On the D5-D1 Conformal Field Theory

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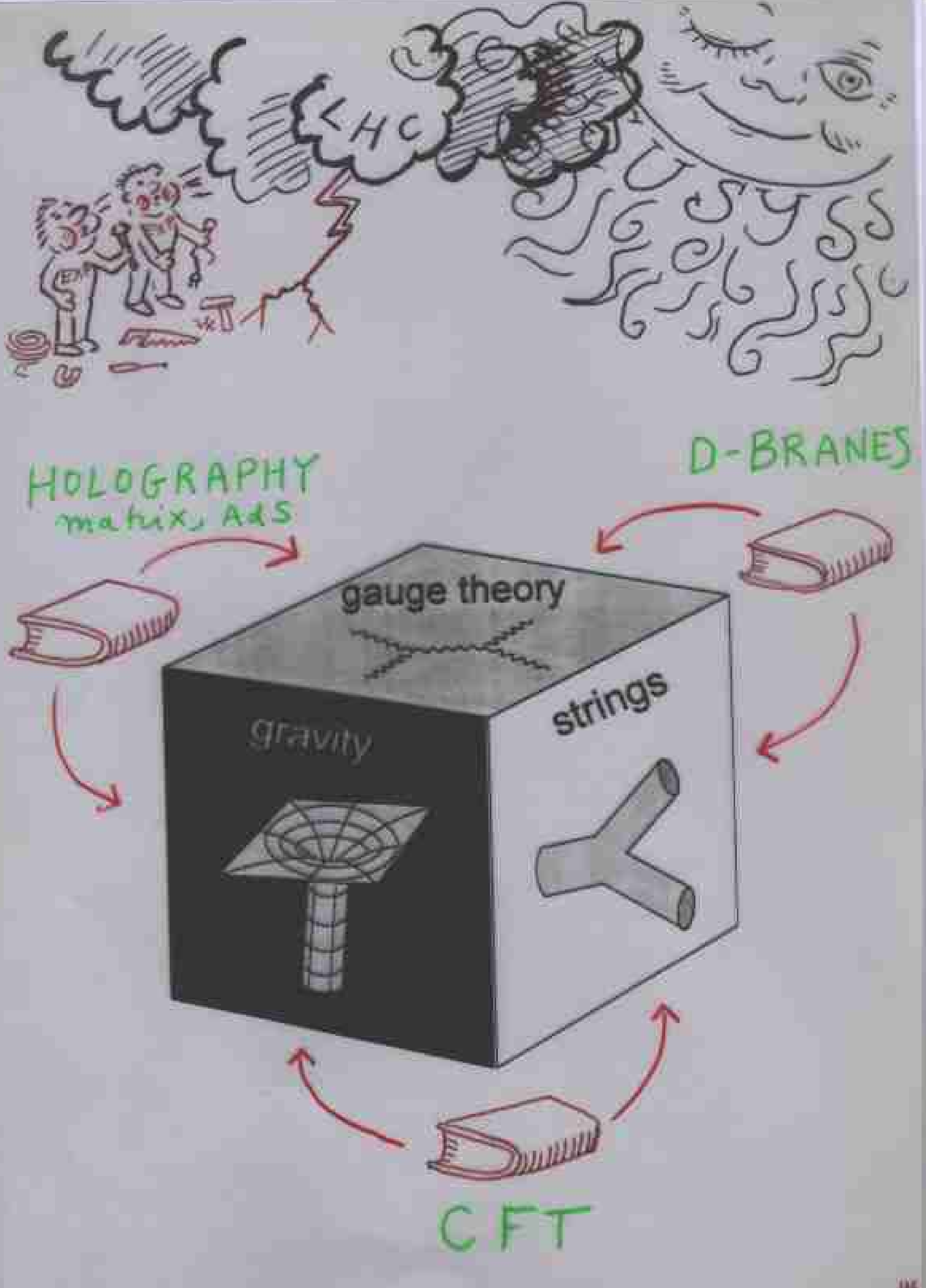
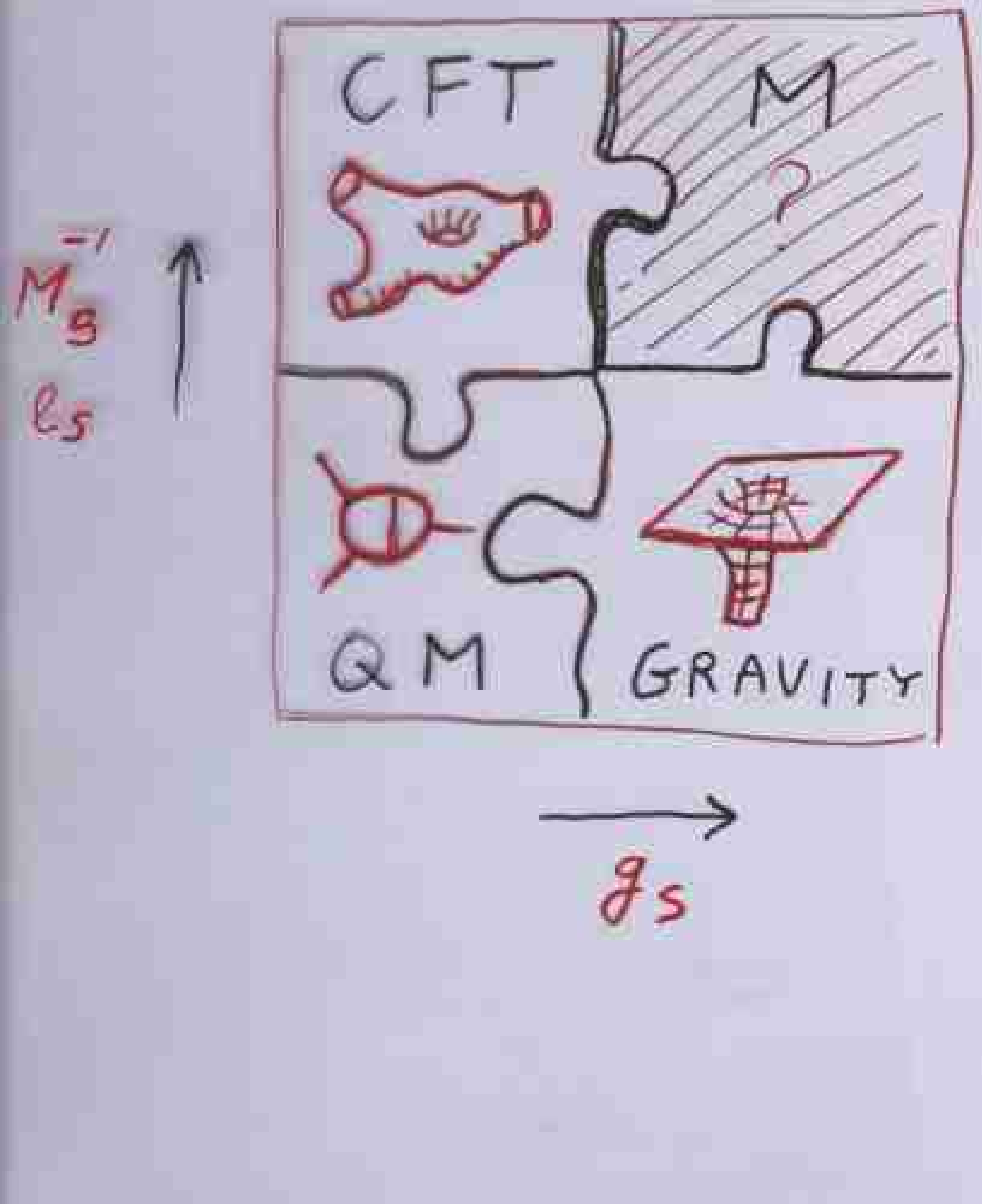
Strings '99
Potsdam


