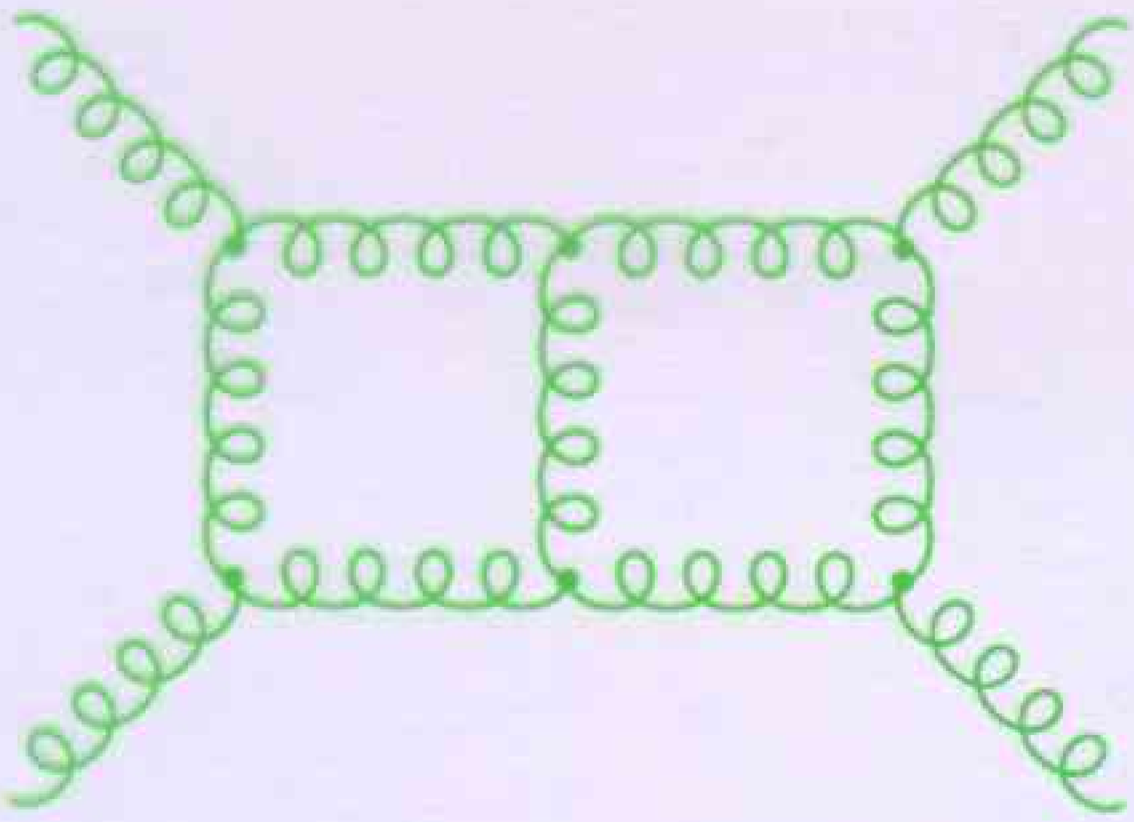
Perturbative Relations between Gravity and Gauge Theory



Strings '99

Z. Bern, L.D., D. Dunbar,

M. Perelstein, J. Rozowsky, B. Yan

hep-ph/9702424; hep-th/9802162, 9809160, 9811140

Multi-loop Scattering Amplitudes: Motivation

Gauge Theory

- Improve perturbative QCD predictions for jet rates & related quantities, to next-to-next-to-leading order (NNLO).

Gravity

- Explore ultraviolet divergences of (super)-gravity.
- String theory $\Rightarrow$ tree-level KLT relations:

Kawai, Lewellen, Tye, 1986

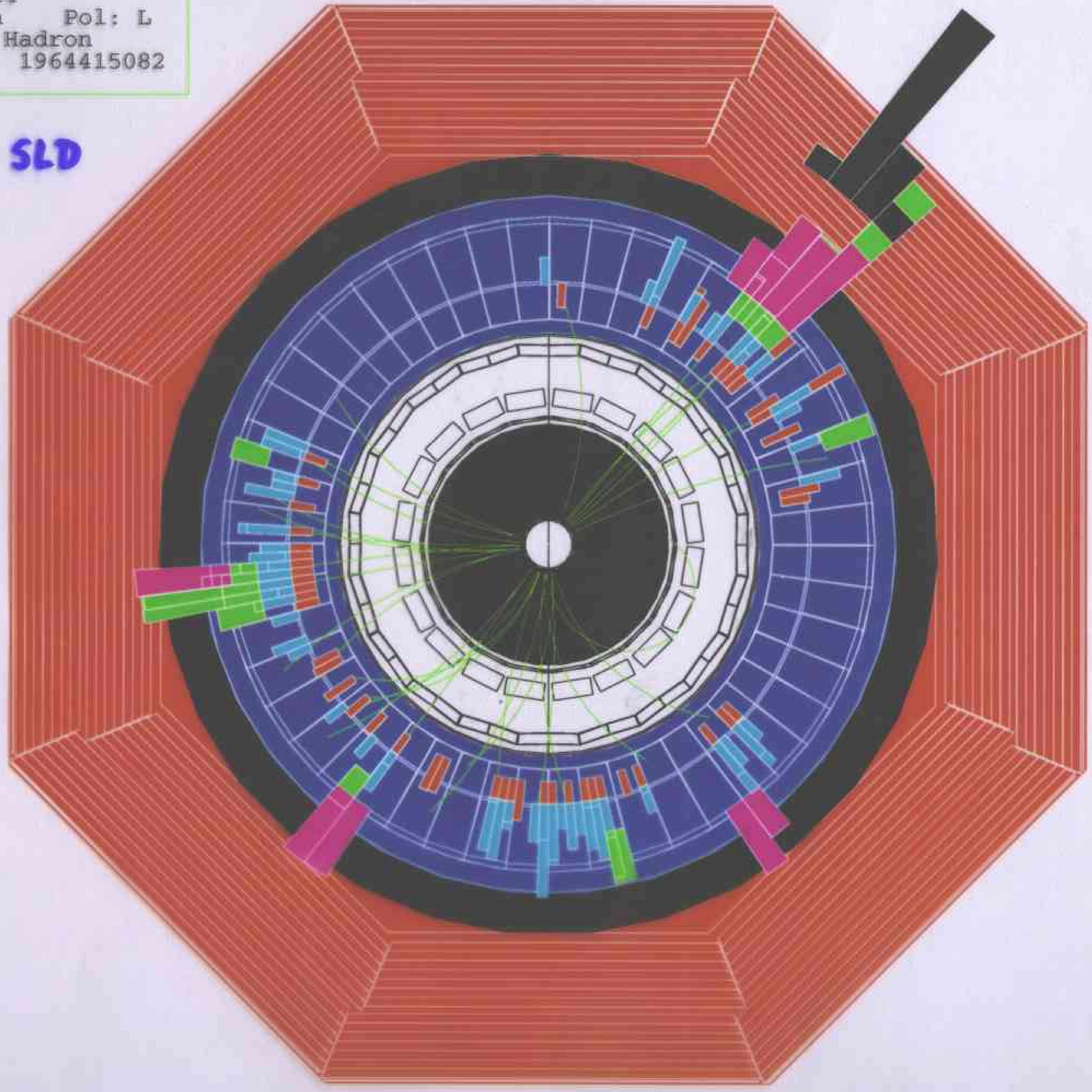
$$\boxed{\text{gravity} \sim (\text{gauge theory})^2}$$

- Are there similar relations at multi-loop level?
- Can we use KLT relations to simplify gravity calculations?

