp(1)$ と思うと, $Sp(N_c)$ N_f flavor $8_1 \cdots 2N_f$
$\Updownarrow$
$Sp(N_f - N_c - 3)$, N_f flavor. Q^c + meson.

ii) ☆ $J(\mu_1 \cdots \mu_f) = J(\mu_1^{-1} \cdots \mu_f^{-1})$.

はじめの config : 1コ
i) は ${}_8C_4 = 70$ とおり. $\Big\}$ 72コ. $\dfrac{W(E_7)}{W(A_7)} = 72$.
ii) は 1コ

この prefactor $\dfrac{1}{(t\mu_i^{-1}\mu_j^{-1})_{t/y, ty}}$ は何か? → $\boxed{\text{丙}}$ に戻る

$\boxed{\text{丁}}$ $W = X^{ij} M_{ij}$
$Y^{ij}| = M_{ij}$ において Seiberg dual をすると

一部の M は boundary elementary field になる.

$\begin{cases} W = \overset{2d}{X} \overset{2d}{M} \\ \overset{2d}{Y}| = M \end{cases}$ とするのは 単に $X| = \frac{\partial W}{\partial M} = 0$, $Y = M$: neumann と等価.

index の level では

$$\frac{1}{(tz)_{ty, t/y}} \cdot \Gamma_{ty, t/y}(tz^{-1}) = \frac{1}{(tz^{-1})_{ty, t/y}}$$

← M は別にもはや bdry に いない.

⤳ IR では E_7-inv. bdry condition になるのであろう.

(X, Y)
28コ 28コ ← E_7 の half-hyper.

E_7 のはず.
SU(8) SU(8)
$SU(4) \times SU(4) \times U(1)$

Seiberg-Dual では
$W = c(X\overset{6}{B} + \tilde{X}\overset{6}{\tilde{B}})$
$+ c' YM$
$\;\;\;\uparrow\;\;\;\;\;\;12$
$\;\;\;$ c_a 関数.

$W = cXM$ $\xrightarrow{\text{Seib. dual}}$
$\;\;\;\;\;28\;28$

~~$\mu_i^{-1} v$, μ_i~~

$c = c'$ 位になるくまで流すと,
dual frame で
$SU(4) \times SU(4) \times U(1) \to SU(8)$

6.1 An application of SCI

The SCI can be used to decide the behavior of a confusing supersymmetric gauge theory. Consider the $\mathrm{SU}(2)$ theory with one chiral superfield Q in 4, i.e. the 3-index symmetric traceless tensor, or equivalently the spin $3/2$ representation [23]. The global anomaly is absent, so it is OK to consider this theory. The one-loop beta function is the same as $N_f = 5$ flavors of doublets. So this is asymptotically free. Under the anomaly-free R-symmetry, $R(Q) = 3/5$.

The basic gauge-invariant chiral superfield is $U := Q^4$ with all indices contracted; there is unique such thing. One can entertain two possibilities for the IR behavior of the theory at this point:

1. It is given just by U as a chiral scalar, rather as in the 'deformed moduli' case.
2. It is a nontrivial superconformal theory.

We now know the answer is the second. But historically, this conclusion was reached in a complicated process [24, 25, 26].

As a support for the first possibility, the authors of [23] computed the 't Hooft anomaly for the anomaly-free $\mathrm{U}(1)_R$ symmetry, and compared the values in the IR and in the UV. They magically agreed. That did not prove the description 1, but at least was a support.

Gradually, it was noticed that the description 2 is more plausible [24, 25]. The definitive argument in favor of the latter came after the SCI was introduced. In [26] the author computed the SCI in the UV gauge description and that in the proposed IR free description. They were clearly different, therefore the choice 1 was ruled out.

7 SO SQCD

Let us now consider $\mathrm{SO}(N)$ SQCD with N_f flavors Q_i, $i = 1, \ldots, N_f$ in the vector representation. There are a lot of surprises in this case.