LHC
HOLOGRAPHY
matrix, AdS
D-BRANES
gauge theory
gravity
strings
CFT

MAGIC SQUARE



Big Questions:

- interpretation of g_s ?
- symmetries T-duality ?
- RR fields ?
- theory of gravity ?
  sum over metrics
- # of states ?
  holography

D5-D1 BRANE SYSTEM

dictionary

SPACETIME	CFT_{1+1}
g_s	α'_{CFT}
T	geom.
U	T
RR-fields	B-field
geometry thermodyn.	states stat. mech
questions?	answers!

IIB string on K3 (or T^4)

charge lattice of 6d strings

$$\boxed{\Gamma^{5,21} = \Gamma^{1,1}_{NS} \oplus \Gamma^{1,1}_{RR} \oplus \Gamma^{3,19}_{RR}}$$

F1 - NS5 — $\underbrace{\text{D1-D5 — D3}}$

D-brane charges: $H^{\cdot}(K3) \cong \Gamma^{4,20}$

moduli space (locally)

$$O(5,21)/O(5)\times O(21)$$

duality group

$$U = O(5,21;\mathbb{Z})$$

$v \in \Gamma^{5,21}$ $\xrightarrow{U}$ $v = (Q_1, Q_5) \in \Gamma^{1,1}_{RR}$

$v^2 > 0$ D1-D5 brane

"NEAR HORIZON" LIMIT

D1–D5 string $\simeq$ 1+1 SCFT

$\Rightarrow$ σ-model on $\mathcal{M}_v(K3)$

instanton moduli space
topol. charge v

$$v = ch(E)\,\hat{A}^{1/2}(K3)$$

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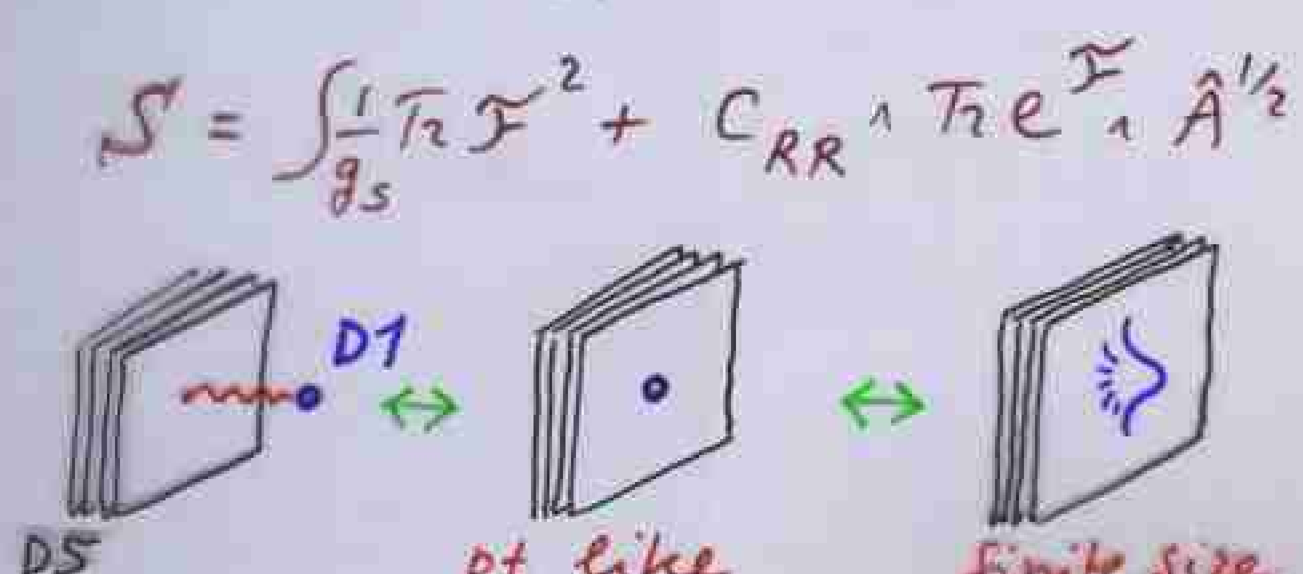
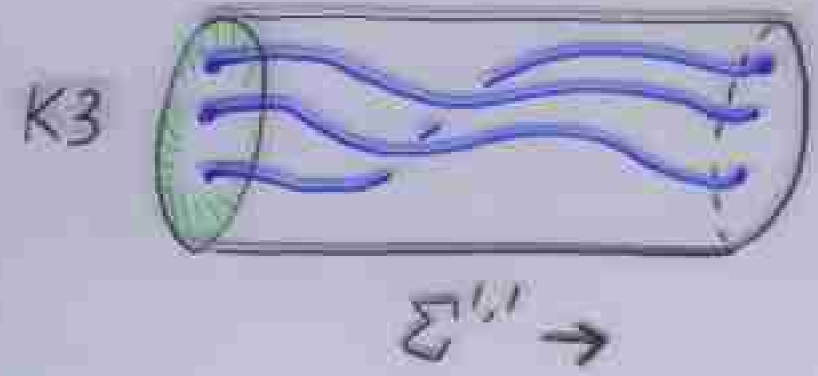
1) D5 $\triangleleft$

$U(Q_5)$ SYM_{5+1} with Q_1 instantons

$$S = \int \frac{1}{g_s} \mathrm{Tr}\,\mathcal{F}^2 + C_{RR} \wedge \mathrm{Tr}\,e^{\mathcal{F}} \wedge \hat{A}^{1/2}$$



adiabatic limit vol K3 $\ll$ vol Σ



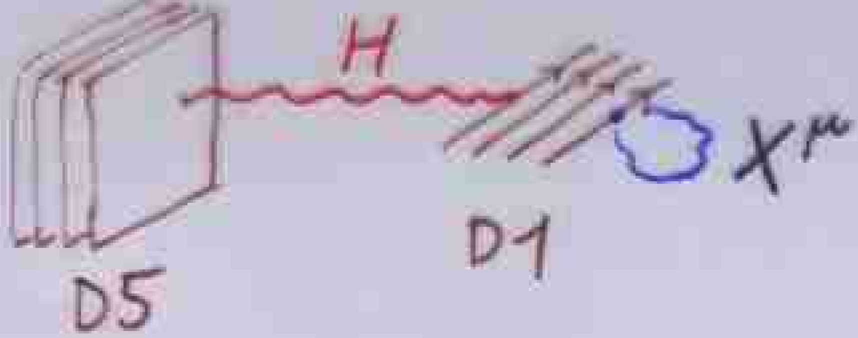
instanton strings $F_+ = 0$

map: $\Sigma \longrightarrow \mathcal{M}_v(K3)$

Space-time	CFT
NS B-field $\mathcal{F} = F - B$	metric $F_+ = B_+$
g_s $\int \frac{1}{g_s} \mathrm{Tr}\, \mathcal{F}^2$	α'_{CFT} $\int \frac{1}{\alpha'} (\partial x)^2$
RR fields $\int C_{RR} \mathrm{Tr}\, e^{\mathcal{F}}$	B-fields observables in Donaldson theory