Run 12637, EVENT 6353
8-JUL-1992 10:14
Source: Run Data Pol: L
Trigger: Energy Hadron
Beam Crossing 1964415082

$e^+e^- \rightarrow Z^0 \rightarrow 3$ jets

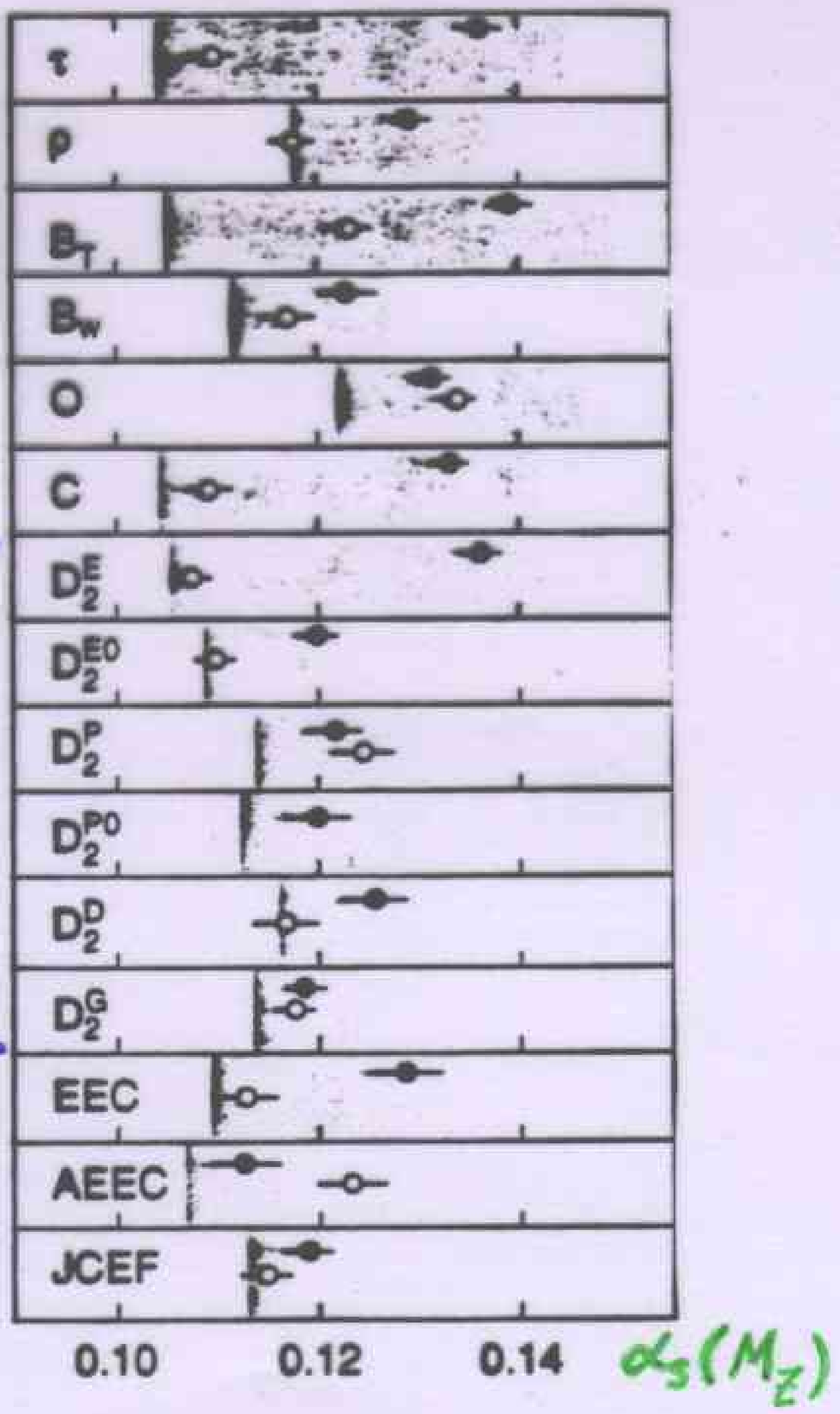
SLD



Motivation for NNLO Computations

15 measurements of $\underline{\alpha_s}$ in e^+e^-

~3 jet rates



SLD data

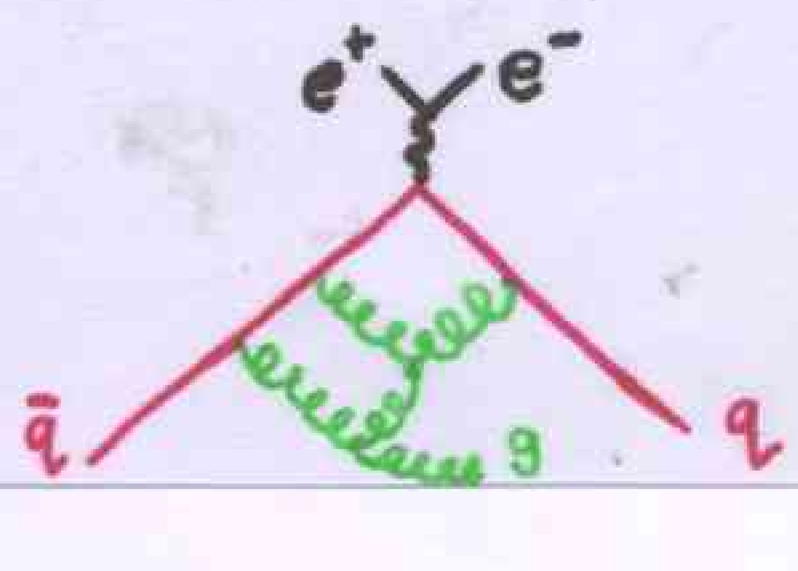
- $\bullet$ ≡ 'physical' scale ($\mu = M_Z$)
- $\circ$ ≡ 'experimentally-optimized' scale (fit for μ)

$$O_i = A_i \frac{\alpha_s(\mu)}{\pi} + \left[B_i + 2\beta_0 \ln(\mu^2/s) A_i\right] \left(\frac{\alpha_s(\mu)}{\pi}\right)^2$$

LO

$$+ \left[??? \qquad \right] \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 + \cdots$$

NNLO requires 2 loops



UV Divergences of (Super-) Gravity

D = 4

gravity theory	first known divergence
N = 0 + matter	L = 1
N = 0 pure	L = 2
N ≥ 1	? (L > 2)

't Hooft, Veltman ...

Goroff, Sagnotti; van de Ven

Grisaru, van Nieu., Vermaseren; Grisaru

D > 4

N = 8 in D = 8 diverges at L = 1

Green Schwarz Brink

- Also some constraints from superspace power counting

Howe Stelle Townsend

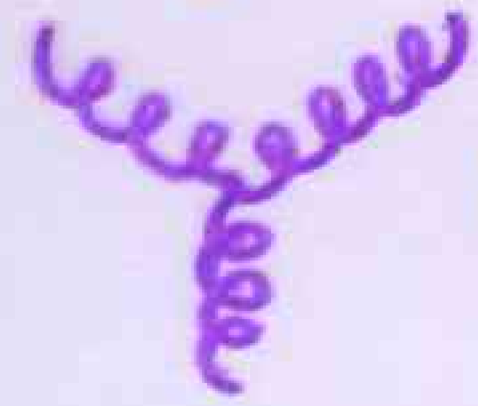
$$\text{Gravity} \overset{?}{\sim} (\text{Yang-Mills})^2$$

spin 2   spin 1

- **Obscure at Lagrangian level:**

$$\mathcal{L}_{grav} = -\frac{2}{\kappa^2}\sqrt{g}\,R \qquad\qquad \mathcal{L}_{YM} = -\frac{1}{4g^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

$\Downarrow$ $\Downarrow$

 ∂^2  ∂

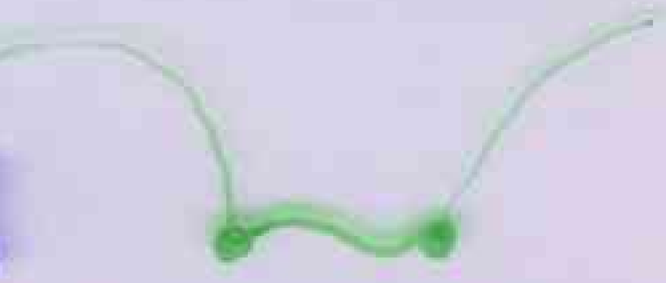
 ∂^2 ?   