7.1 $\mathrm{SO}(3)$ with one flavor

The simplest nontrivial example is $\mathrm{SO}(3)$ with $N_f = 1$ flavor Q. Giving a vev, $\mathrm{SO}(3)$ breaks to $\mathrm{SO}(2)$. Therefore, generically, the low energy theory contains a massless Abelian gauge field W_α. It also has a massive W-bosons and the 't Hooft-Polyakov monopoles.

Its coupling depends on the vev. The gauge-independent combination of the vev is $u := Q \cdot Q$. The low-energy Lagrangian would have the form

$$\propto \int d^2\theta \tau(u) W_\alpha W^\alpha + cc. \tag{7.1.1}$$

We have to determine $\tau(u)$ as a locally holomorphic function of u.

When $|u|$ is large, the coupling is weak, and the running can be computed, and we have

$$\tau(u) = -\frac{1}{2\pi i} \log \frac{u}{\Lambda^2} + \cdots \tag{7.1.2}$$

where Λ^4 is the one-instanton factor.

The anomaly-free R-symmetry is easily computed: $R(Q) = 0$. This is not very useful. The more useful is the anomalous $\mathrm{U}(1)$ symmetry rotating $Q \to e^{i\varphi}Q$. This shifts θ by 4φ. Equivalently Λ^4 is charge 4. Then the $\mathbb{Z}_4$ subgroup remains a conserved symmetry. There is also a corresponding anomalous conservation law. In components it is

$$\partial_\mu J^\mu_Q \propto \mathrm{tr}\, F\tilde{F} \tag{7.1.3}$$

and in terms of the superfield we have

$$\bar{D}^2 Q^\dagger Q = 4\frac{1}{16\pi^2}\,\mathrm{tr}\, WW. \tag{7.1.4}$$

An important clue to solve this system is to relate it to pure $\mathrm{SU}(2)$ by adding $mQ^2/2$. Then we should just have two vacua, as we already saw. In this case, the anomalous transformation law is

$$\bar{D}^2 Q^\dagger Q = 4\frac{1}{16\pi^2}\,\mathrm{tr}\, WW + mQ^2 \tag{7.1.5}$$

and therefore

$$\langle\frac{1}{16\pi^2}\,\mathrm{tr}\,\lambda\lambda\rangle = \langle\frac{1}{4}mQ^2.\rangle \tag{7.1.6}$$

Exercise. Confirm this logarithmic running, under the normalization specified below.

When $|u|$ is small enough, the leading term would have negative imaginary part, making the system unstable. Something needs to happen there. We are going to invoke the Abelian duality.

7.2 Physical and mathematical digression

The Abelian theory has the $\mathrm{SL}(2,\mathbb{Z})$ duality, generated by the exchange of (e, m):

$$S: \quad (e, m) \mapsto (-m, e) \tag{7.2.1}$$

and the shift of the theta angle by 1, which acts on the charges by the Witten effect

$$T: \quad (e, m) \mapsto (e + m, m). \tag{7.2.2}$$

In general, a duality action is given by a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \tag{7.2.3}$$

acting on (e, m) on the right and on τ as

$$\tau \mapsto \tau_M = \frac{d\tau + b}{c\tau + a}. \tag{7.2.4}$$

Now, the normalization of the Abelian gauge field we use is as follows. We declare that the W-bosons to have the electric charge $e = 2$ and that the 't Hooft-Polyakov monopoles to have the magnetic charge $m = 2$, so that the Dirac quantization condition is given by $em \in 2\mathbb{Z}$.

This somewhat unconventional choice[8] is to allow an external electric source from $\mathrm{SU}(2)$'s fundamental representation, and an external static magnetic source whose charge is half that of a 't Hooft-Polyakov monopole, which is compatible with the dynamical fields in the Lagrangian. They count as electric charge 1 or magnetic charge 1, respectively. One cannot introduce a static electric source of charge 1 and a static magnetic source of charge 1 in a simple manner, since the pair violates the Dirac quantization condition.