2) D1 ◁≡

$U(Q_1)$ SYM_{1+1} with Q_5 hypers



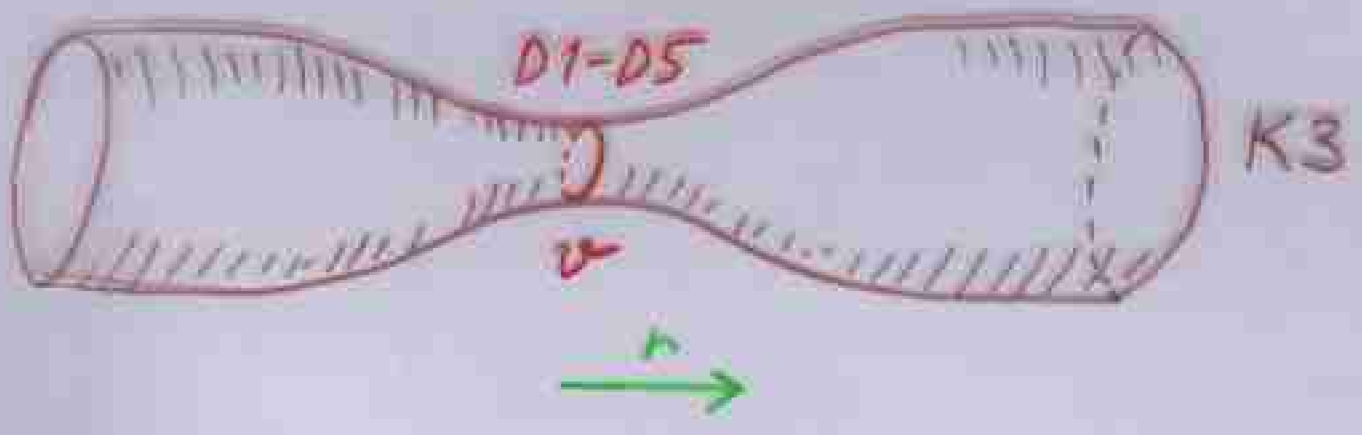
IR: σ-model on Higgs branch

$$[X^\mu, X^\nu]_+ + \bar{H}\sigma^{\mu\nu}H = 0 \Big/ U(Q_1)$$

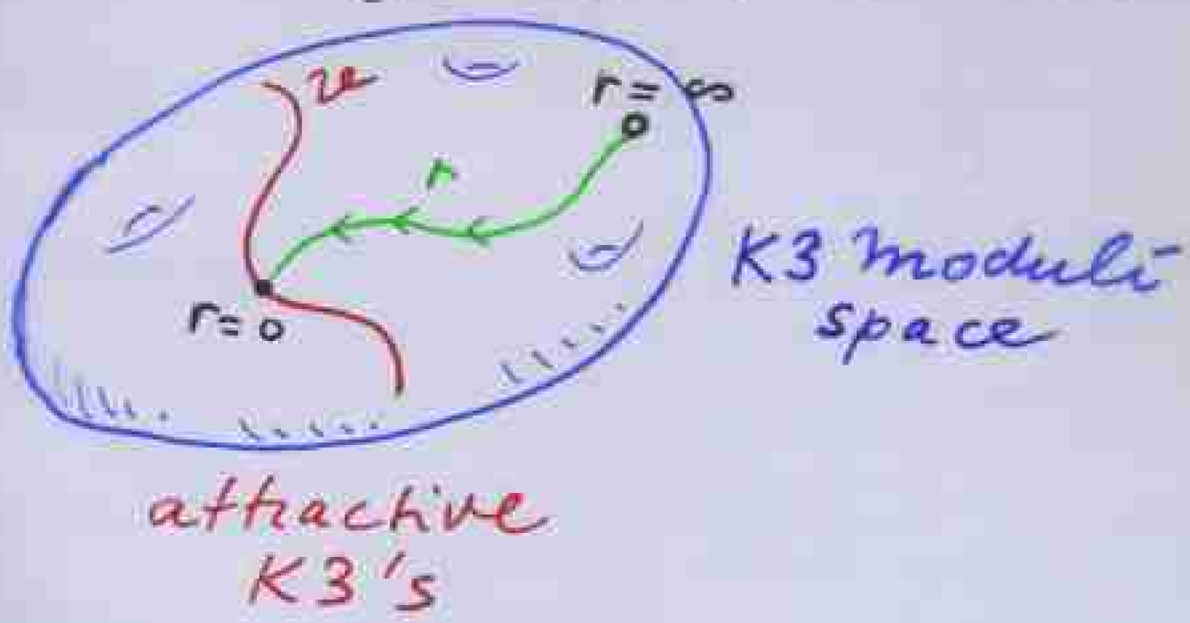
ADHM description of $\mathcal{M}_v(\mathbb{R}^4)$

B-field: non-com $\mathbb{R}^4$

ATTRACTIVE MODULI



dynamical system on K3 mod.



gradient flow

$$m_{BPS}^2 = v_L^2 = v^2 + v_R^2 \geq v^2$$

fixed pt

$$v_R = 0$$

attractive K3's

$$O(4,21)/O(4)\times O(21)$$

residual U-duality

$$U_v = \{ g \in U ;\ g\cdot v = v \}$$
$$\subset O(v^\perp) = O(4,21;\mathbb{Z})$$

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example: D1-D5
pert, no RR fields

$$O(3,20)/O(3)\times O(20)$$

general HK structure	3×19
volume (K3) fixed $= \frac{Q_1}{Q_5}$	$-$
B-field is SD	3
	3×20

match with HK on $\mathcal{M}_v$

SCFT on $\mathcal{M}_v$ = instanton strings

$\mathcal{N}=(4,4)$ SUSY $\quad c = 6k$

$\mathcal{M}_v$ is simple HK $(h^{2,0}=1)$

dim $4k$, $\quad k = 1 + \frac{1}{2}v^2$

smooth compactification including sheaves if v prim.

$$\mathcal{M}_v \overset{\text{diff}}{\simeq} \mathrm{Hilb}^k K3$$

$$\Big\downarrow \text{resolution}$$

$$S^k K3 = (K3)^k / S_k$$

orbifold CFT

$$\boxed{H^2(\mathcal{M}_v) \simeq \Gamma^{3,20} = \Gamma^{3,19} \oplus \mathbb{Z}\, w}$$