 ∂^2 ??   –

$\vdots$

KLT tree-level string relations

gluon:
[+ Chan-Paton T^a]



$$V_{open}^{(x)} = \epsilon^\mu \partial_x X^\mu e^{ik\cdot X(x)}$$

graviton:

$$V_{closed} = \epsilon^{\mu\nu} \partial_z X^\mu \partial_{\bar z} X^\nu e^{ik\cdot X(z,\bar z)}$$

$$\underset{\text{(up to 0-mode)}}{=} V_{open}(z) \cdot \bar V_{open}(\bar z)$$

for $\epsilon_\pm^{\mu\nu} = \epsilon_\pm^\mu \bar\epsilon_\pm^\nu$

$$A_n = \int \frac{dx_1 \ldots dx_n}{V_{abc}(x)} \cdot \prod_{1\le i<j\le n} |x_i - x_j|^{k_i\cdot k_j} e^{\sum_{i<j}\left[\frac{\epsilon_i\cdot\epsilon_j}{(x_i-x_j)^2} + \frac{k_i\cdot\epsilon_j - k_j\cdot\epsilon_i}{(x_i-x_j)}\right]}\Bigg|_{m.l.}$$

dress with Chan-Paton

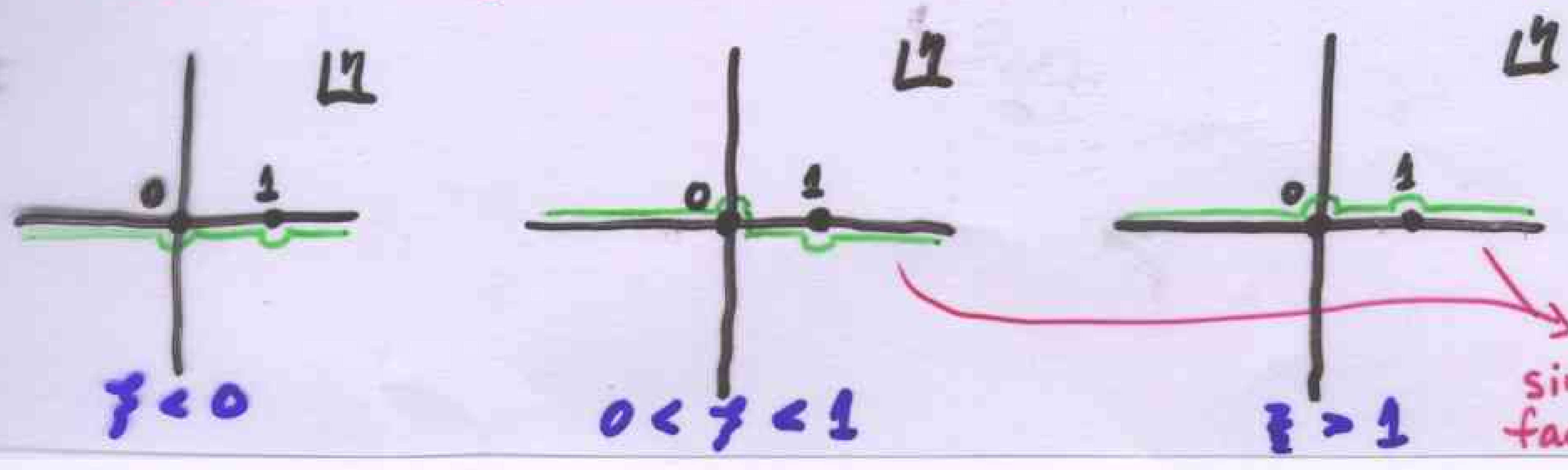
$$M_n = \int \frac{d^2 z_1 \ldots d^2 z_n}{|V_{abc}(z)|^2} \cdot \prod_{1\le i<j\le n} (z_i - z_j)^{k_i\cdot k_j} e^{\sum_{i<j}\left[\frac{\epsilon_i\cdot\epsilon_j}{(z_i-z_j)^2} + \frac{k_i\cdot\epsilon_j - k_j\cdot\epsilon_i}{(z_i-z_j)}\right]}\Bigg|_{m.l.}$$

$$\cdot \prod_{1\le i<j\le n} (\bar z_i - \bar z_j)^{k_i\cdot k_j} e^{\sum_{i<j}\left[\frac{\bar\epsilon_i\cdot\bar\epsilon_j}{(\bar z_i-\bar z_j)^2} + \frac{k_i\cdot\bar\epsilon_j - k_j\cdot\bar\epsilon_i}{(\bar z_i-\bar z_j)}\right]}\Bigg|_{m.l.}$$

n-3 real / complex integrations

contour integral deformations $\Rightarrow$ $z, \bar z \to \eta, \xi$

$\underline{n=4}$:



KLT Relations in field theory limit

$\alpha' s_{ij} \to 0 \qquad s_{ij} = 2k_i \cdot k_j$

$$M_4^{\text{tree}}(1,2,3,4) = -i\, s_{12}\, A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)$$

$$\begin{aligned} M_5^{\text{tree}}(1,2,3,4,5) &= i\, s_{12} s_{34} A_5^{\text{tree}}(1,2,3,4,5) A_5^{\text{tree}}(2,1,4,3,5) \\ &\quad + i\, s_{13} s_{24} A_5^{\text{tree}}(1,3,2,4,5) A_5^{\text{tree}}(3,1,4,2,5) \end{aligned}$$

$\cdots$

- **Hold for <u>any</u>**

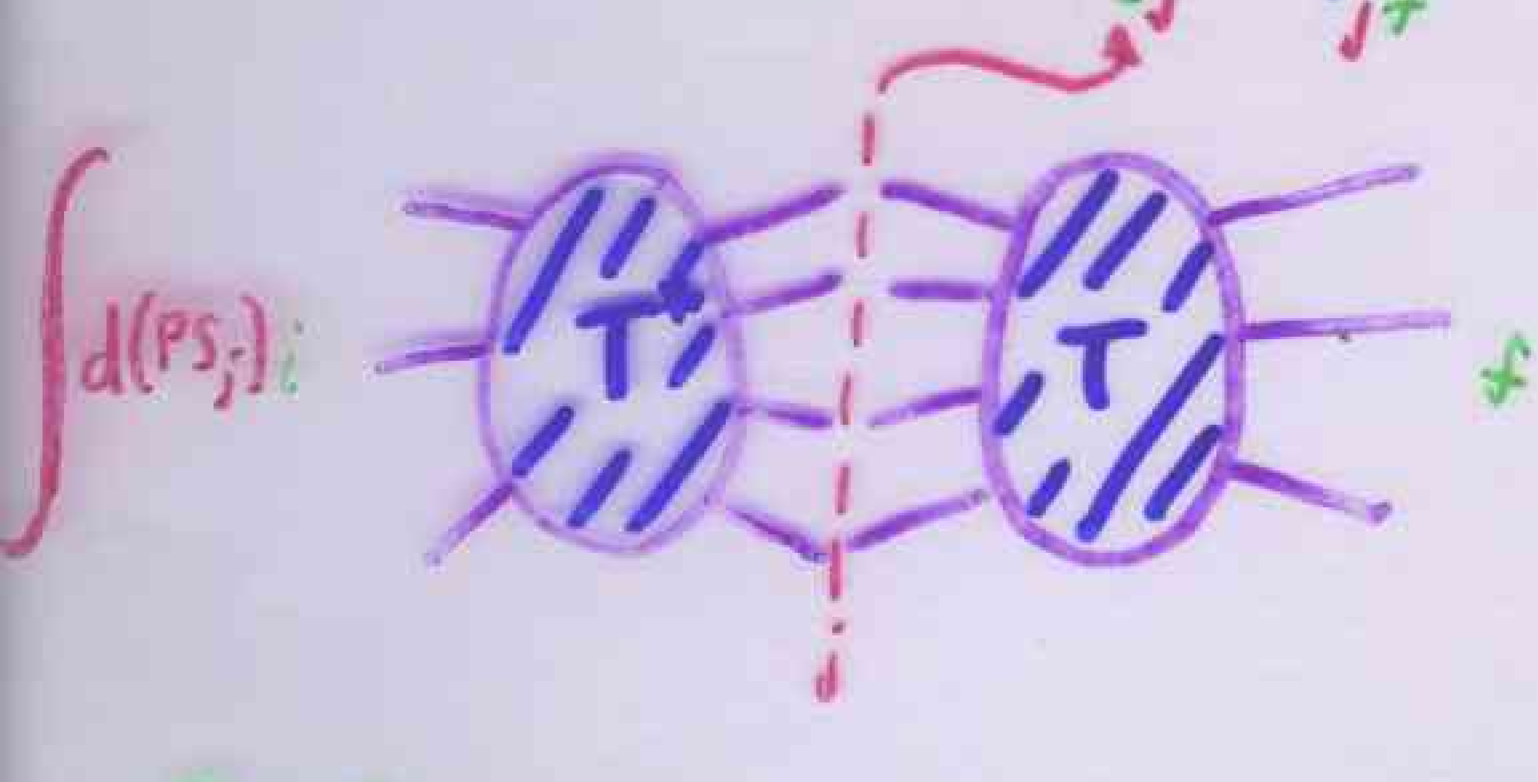
$$|N=8 \text{ SUGRA state}\rangle = |N=4 \text{ YM state}\rangle \otimes |N=4 \text{ YM state}\rangle$$