Our infrared Abelian theory is embedded in a bigger $\mathrm{SO}(3)$ theory. In our normalization, odd charges can only come from external sources. Therefore we can restrict the duality transformation of the Abelian theory from the full $\mathrm{SL}(2, \mathbb{Z})$ to the subgroup $\Gamma(2)$ specified by the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2. \tag{7.2.5}$$

The coupling τ modulo the action of $\Gamma(2)$ has a nice geometrical representation. Take a torus E_τ obtained by identifying $z \sim z + 1 \sim z + \tau$ of the complex plane. The group $\mathrm{SL}(2, \mathbb{Z})$ is the group of the change of the basis of the torus. The subgroup $\Gamma(2)$ is the group that fixes the four points P, Q, R and S on the torus invariant under $z \mapsto -z$. Now, take the quotient of the torus E_τ by $z \mapsto -z$. This makes the torus a double cover of a sphere, whose coordinate we call x, together with four branch points P, Q, R and S. Without loss of generality we can put P at $x = 0$, Q at $x = 1$, R at $x = \infty$. Then the position $x = \lambda$ of S is the only remaining freedom. From the construction it corresponds 1:1 to the coupling τ up to the action of $\Gamma(2)$. The original torus is given by the double cover

$$y^2 = x(x-1)(x-\lambda). \tag{7.2.6}$$

When the coupling is very weak, an explicit computation shows that

$$\lambda \sim e^{2\pi i \tau}. \tag{7.2.7}$$

When λ is very small, one can redefine the coordinate on the sphere as $x' = 1/x$. Then the torus becomes

$$y'^2 = x'(x'-1)(x'-1/\lambda). \tag{7.2.8}$$

Note that this change of variables is *not* in $\Gamma(2)$ and in fact corresponds to the S transformation. Correspondingly, we have

$$\lambda \sim e^{-2\pi i \tau_S}. \tag{7.2.9}$$

Similarly, when $\lambda \to 1$, we can change the coordinate system, and we have another dual coupling τ_{ST}.

[8] This choice leads to the form of the Seiberg-Witten curve which I wrote to be 'not very well motivated' in [27]. But as we will see below it is well motivated and useful.

7.3 $\mathrm{SO}(3)$ with one flavor, continued

After this digression, we see that our aim is to fix $\lambda(u)$ as a meromorphic function. The leading behavior is

$$\lambda(u) \sim e^{2\pi i \tau(u)} \sim c\frac{u}{\Lambda^2} \tag{7.3.1}$$

which is just a funny way of expressing the one-loop running. The full solution would be given by adding correction terms.

Suppose the full solution is found. As long as $\lambda(u)$ is neither 0, 1 nor ∞, we just have an Abelian gauge multiplet and that is it. Let us add the mass term mQQ to the original chiral Q. This is a term mu in the infrared description. This is linear in u, and therefore, these points do not survive as supersymmetric vacua.

Let us say $\lambda(u) = 0$ at $u = u_*$. Write $\lambda(u) = c(u - u_*) + \cdots$. From (7.2.9), we see that

$$\tau_S(u) = +\frac{1}{2\pi i}\log c(u - u_*). \tag{7.3.2}$$

This is an infrared free logarithmic running of the Abelian coupling, which can be reproduced assuming that there is a charged particle of mass proportional to $|u - u_*|$. Since this is the dual coupling, this particle is a monopole. The charge can be fixed by carefully following the conventions; we find that this is due to a charge-2 monopole. Let us denote by $q_\pm$ these monopole chiral multiplets. Then we have the superpotential

$$W \sim (u - u_*)q_+q_-. \tag{7.3.3}$$

Let us add $mQQ = mu$ again. The total superpotential is

$$W \sim mu + (u - u_*)q_+q_-. \tag{7.3.4}$$

Taking the variations, one find that $q_\pm$ gets a vev and $u = u_*$. Since the magnetic particles condense, this is a dual Higgs mechanism and the original electric charge is confined. From the Konishi equation (7.1.6) we see

$$\langle\lambda\lambda\rangle \simeq mu_*. \tag{7.3.5}$$

The same analysis can be carried out when $\lambda(u) = 1$ or $\lambda(u) = \infty$ at $u = u_*$. In each case, we find that a dyon of charge 2 or a electric particle of charge 2 has a mass proportional to $|u - u_*|$. Adding mQ^2 to the superpotential, there is either the oblique confinement or the Higgs mechanism, and the relation (7.3.5) holds.