$H^2(K3)$

$w \sim$ diagonal, $w^2 = -v^2$

$x_i = x_j$

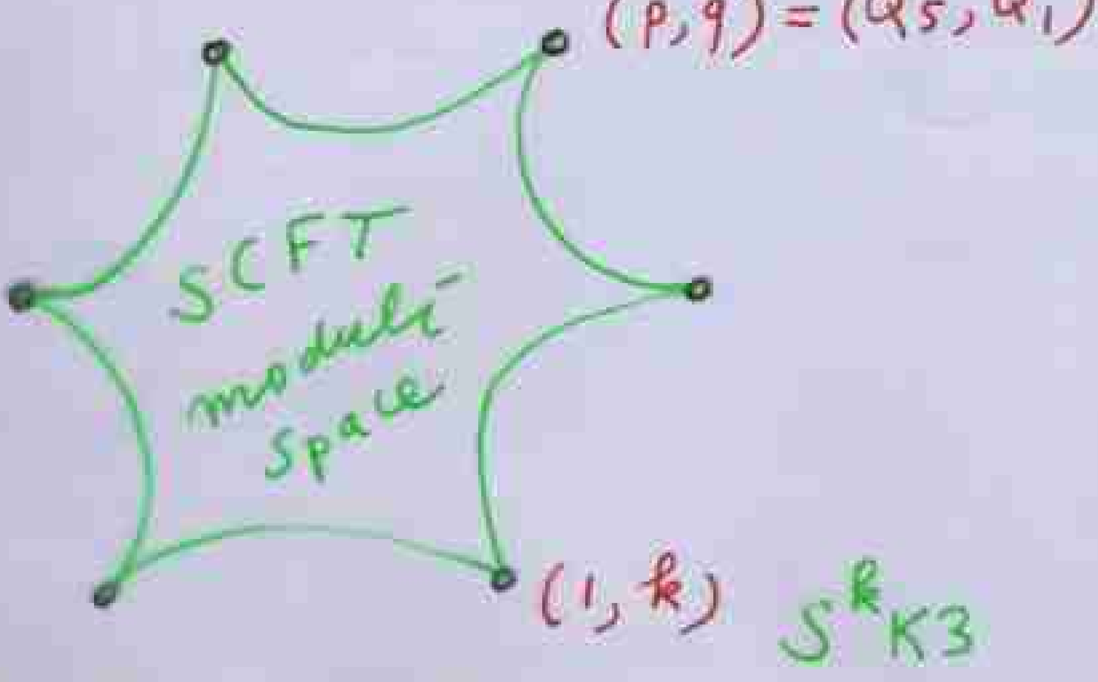
(Beauville "intersection product")

local HK moduli

$$O(3,20)/O(3) \times O(20)$$

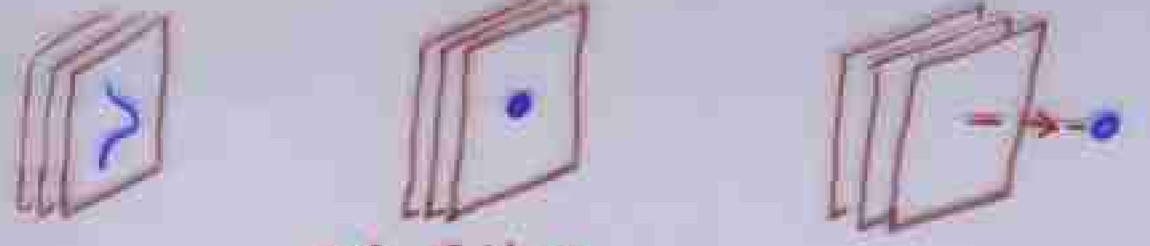
sym. prod. orbifold

$w_L = 0$ (size of blow-up)