$$256 \qquad\qquad 16 \quad \times \quad 16$$

Amplitude Reconstruction via Unitarity

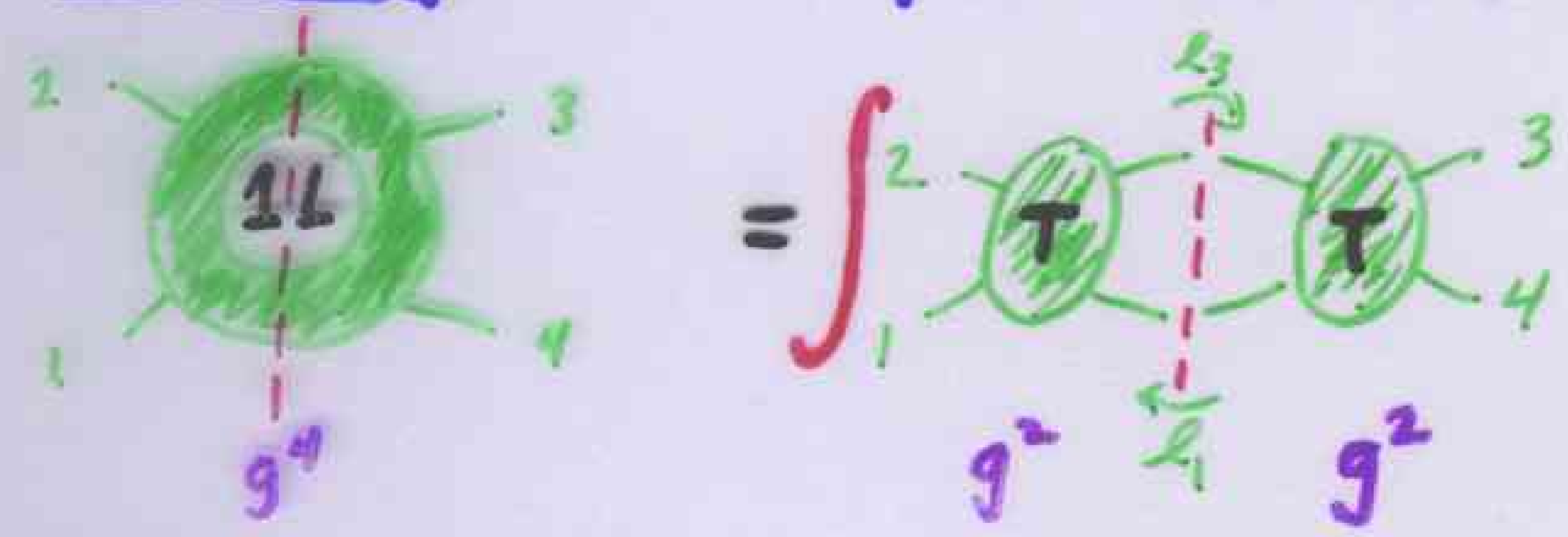
$$1 = S^\dagger S = (1 - iT^\dagger)(1 + iT)$$

$$\Rightarrow 2\,\mathrm{Im}\,T_{if} = T_{ij}^{*}\,T_{jf}$$

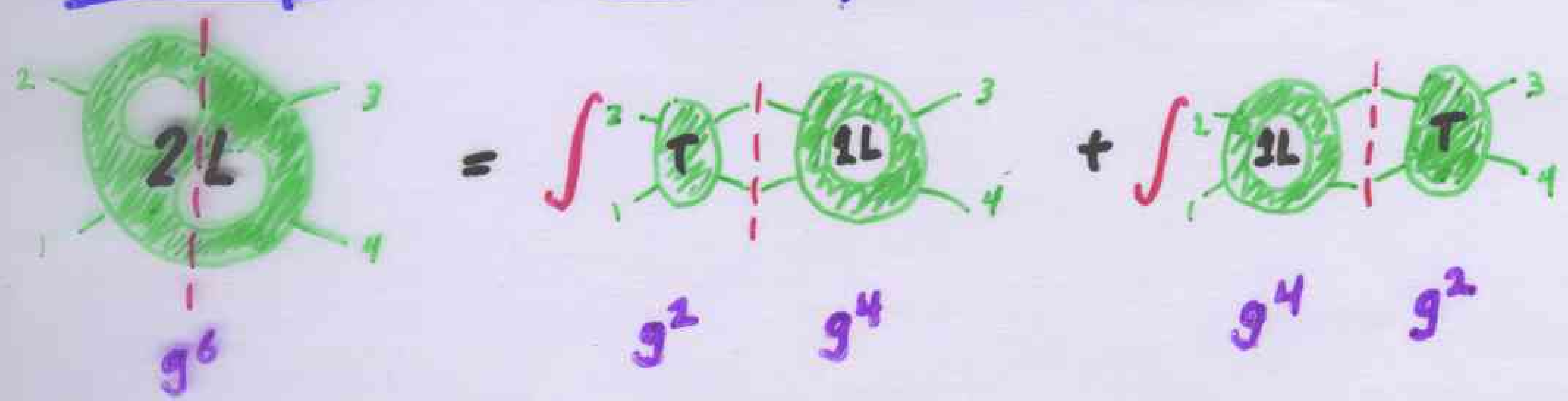


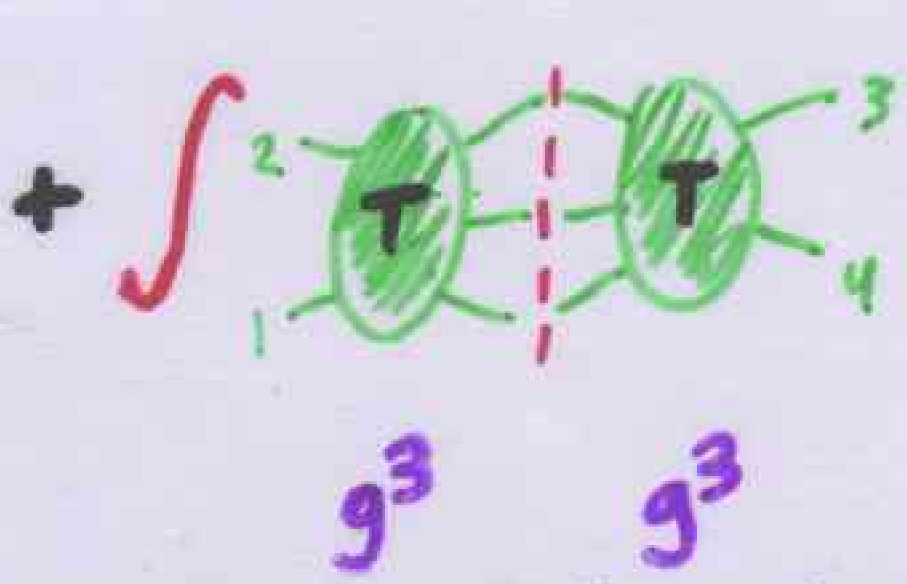
Perturbative Implications

- **One-loop** $\Rightarrow$ 2-particle cuts only, tree amplitudes:



- **Two-loops** $\Rightarrow$ 2- and 3- particle cuts:





- Find amplitudes which match all cuts, then argue for uniqueness.