Now, we know from the analysis of pure $\mathrm{SU}(2)$ that there are only two vacua. Therefore, when u is finite, there are exactly two points where $\lambda(u) = 0, 1, \infty$. We also know that $\langle\lambda\lambda\rangle \propto \pm\Lambda^3_{\text{pure}}$, and we have the matching $\Lambda^3_{\text{pure}} = m\Lambda^2$.

This is sufficient to fix $\lambda(u)$ uniquely to be

$$\lambda(u) = \frac{u}{2\Lambda^2} + \frac{1}{2} \tag{7.3.6}$$

with the torus given by

$$y^2 = x(x-1)(x - \frac{u}{2\Lambda^2} - \frac{1}{2}). \tag{7.3.7}$$

By a slight change of the variables we can also write

$$y^2 = (x^2 - \Lambda^4)(x-u). \tag{7.3.8}$$

7.4 $\mathrm{SO}(N)$ with N_f flavors, N_f sufficiently large

Let us consider $\mathrm{SO}(N)$ theory with N_f chiral fields Q^i in the vector representation. The flavor symmetry is $\mathrm{SU}(N_f)$. The one-instanton factor is

$$\eta = \Lambda^{3(N-2)-N_f}. \tag{7.4.1}$$

Under the anomaly-free R-symmetry, $R(Q) = 1 - (N-2)/N_f = (N'-2)/N_f$ where we define $(N-2) + (N'-2) = N_f$.

Close to the maximum $N_f \sim 3(N-2)$, the theory is in the weakly-coupled conformal phase. As always, we consider lowering N_f gradually. Again as always, there is the Seiberg dual description, where the gauge group is $\mathrm{SO}(N')$, with N_f flavors q_i and a gauge singlet M^{ij} in $N_f \times N_f$ symmetric, with the superpotential

$$W = q_i q_j M^{ij}. \tag{7.4.2}$$

We see that $R(q) = (N-2)/N_f$. It is straightforward to check the agreement of 't Hooft anomalies.

Exercise. Carry this out.

The mapping of the chiral operators has some surprise:

$$\begin{array}{ccc} \text{original} & & \text{dual} \\ \hline M^{ij} := Q^i Q^j & \leftrightarrow & M^{ij} \\ \epsilon_{N_f}\epsilon_N Q^N & \leftrightarrow & \epsilon_{N'} W_\alpha W^\alpha q^{N'-4} \\ \epsilon_{N_f}\epsilon_N W_\alpha W^\alpha Q^{N-4} & \leftrightarrow & \epsilon_{N'} q^{N'} \end{array} \tag{7.4.3}$$

where epsilon symbols are used to contract indices for the second line and for the third line. Only with the insertion of $W_\alpha W^\alpha$, the R-charge and the flavor symmetry structure match.

7.5 Digression: generalized Konishi anomaly

But this begs the question: why didn't we have to consider such chiral scalar operators involving $W_\alpha W^\alpha$ in $\mathrm{SU}(N)$ and $\mathrm{Sp}(N)$ SQCD? After all, we should have listed every gauge-invariant scalar chiral operator, without asking whether they are composed of scalar chiral operators.[9] For example, what happens to $\epsilon_N (W_\alpha W^\alpha Q) Q^{N-1}$ in the $\mathrm{SU}(N)$ SQCD?

[9]The content of this section is based on an unpublished discussion with Futoshi Yagi in 2007, following the joint paper [28] with Ookouchi, Kawano and the lecturer.