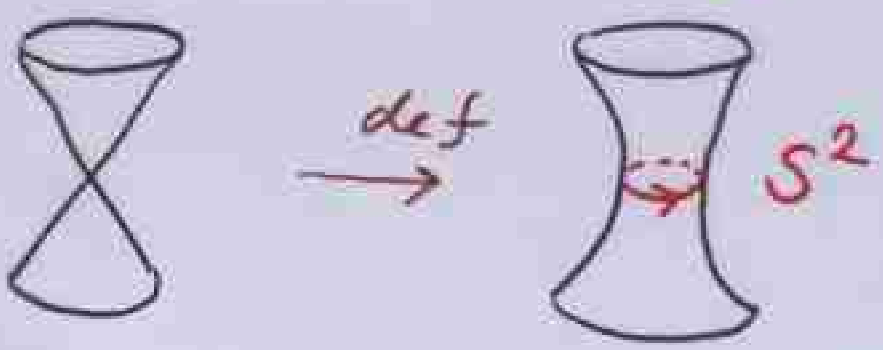
SINGULARITIES in CFT

D1 can escape



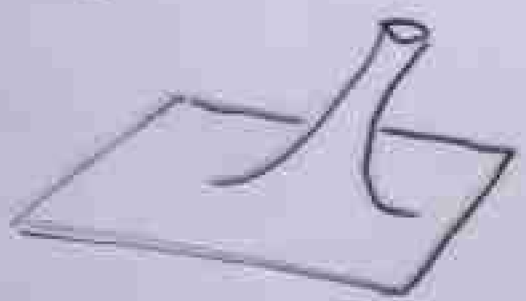
pt-like
instanton

local model $\mathbb{R}^4/\mathbb{Z}_2$



$$\text{ampl} \sim \sum_{\text{instantons}} a_n e^{-n(A+i\theta)}$$

non-compact direction in
field space ~ Liouville



SCFT on $S^k X$

1st vs 2nd quantization

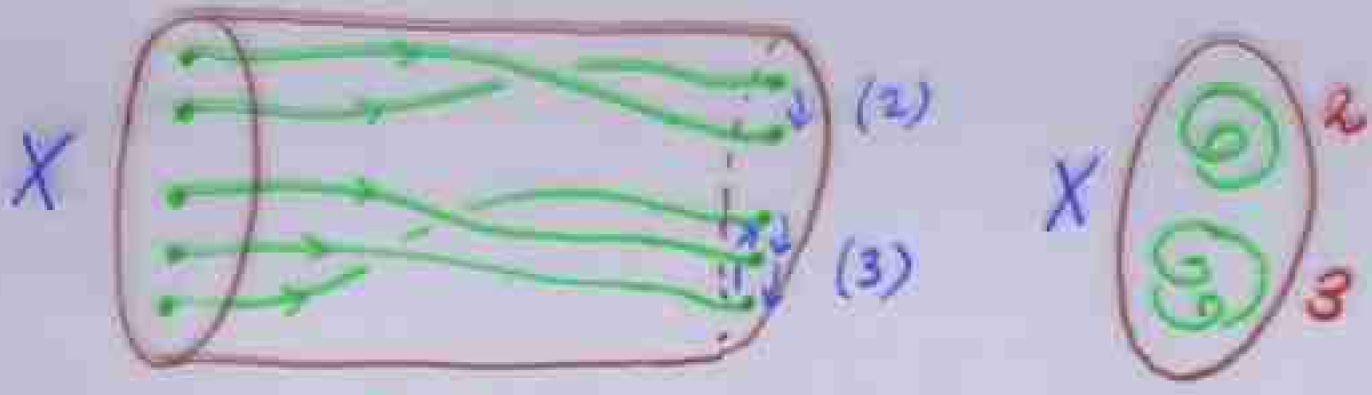
$$
\begin{tikzcd}
\boxed{\begin{array}{c} X \\ \text{manifold} \end{array}} \arrow[r, "Q1"] \arrow[d, "Q2"] & \boxed{\begin{array}{c} \mathcal{H} \\ \text{Hilbert space} \end{array}} \arrow[d, "Q2"] \\
\boxed{\begin{array}{c} S^{\cdot}X = \sum S^k X \\ \text{configuration space} \end{array}} \arrow[r, dashed, "Q1"] & \boxed{\begin{array}{c} S^{\cdot}\mathcal{H} = \oplus S^k \mathcal{H} \\ \text{Fock space} \end{array}}
\end{tikzcd}
$$

$$Q_1 Q_2 \overset{?}{=} Q_2 Q_1$$

NO: long strings $\Rightarrow$ extra dim

DLCQ on $X \times \mathbb{R}^{1,1}$

LONG STRINGS



$\sigma \rightarrow$

twisted sectors

$$x(\sigma + 2\pi) = g \cdot x(\sigma)$$

$$g \in S_k$$

$\Rightarrow$ gas of strings of "length"

$n = 1, 2, \ldots$

Fock space

$$S^{\cdot}\Big(\bigoplus_{n \geq 1} \mathcal{H}_n(X)\Big)$$

$$L_0 - \bar{L}_0 \equiv 0 \pmod{n}$$

DISCRETE TORSION

orbifold CFT: discrete B-field

$$H^2(S_k, U(1)) \simeq \mathbb{Z}_2 \qquad k \geq 4$$

central (spin) extension