2-particle cutting equation for N=4 YM

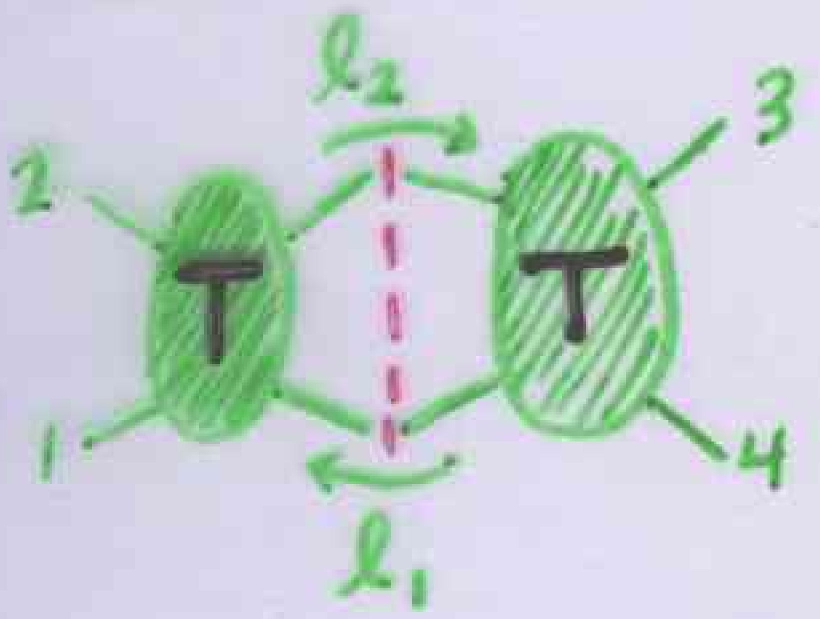
Bern
Rozowsky
Yan

$$\sum_{s_1, s_2 \in N=4} A_4^{\text{tree}}(1,2,\ell_2^{s_2},-\ell_1^{s_1}) \times A_4^{\text{tree}}(\ell_1^{s_1},-\ell_2^{s_2},3,4)$$

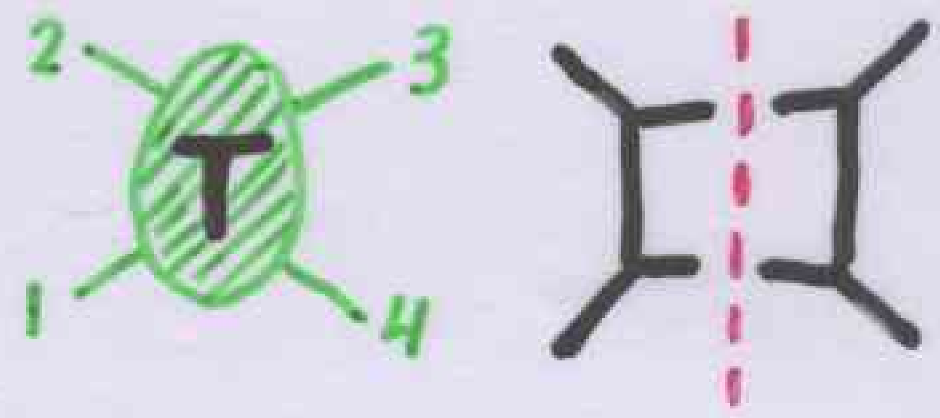
$$= -i \frac{s_{12}\, s_{23}}{(\ell_1-k_1)^2(\ell_2-k_3)^2} A_4^{\text{tree}}(1,2,3,4)$$

GRAPHICALLY:

$\sum_{N=4}$



$= -i\, s_{12}\, s_{23}$



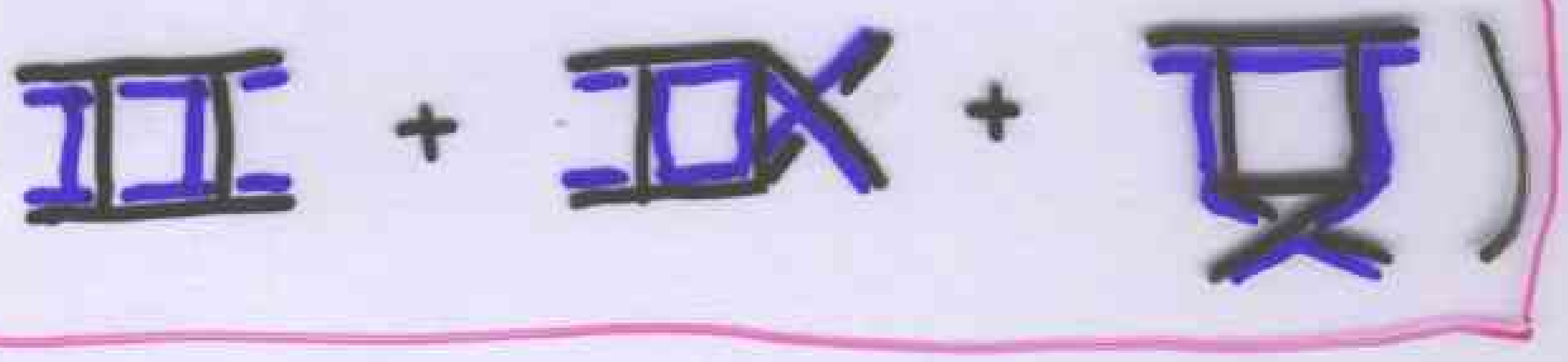
cut scalar box integral

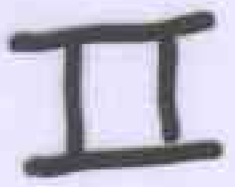
• **Derivable from** e.g. one-loop amplitude by cutting ("backwards")

Green
Schwarz
Brink

One-loop N=4 Amplitude, Restoring Color Factors

& imposing consistency with 2-particle cuts:

$$\boxed{\mathcal{A}_4^{\text{1-loop},\, N=4} = i g^4 \left(s t A_4^{\text{tree}}\right)\left(\right.}$$ 

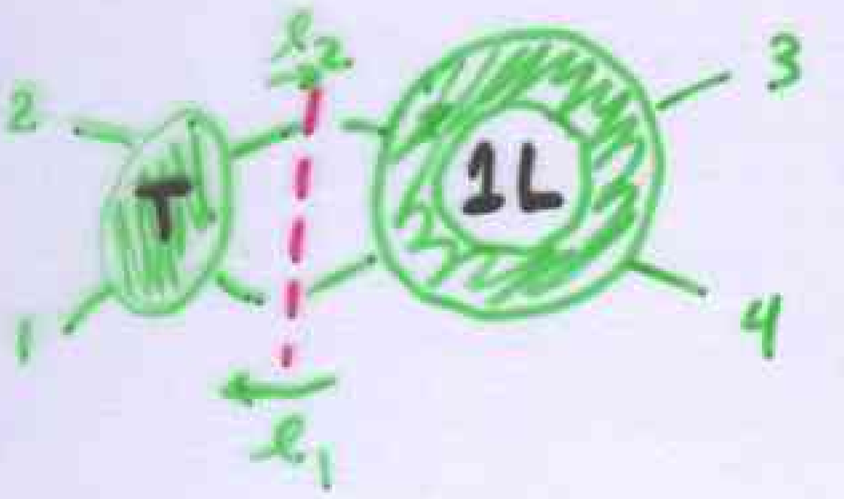
where  $= \displaystyle\int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2 (\ell - k_1)^2 (\ell - k_1 - k_2)^2 (\ell + k_4)^2}$, ...