The simplest case to consider is $\operatorname{tr} W_\alpha W^\alpha$ itself. For this, we already know the answer. The Konishi anomaly, the anomalous transformation law under $Q \to e^{i\varphi}Q$, says

$$\bar{D}^2 Q^\dagger Q = Q\frac{\partial W}{\partial Q} + 2C(R)\frac{1}{16\pi^2}\operatorname{tr} W_\alpha W^\alpha. \tag{7.5.1}$$

Therefore, $\operatorname{tr} W_\alpha W^\alpha$ is essentially $Q\frac{\partial W}{\partial Q}$, and doesn't have to be treated independently.

This relation can be generalized even further [29], by considering the infinitesimal variation $\delta Q = \epsilon f(Q)$, to which the corresponding equation is

$$\bar{D}^2 Q^\dagger f(Q) = f(Q)\frac{\partial W}{\partial Q} + \frac{1}{16\pi^2}\operatorname{tr}_R W_\alpha W^\alpha \frac{\partial f(Q)}{\partial Q}. \tag{7.5.2}$$

This allows us to kill $\epsilon_N (W_\alpha W^\alpha Q) Q^{N-1}$ in $\mathrm{SU}(N)$ SQCD by considering the variation for $\tilde{Q}$ given by

$$\delta\tilde{Q} = \epsilon_N Q^{N-1}. \tag{7.5.3}$$

Note that the transformation on $\tilde{Q}$ is used to kill an operator with Q and W_α. Similarly, we can kill the operator $\epsilon_N W_\alpha W^\alpha Q^{N-4}$ in an $\mathrm{SO}(N)$ theory with Q in the vector representation, if there is a chiral field Ψ in the chiral spinor representation. For example, for $\mathrm{Spin}(10)$ theory with one Ψ and a number of Q's, we can consider the variation

$$\delta\Psi = \not{Q}^6 \Psi, \tag{7.5.4}$$

whose generalized Konishi implies

$$\bar{D}^2 \Psi^\dagger \Psi \sim \operatorname{tr}_{\text{chiral spinor}} \not{Q}^6 \Gamma^{ij} W^\alpha_{ij} \Gamma^{kl} W_{kl\alpha} \sim \epsilon_N W_\alpha W^\alpha Q^{N-4}. \tag{7.5.5}$$

7.6 $\mathrm{SO}(N)$ with N_f flavors, $N_f \geq N = 1$

Let us come back to the original discussion. Lowering N_f, eventually we go out of the conformal window, and the dual description becomes infrared free. Nothing of note happens up to and including $N_f = N$, for which the dual is $\mathrm{SO}(N' = 4)$, with the superpotential

$$W = q_i q_j M^{ij} \tag{7.6.1}$$

as always.

Decoupling one flavor, we get to $N_f = N - 1$, $N' = 3$. It is straightforward to see that a new term can be generated by the one-instanton configuration in the broken gauge group, and we get

$$W = q_i q_j M^{ij} + \tilde{\Lambda}^{6-2(N-1)} \det M. \tag{7.6.2}$$

7.7 $\mathrm{SO}(N)$ with N_f flavors, $N_f = N - 2$

We now add the term $\delta W = mM^{N-1,N-1}$. Close to the origin of M, we see that it gives a vev to q_{N-1}, breaking $\mathrm{SO}(3)$ to $\mathrm{SO}(2)$. The remaining superpotential is $W = q_i q_j M^{(N-2)\times(N-2)}$. Note that q_i here are in the two-dimensional representation of $\mathrm{SO}(2)$, in addition to being in the fundamental of $\mathrm{SU}(N-2)$. As field charged under $\mathrm{U}(1)$, it can be written as

$$W = q_i^+ q_j^- M^{ij}. \tag{7.7.1}$$

(There can be a complicated instanton correction $f(\det M/\Lambda^{2(N-2)})$ multiplying the whole expression, with $f(0) \neq 0$. We drop these coefficients below.)

There is another region where something happens. To see this, we imitate our analysis of $\mathrm{SO}(3)$ with one flavor, which was already given. Note that for the rest of the analysis of $N_f = N - 2$, we directly work in the original description.