$$\mathbb{Z}_2 \to \hat{S}_k \to S_k$$

$$\hat{t}_i \hat{t}_j = -\hat{t}_i \hat{t}_j$$

g-twisted sector: proj on ε_g

$$\varepsilon_g(h) = [\hat{g}, \hat{h}] \in \mathbb{Z}_2$$

result

$$\mathcal{H}_n(X) = \begin{cases} \text{BOSON} & n \text{ odd} \\ \text{FERMION} & n \text{ even} \end{cases}$$

"SUPERSYMMETRY"

spacetime: discrete RR flux

ELLIPTIC GENUS

topological $\mathcal{M}_{\sigma} \simeq S^k K3$

counts BPS states

$$Z_k(\tau, z) = \mathrm{Tr}\left[(-1)^F q^{L_0} y^{J_0} \bar{q}^{\bar{L}_0}\right]$$

$$q = e^{2\pi i \tau}, \; y = e^{2\pi i z}$$

Compute in orbifold [DMVV]

$$\sum Z_k p^k = \prod_{\substack{n>0 \\ m, \ell}} \left(1 - p^n q^m y^\ell\right)^{-c(nm, \ell)}$$

Space-time interpretation?

w. J. Maldacena, G. Moore, E. Verlinde.

MALDACENA DUALITY

$$\boxed{\begin{array}{c} D1\text{-}D5 \quad SCFT_{1+1} \\ \text{IIS} \\ \text{IIB string on } AdS_3 \times S^3 \times K3 \end{array}}$$

holographic

CFT  AdS-string

Chern-Simons level k

$$(SL_2 \times SU_2)_L \times (SL_2 \times SU_2)_R$$

SL_2 : hyperbolic metric : AdS^3

SU_2 : S^3 gauge fields

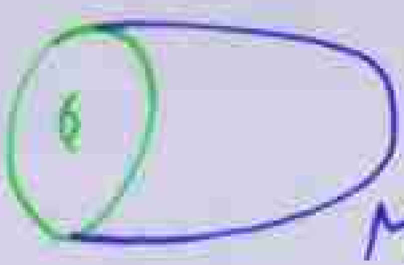
+ matter multiplets

elliptic genus:

Euclidean partition funct

T^2

Space-time $\quad \partial M^3 = T^2$



M^3, hyperbolic

∞-volume: D-brane

(finite vol: ∞ # of M^3, e.g.)

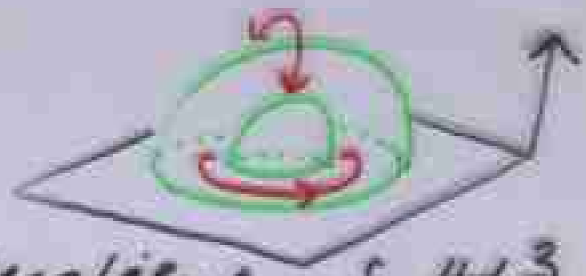
knot compl.