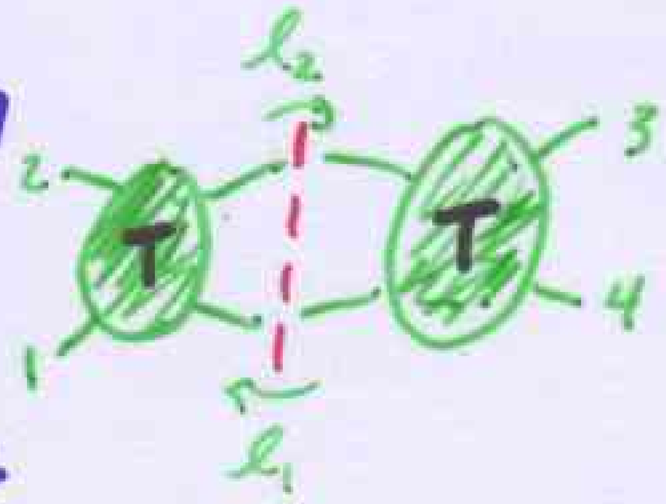
 $= \delta^{ab}$

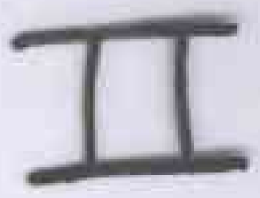
 $= f^{abc}$

2-loops: Iterate!

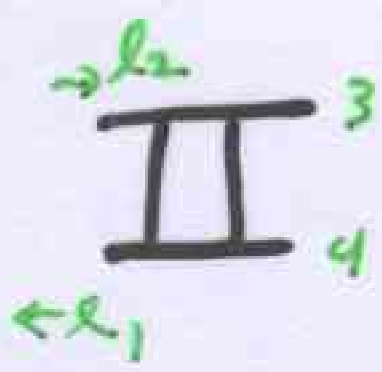
BRY

$\displaystyle\sum_{s_1, s_2}$ 

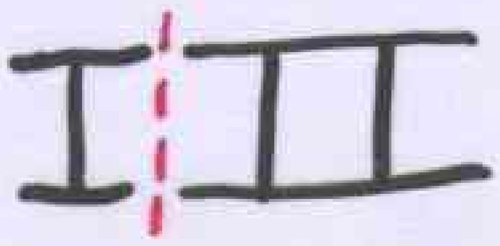
$= \displaystyle\sum_{s_1, s_2}$ 

$\times (-i) s_{34}\, s_{\ell_2 3}$ 

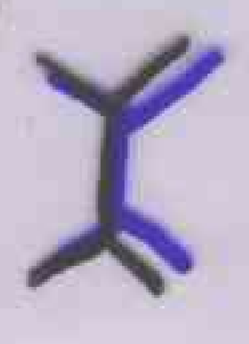
$=$ 

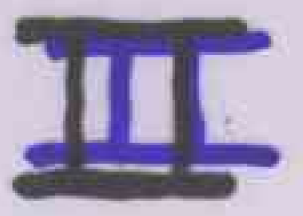
$\displaystyle\times \frac{(-i) s_{12}\, s_{23}\ (-i)\, s_{34}\, \cancel{s_{\ell_2 3}}}{(\ell_1 - k_1)^2\ \cancel{(\ell_2 - k_3)^2}}$ 

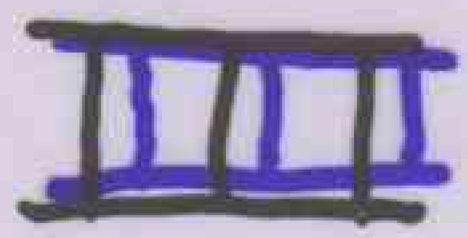
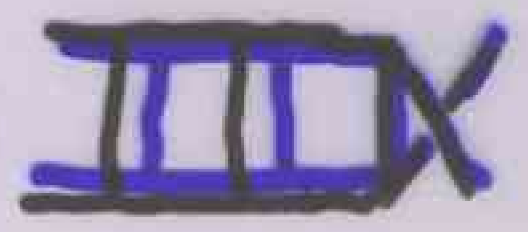
$= - s_{12}^2\, s_{23}$ 

$\times$ 

All-loop 4-point Amplitudes in $\mathcal{N}=4$ SYM

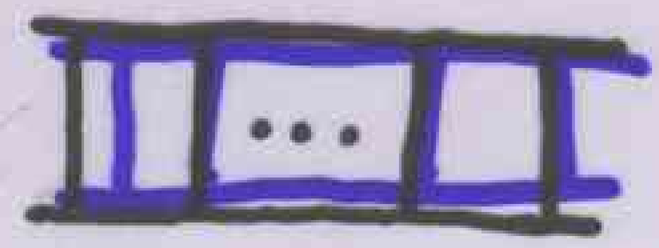
$$A_4^{tree} = g^2 \frac{st A_4^{tree}}{s} \Big($$  $+$  $\Big)$

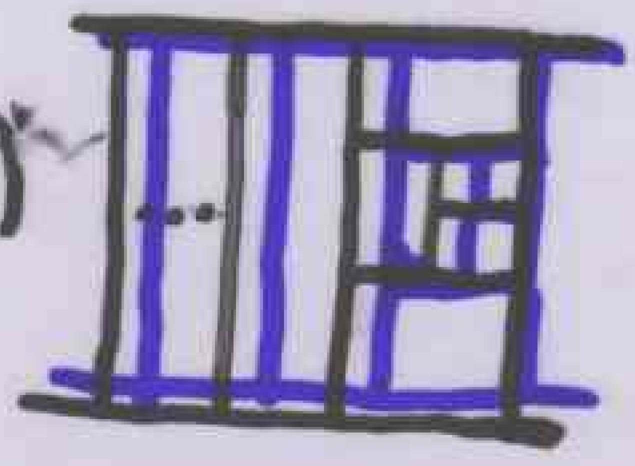
$$A_4^{1\text{-}loop} = i g^4 \, st A_4^{tree} \Big($$  $+$  $+$  $\Big)$

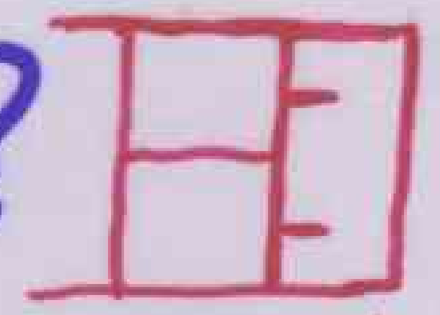
$$A_4^{2\text{-}loop} = -g^6 \, st A_4^{tree} \Big(s$$  $+ s$ 

$+ s$  $+ s$ 

$+ \text{cyclic}(2,3,4) \Big)$

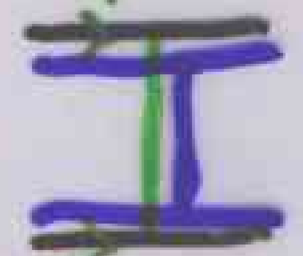
$$A_4^{L\text{-}loop} = i^L g^{2(L+1)} st A_4^{tree} \Big(s^{L-1}$$ 

$+ \cdots + M(l_i, k_i)$ 

$+ ??$  $+ \cdots \Big)$

no 2-particle cuts

$M(l_i, k_i)$ determined recursively by "rung rule"