The one-instanton factor is

$$\eta = \Lambda^{3(N-2)-(N-2)} = \Lambda^{2(N-2)}; \tag{7.7.2}$$

note that this is valid only for $N \geq 4$. The field Q is neutral under the anomaly-free R-charge, and there is an unbroken symmetry $\mathbb{Z}_{2(N-2)}$ on Q.

Giving generic vevs, we get low energy $\mathrm{SO}(2)$ gauge field. We need to determine its coupling as a function of $M_{ij} := \det Q_i \cdot Q_j$. This M_{ij} is in a representation of $\mathrm{SU}(N-2)_F$. The flavor symmetry says that the coupling can only depend on $U = \det M$, on which the unbroken $\mathbb{Z}_{2(N-2)}$ acts trivially.

Since we have a dynamical field in the vector representation, we can consider static electric sources in the spinor representation and also static magnetic sources which is half the charge of the dynamical 't Hooft-Polyakov monopole. Accordingly, the dynamical duality group is $\Gamma(2)$ again, and the coupling can be usefully represented in terms of $\lambda(U)$, or equivalently in terms of the equation of the torus.

We can give a big vev to $Q_{2,\ldots,N-1}$ and think of the Higgsed theory as $\mathrm{SO}(3)$. From this consideration we see

$$\lambda(U) \sim \frac{U}{\Lambda^{2(N-2)}} \tag{7.7.3}$$

in the large U region. We already know from the decoupling argument from $N_f = N - 1$ that at $U = 0$ we have $N - 2$ pairs of fields $q_i^\pm$ of charge ± 1, making the coupling at $U = 0$ to go to zero there.

This forces the equation above to be in fact essentially exact, and the curve is given by

$$y^2 = x(x - \Lambda^{2(N-2)})(x - U). \tag{7.7.4}$$

We already discussed the physics when $U \to \infty$ and $U \to 0$. There is another singular locus where $U = \Lambda^{2(N-2)}$. Note that this is not a point but a complex codimension-1 subspace in the space of M^{ij}. The fact that $\lambda \to 1$ there means that there is a light dyon $E_\pm$, with the superpotential term

$$W \sim (\det M - \Lambda^{2(N-2)}) E_+ E_-. \tag{7.7.5}$$

7.8 $\mathrm{SO}(N)$ with N_f flavors, $N_f = N-3$

Now we add $\delta W = mQ^{N-2}Q^{N-2} = mM^{N-2,N-2}$ and decouple one flavor. This term forces the vacua to be either on $\det M = 0$ or $\det M = \Lambda^{2(N-2)}$. In the former branch, $q^{\pm}_{N-2}$ condenses, $q_{1,\ldots,N-3}$ and $M^{\text{(N-3)}\times\text{(N-3)}}$ remain massless, with the superpotential

$$W \sim q_i q_j M^{ij} \tag{7.8.1}$$

where the indices are now $i,j = 1,\ldots,N-3$. This q_i is naturally identified with

$$q_i = \epsilon_{N_f=N-3}\epsilon_N W_\alpha W^\alpha Q^{N-4}. \tag{7.8.2}$$

In the latter branch, $E^{\pm}$ condenses. We find that we have the Affleck-Dine-Seiberg superpotential

$$W \sim \frac{\Lambda^{3(N-2)-(N-3)}}{\det M}. \tag{7.8.3}$$

7.9 $\mathrm{SO}(N)$ with N_f flavors, $N_f = N-4$

Let us add $\delta W = mQ^{N-3}Q^{N-3} = mM^{N-3,N-3}$ and decouple another flavor. From the former branch, we just condense $q_{N-3} = \pm\sqrt{m}$, eliminating $M^{i,N-3}$ and the rest of q_i. We obtain two branches of vacua, parameterized by $M^{(N-4)\times(N-4)}$, with zero superpotential.