S^3

problems with entropy

M^3 unique upto diffs

Solid torus



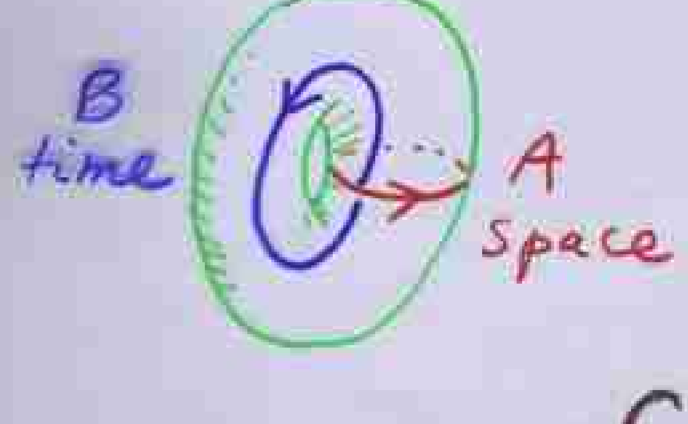
C

contractible cycle

$\mathbb{Z}$ quotient of $\mathbb{H}^3$

conformal structure on T^2 τ

M^3 determined by cycle C

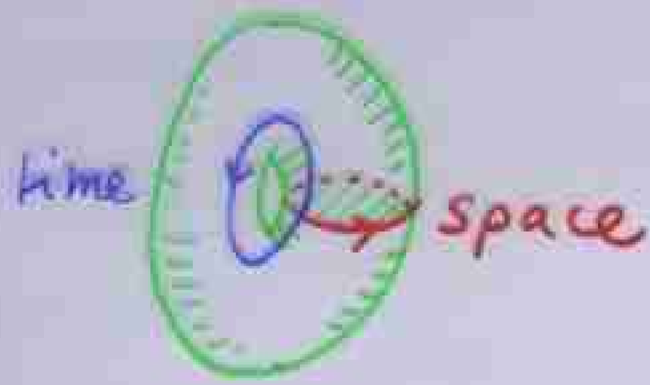


homology basis

$$C = cA + dB$$

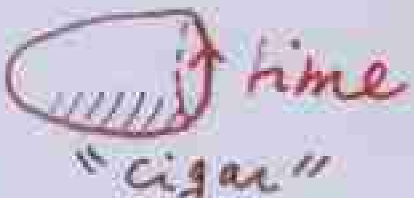
$$(c, d) = 1$$

$C = A$ thermal gas $(c,d) = (1,0)$



$C = B$ BTZ Euclid. BH

$(c,d) = (0,1)$



expect general form

$$Z^{CFT}(T^2) = \sum_{(c,d)=1} Z^{string}(M^3_{c,d})$$

coset $\Gamma \backslash \Gamma_\infty$, $\Gamma = SL(2,\mathbb{Z})$

$\Gamma_\infty = \mathbb{Z} = \langle T \rangle$

Thm (Moore, DV)

$$\tilde{\tilde{Z}}_k(\tau, z) = 2\pi\, i^{-\frac{1}{2}} \sum_{(c,d)=1} \sum_{j=\frac{1}{2}}^{k/2} \sum_{km-j^2<0}$$

$$\tilde{c}(km-j^2; S^k K3) \times (c\tau+d)^{-3}$$

$$\times \exp\left[-2\pi i \left(\frac{j^2}{k} - m\right) \frac{a\tau+b}{c\tau+d}\right]$$

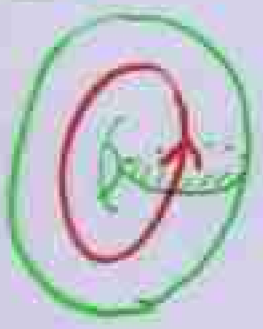
$$\times \exp\left[-2\pi i k \frac{cz^2}{c\tau+d}\right]$$

$$\times \theta_{j,k}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$$

$$\tilde{\tilde{Z}} = \left(\partial_\tau - \frac{1}{k}\partial_z^2\right)^{3/2} \cdot Z$$

INTERPRETATION

$\sum_{(c,d)=1}$	Sum over gravitational instantons $M^3_{c,d}$
$\sum_{j=0}^{k/2}$	Sum over integrable $SU(2)$ spin j reps



$\sum_{km-j^2<0} \tilde{c}(km-j^2)$	matter contribution mass m, spin j

particle states that donot create a BH.

$\boxed{(c\tau + d)^{-3}}$

$\tilde{Z}$ has $SL(2,\mathbb{Z})$ wt 3

string wavefunction

$$\tilde{Z}(\tau, z)\, d\tau\, dz$$

($\frac{1}{2}$-density)

$$\boxed{\exp\left[-2\pi i \left(\frac{j^2}{k} - m\right)\frac{a\tau + b}{c\tau + d}\right]}$$

classical Sugra action (reg)

limit $\bar{\tau} \to \infty$, for $M^3_{c,d}$

$$\boxed{\exp\left[-2\pi i\, k\, \frac{c z^2}{c\tau + d}\right] \theta_{j,k}\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right)}$$