$\Rightarrow (l_1 + l_2)^2$

Multi-loop N=8 SUGRA:

Use KLT to promote $N=4$ **2-particle cutting eqn. to** $N=8$

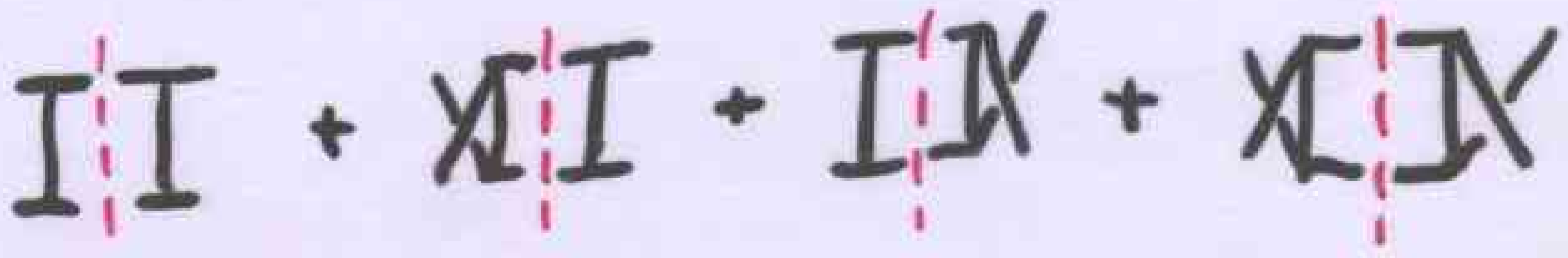
$$\sum_{N=8\ (256)} M_4^{tree}(-\ell_1^{s_1}, 1, 2, \ell_2^{s_2}) \times M_4^{tree}(-\ell_2^{s_2}, 3, 4, \ell_1^{s_1})$$

$$= -s_{12}^2 \left(\sum_{N=4\ (16)} A_4^{tree}(-\ell_1^{s_1}, 1, 2, \ell_2^{s_2}) \times A_4^{tree}(-\ell_2^{s_2}, 3, 4, \ell_1^{s_1}) \right)$$

$$\times \left(\sum_{N=4\ (16)} A_4^{tree}(-\ell_1^{s_1}, 2, 1, \ell_2^{s_2}) \times A_4^{tree}(-\ell_2^{s_2}, 4, 3, \ell_1^{s_1}) \right)$$

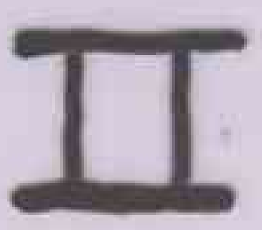
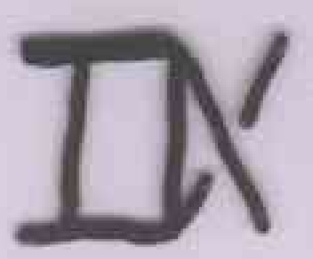
$$= \left[s_{12} s_{23} A_4^{tree}(1,2,3,4)\right]^2 \frac{s_{12}^2}{(\ell_1-k_1)^2(\ell_1-k_2)^2(\ell_2-k_3)^2(\ell_2-k_4)^2}$$

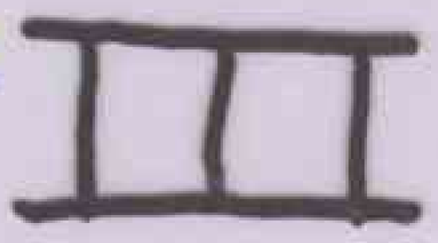
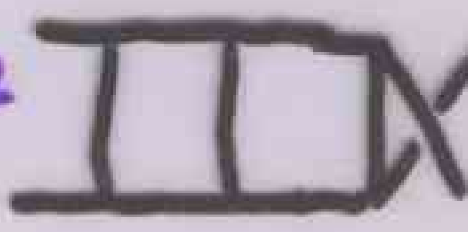
$$= \underbrace{\left[s_{12} s_{23} A_4^{tree}(1,2,3,4)\right]^2}_{i\,stu\, M_4^{tree}} \underbrace{\left[\frac{1}{(\ell_1-k_1)^2} + \frac{1}{(\ell_1-k_2)^2}\right]\left[\frac{1}{(\ell_2-k_3)^2} + \frac{1}{(\ell_2-k_4)^2}\right]}$$

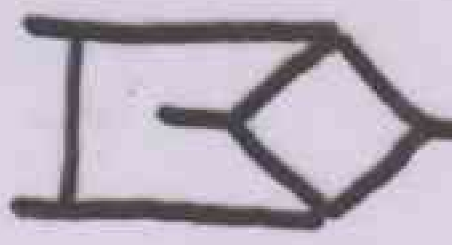
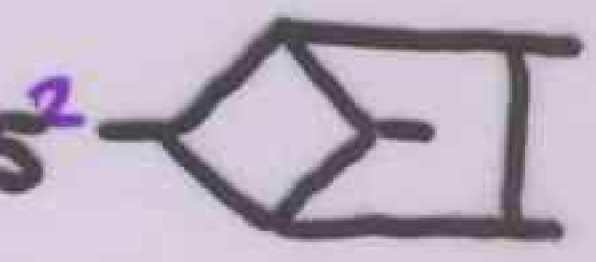


All-loop 4-point Amplitudes in $N=8$ SUGRA

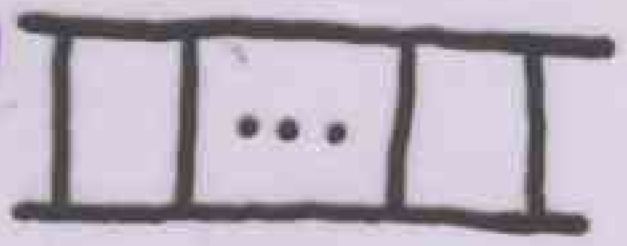
$$\mathcal{M}_4^{\text{tree}} = i\left(\frac{\kappa}{2}\right)^2 \left(\frac{s\,t\,A_4^{\text{tree}}}{s}\right)^2 \Big($$  $+$  $\Big)$

$$\mathcal{M}_4^{\text{1-loop}} = -\left(\frac{\kappa}{2}\right)^4 \left(s\,t\,A_4^{\text{tree}}\right)^2 \Big($$  $+$  $+$  $\Big)$

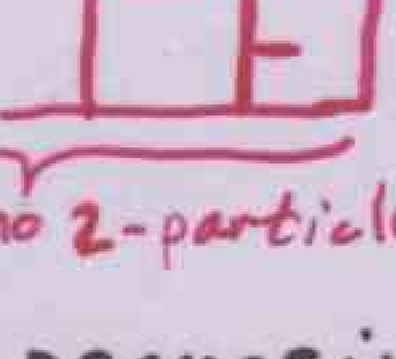
$$\mathcal{M}_4^{\text{2-loop}} = -i\left(\frac{\kappa}{2}\right)^6 \left(s\,t\,A_4^{\text{tree}}\right)^2 \Big(s^2$$  $+\, s^2$ 

$+\, s^2$  $+\, s^2$ 

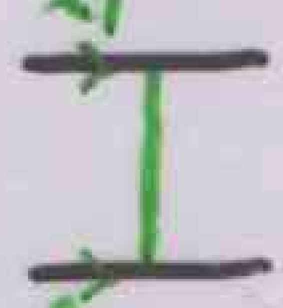
$+ \text{ cyclic}(2,3,4) \Big)$

$$\mathcal{M}_4^{L\text{-loop}} = i^{L+1} \left(\frac{\kappa}{2}\right)^{2(L+1)} \left(s\,t\,A_4^{\text{tree}}\right)^2 \Big(s^{2(L-1)}$$ 