From the latter branch, we get a behavior familiar from the analysis of $\mathrm{SU}(N)$ and $\mathrm{Sp}(N)$: we just get the ADS superpotential

$$W = \pm\left[\frac{\Lambda^{3(N-2)-(N-4)}}{\det M}\right]^{1/2}. \tag{7.9.1}$$

In the end we found *four* branches. They can also be understood as follows: giving a generic vev to Q, $\mathrm{SO}(N)$ is broken to pure $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$, and the one-instanton factor of either is $\Lambda'^6 = \Lambda^{3(N-2)-(N-4)}/\det M$. Each of $\mathrm{SU}(2)$ can have $W = \epsilon_{1,2}\Lambda'^3$ where $\epsilon_{1,2} = \pm 1$, with the total superpotential

$$W = (\epsilon_1+\epsilon_2)\Lambda'^3 = (\epsilon_1+\epsilon_2)\left[\frac{\Lambda^{3(N-2)-(N-4)}}{\det M}\right]^{1/2}. \tag{7.9.2}$$

7.10 $\mathrm{SO}(N)$ with N_f flavors, $N_f < N-4$

Let us add $\delta W = mQ^{N-4}Q^{N-4} = mM^{N-4,N-4}$ and decouple another field. The branch with $W=0$ is eliminated, while from the branch with $W \neq 0$ we get the standard ADS superpotential

$$W = \left[\frac{\Lambda^{3(N-2)-(N-5)}}{\det M}\right]^{1/3}. \tag{7.10.1}$$

From this point on, things regularize, and we just have the ADS superpotential

$$W = \left[\frac{\Lambda^{3(N-2)-N_f}}{\det M}\right]^{1/(N-N_f-2)} \tag{7.10.2}$$

up to and including $N_f = 1$. When $N_f = 0$ we have the pure $\mathrm{SO}(N)$ Yang-Mills.

7.11 $\mathrm{Spin}(N)$, $\mathrm{SO}(N)_+$ and $\mathrm{SO}(N)_-$

So far in this section we didn't distinguish the three possible choices, $\mathrm{Spin}(N)$, $\mathrm{SO}(N)_+$ and $\mathrm{SO}(N)_-$. In this last section we see how they are mapped under the Seiberg duality [5]. For this, it is useful to start by considering pure $\mathrm{SO}(4)$ theory with the common coupling $\Lambda_1^6 = \Lambda_2^6 =: \Lambda^6$. The superpotential is

$$W = (\epsilon_1 + \epsilon_2)\Lambda^3. \tag{7.11.1}$$

In the vacua where $W \neq 0$, the 't Hooft line has the perimeter law but the dyonic line is confined. Meanwhile, in the vacua where $W = 0$, the 't Hooft line has the area law while the dyonic line is unconfined. The $\mathrm{SO}(4)_+$ theory has the 't Hooft line and the $\mathrm{SO}(4)_-$ theory has the dyonic line.

Now, consider $\mathrm{SO}(N)$ with N_f flavors Q, whose dual is $\mathrm{SO}(N')$ with N_f flavors q where $N' = N_f - N - 4$, together with mesons M. Give a large vev to M. On the original side, we have completely Higgsed vacua, in which the spinor Wilson line has perimeter law, and both the 't Hooft line and the dyonic line have area law.

On the dual side, the vev to M gives masses to N out of N_f of q, breaking $\mathrm{SO}(N')$ to $\mathrm{SO}(4)$. The branch with $\epsilon_1 + \epsilon_2 \neq 0$ has a runaway superpotential, and the supersymmetric vacua come from the branch with $\epsilon_1 - \epsilon_2 = 0$. Therefore, the spinor Wilson line has the area law, 't Hooft line has the area law, and the dyonic line has the perimeter law.

Comparing the two descriptions, this means that we have $\mathrm{Spin}(N) \leftrightarrow \mathrm{SO}(N')_-$, and by exhaustion we have $\mathrm{SO}(N)_+ \leftrightarrow \mathrm{SO}(N')_+$.

7.12 Summary: SU, Sp, SO Seiberg dualities

Let us now summarize the behaviors we saw in three tables.