$SU(2)$ CS wavefunction

Wilson loop j, on $M^3_{c,d}$

T^2  $M^3_{c,d}$

example: $k = 1$, $\mathcal{M}_{g} = K3$

$$\tilde{Z}_{h}(\tau, z) = \sum_{(c,d)=1} 22 \cdot (c\tau + d)^{-3} \, e^{-\frac{\pi i}{2} \frac{a\tau + b}{c\tau + d} - 2\pi i \frac{c z^2}{c\tau + d}} \cdot \theta_1\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right)$$

$$22 = b^2(K3)$$

mod form $f(\tau)$ of wt w

Fourier expansion

$$f(\tau) = \sum_{n \geq -N} c(n)\, q^{n+\Delta}$$

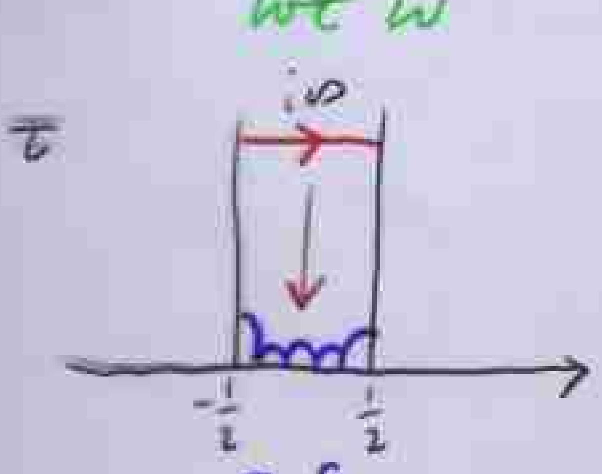
estimate $c(n)$

Cardy formula $\left(N=0,\ \Delta = -\frac{c}{24}\right)$

$$c(n) \sim \exp 2\pi \sqrt{\frac{nc}{6}}$$

Rademacher expansion

$$\left.\begin{array}{l} c(m),\ m+\Delta<0 \\ \text{wt } w \end{array}\right\} \Rightarrow c(n)$$



sum over

$\Gamma_\infty \backslash \Gamma / \Gamma_\infty$

$(c,d)=1$

$0<d<c$

$$c(n) \equiv \sum_{\substack{(c,d)=1 \\ 0<d<c}} \begin{pmatrix} \text{Kloosterman} \\ \text{sum} \end{pmatrix} \begin{pmatrix} \text{Bessel} \\ \text{function} \end{pmatrix}$$

$f(\tau)$ mod form wt $w<0$

$\partial_\tau^{1-w} f(\tau)$ mod of wt $2-w>0$

Poincaré series

$f(\tau)$ mod form wt $w>2$

pole q^{-n}

$$f(\tau) = \sum_{(c,d)=1} (c\tau+d)^{-w} e^{-2\pi i n \frac{a\tau+b}{c\tau+d}}$$

$$= \sum_{g \in \Gamma/\Gamma_\infty} (q^{-n})|g$$

elliptic genus $S^k K3$

$$\tilde{Z}_k = \sum_{j=0}^{k/2} h_j(\tau)\, \theta_{j,k}(\tau, z)$$

$$\qquad\qquad\qquad -\tfrac{1}{2} \qquad\qquad \tfrac{1}{2}$$

Jacobi form wt 0, index k

$$\partial_\tau^{3/2} h_j = \tilde{h}_j = \sum \tilde{c}_j(n) q^n$$

$$\tilde{Z}_k = \sum_j \tilde{h}_j \, \theta_{j,k} \qquad \text{wt } 3$$

use Rademacher expansion for $\tilde{h}_j$

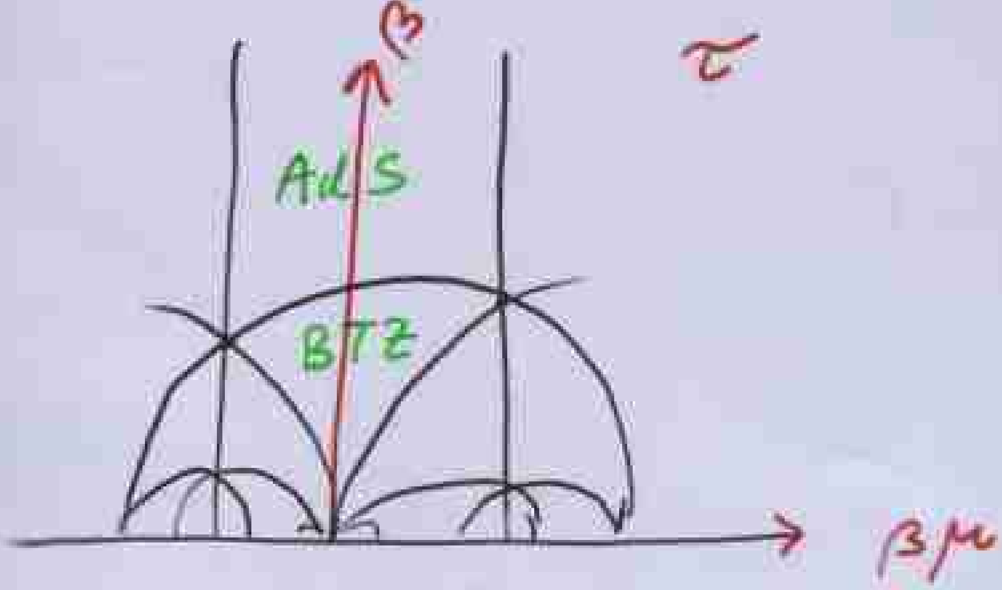
physics

$$Z(\tau, z) = \sum c(m, \ell)\, q^m y^\ell$$

microcan. states

gravity: thermodynamics

phase diagram



large k limit

$$\frac{F_k}{k} \sim \frac{a\tau + b}{c\tau + d}$$

1st order phase transitions