Acknowledgement

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N_f	generic unbroken	dual	behavior
$3N$	$-$	$\mathrm{SU}(2N)$	
$3N-1$	$-$	$\mathrm{SU}(2N-1)$	superconformal
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2N$	$-$	$\mathrm{SU}(N)$	superconformal, selfdual
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3/2)N$	$-$	$\mathrm{SU}(N/2)$	superconformal
$(3/2)N-1$	$-$	$\mathrm{SU}(N/2-1)$	IR free with $W = q\tilde{q}M$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$N+2$	$-$	$\mathrm{SU}(2)$	IR free with $W = q\tilde{q}M$
$N+1$	$-$	$\mathrm{SU}(1)$	$W = BM\tilde{B} + \det M$
N	$-$	$-$	$\det M - B\tilde{B} = \Lambda^{2N}$
$N-1$	$\mathrm{SU}(1)$	$-$	ADS superpotential, computable
$N-2$	$\mathrm{SU}(2)$	$-$	ADS superpotential, 2 branches
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$\mathrm{SU}(N-1)$	$-$	ADS superpotential, $N-1$ branches
0	$\mathrm{SU}(N)$	$-$	N vacua

Table 1: Behavior of SU SQCD.

N_f	generic unbroken	dual	behavior
$3(N+1)$	$-$	$\mathrm{Sp}(2N+1)$	
$3N+2$	$-$	$\mathrm{Sp}(2N)$	superconformal
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2N+2$	$-$	$\mathrm{Sp}(N)$	superconformal, selfdual
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3/2)(N+1)$	$-$	$\mathrm{Sp}((N-1)/2)$	superconformal
$(3/2)(N+1)-1$	$-$	$\mathrm{Sp}((N-1)/2-1)$	IR free with $W=qqM$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$N+3$	$-$	$\mathrm{Sp}(1)$	IR free with $W=qqM$
$N+2$	$-$	$-$	$W=\mathrm{Pf}\,M$
$N+1$	$-$	$-$	$\mathrm{Pf}\,M=\Lambda^{2(N+1)}$
N	$-$	$-$	ADS superpotential, computable
$N-1$	$\mathrm{Sp}(1)$	$-$	ADS superpotential, 2 branches
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$\mathrm{Sp}(N-1)$	$-$	ADS superpotential, N branches
0	$\mathrm{Sp}(N)$	$-$	$N+1$ vacua

Table 2: Behavior of Sp SQCD.

N_f	generic unbroken	dual	behavior
$3(N-2)$	–	$\mathrm{SO}(2N-2)$	
$3N-7$	–	$\mathrm{SO}(2N-3)$	superconformal
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2N-4$	–	$\mathrm{SO}(N)$	superconformal, selfdual
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3/2)(N-5)$	–	$\mathrm{SO}((N-7)/2)$	superconformal
$(3/2)(N-5)-1$	–	$\mathrm{SO}((N-7)/2-1)$	IR free with $W = qqM$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	–	$\mathrm{SO}(4)$	IR free with $W = qqM$
$N-1$	$\mathrm{SO}(1)$	$\mathrm{SO}(3)$	IR free with $W = qqM + \det M$
$N-2$	$\mathrm{SO}(2)$	$\mathrm{SO}(2)$	Coulomb phase, $y^2 = x(x-\Lambda^{2(N-2)})(x-U)$
$N-3$	$\mathrm{SO}(3)$	$\mathrm{SO}(1)$	one branch with $W = qqM$, another with ADS
$N-4$	$\mathrm{SO}(4)$	–	two branches parameterized by M, two more w
$N-5$	$\mathrm{SO}(5)$	–	ADS with three branches
$N-6$	$\mathrm{SO}(6)$	–	ADS with four branches
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$\mathrm{SO}(N-1)$	–	ADS superpotential, $N-3$ branches
0	$\mathrm{SO}(N)$	–	$N-2$ vacua

Table 3: Behavior of SO SQCD.

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