$+ \cdots + [M(\ell_i, k_i)]^2$ 

$+\,?$  $+ \cdots \Big)$

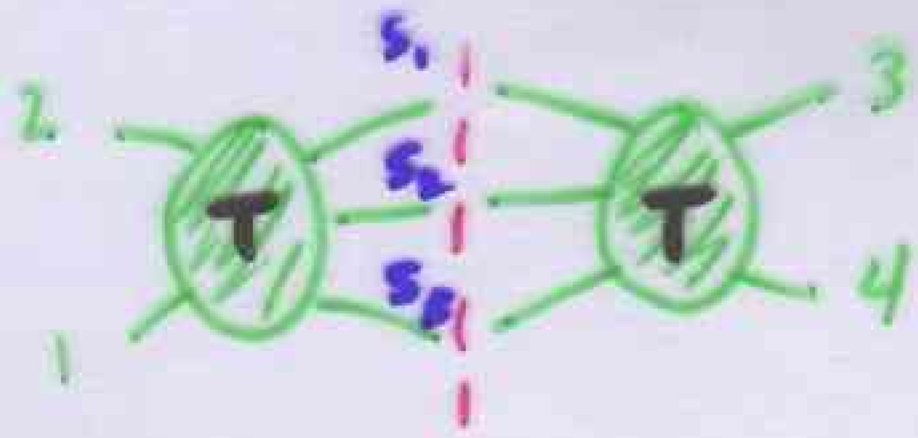
no 2-particle cuts

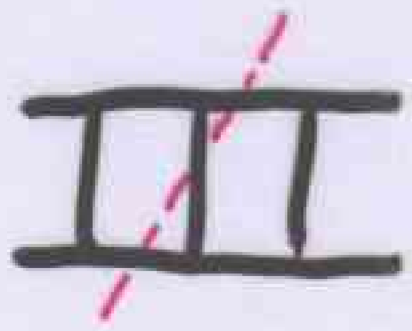
$M(\ell_i, k_i)$ determined recursively by "rung rule" 

$\Rightarrow [(\ell_1+\ell_2)^2]^2$

2-loop results cross-checked by $\overset{D=4}{\downarrow}$ 3-particle cuts

$$\sum_{s_1, s_2, s_3 \in N=4}$$



$= s t A_4^{tree} \Big[s$  $+ s$  $+ t$  $\Big]$

KLT 5-point relations again simplify analogous check for N=8 supergravity.

UV Divergences in N=4 SYM and N=8 SUGRA

- **L=2**: Divergences follow from just 2 scalar integrals

$$\sim \int (d^D\ell)^2 \frac{1}{(\ell^2)^7}$$

$\Rightarrow$ $\frac{1}{\epsilon}$ poles in $D = 7, 9, \ldots$

- $D \leq 6$ manifestly finite

- **L>2**: We conjecture $D_{crit}(L)$,
the lowest divergent dimension,
based on the most divergent rung-rule contributions

$$\int (d^D\ell)^L \frac{(\ell^2)^{2(L-2)}}{(\ell^2)^{3L+1}}$$

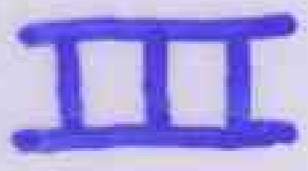
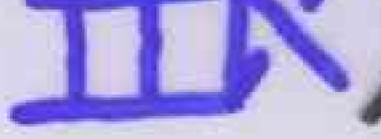
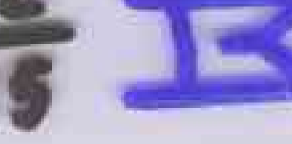
(2 for gravity only)

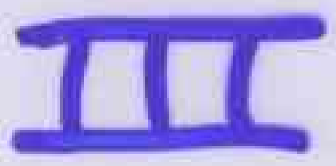
$$\Rightarrow \quad D_{crit}(L) = \frac{6}{L} + 4 \qquad N=4 \text{ SYM}$$

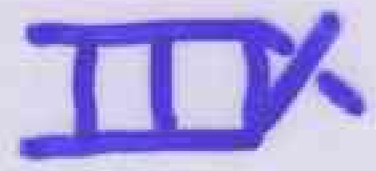
$$D_{crit}(L) = \frac{10}{L} + 2 \qquad N=8 \text{ SUGRA}$$

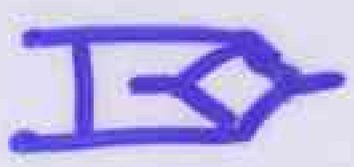
N=4 SYM 2-loop counterterms:

$$T_D\left[\mathrm{tr}(F^4) - \tfrac{1}{4}(\mathrm{tr}F^2)^2\right]$$

$$T_7 = -\frac{g^6\pi}{(4\pi)^7\, 2\epsilon}\Big[s\Big(\tfrac{1}{10}\Big($$  $$+$$  $$\Big) + \tfrac{2}{15}$$  $$\Big) + \text{cyclic}\Big]$$

$$T_9 = -\frac{g^6\pi s}{(4\pi)^9\, 4\epsilon}\Big(\frac{-45s^2+18st+2t^2}{99792}$$ 

$$+\frac{-45s^2+18su+2u^2}{99792}$$ 

$$-\frac{2(75s^2+2tu)}{83160}$$  $$\Big) + \text{cyclic}$$

- Agrees with Marcus & Sagnotti (1985)
  (up to overall $\frac{3}{2}$ for T_7 (?))
  - 43 diagrams
  - overlapping divergences
  - $D=6$ finiteness not manifest

- $L>2$ conjecture agrees with covariant $N=2$ superspace power-counting for $L=3$ ($D_{crit}=6$). Howe Stelle 1984
- But predicts better UV behavior for $L=4$ and $L=5$ ($D_{crit} = 5.5 \to 6$, $5.2 \nearrow$ vs. 5)

N=8 SUGRA 2-loop counterterms:

$$T_D\left[\mathrm{tr}\,R^4 + \cdots\right]$$ ⟿ (Bel-Robinson tensor)^2

$\kappa^2 = 32\pi G_N$

$$T_7 = \left(\frac{\kappa}{2}\right)^6 \frac{\pi}{(4\pi)^7\, 2\epsilon} \; \frac{s^2+t^2+u^2}{3}$$

$$T_9 = \left(\frac{\kappa}{2}\right)^6 \frac{\pi}{(4\pi)^9\, 4\epsilon} \; \frac{(s^2+t^2+u^2)^2}{9072}$$

$$T_{10} = \left(\frac{\kappa}{2}\right)^6 \frac{4}{(4\pi)^{10}\, 12\epsilon} \; \frac{-13\, stu\,(s^2+t^2+u^2)}{25920}$$

$$T_{11} = \left(\frac{\kappa}{2}\right)^6 \frac{\pi}{(4\pi)^{11}\, 48\epsilon} \; \frac{438\,(s^6+t^6+u^6) - 53\, s^2t^2u^2}{5791500}$$

theory was finite at $L=1$ in these dim's (in dim. reg.)

- $L=2$ behavior in $D=5,6$ is already better than predicted by N=4 superspace power counting

(Howe, Stelle, Townsend 1984, 1989)

- We conjecture N=8 SUGRA in $D=4$ only diverges at $L=5$
  - "conventional wisdom" was for divergence at $L=3$

(should be do-able...)

gravity ~ (gauge theory)^2 at Lagrangian level?

Bern, Grant hep-th/9904026

- KLT factorization ⇒ should be able to divide Lorentz indices in $\mathcal{L}_{grav}(h_{\underset{LR}{\mu\nu}})$ into left indices and right indices.

- Obstructions: in $\mathcal{L}^{EH} = \sqrt{-g}\,R$

(a) traceless projection in propagator,

$$P_{\mu\nu;\alpha\beta} = \frac{1}{2}\left(\underbrace{\eta_{\underset{LL}{\mu\alpha}}\,\eta_{\underset{RR}{\nu\beta}}}_{} + \underbrace{\eta_{\mu\beta}\,\eta_{\nu\alpha}}_{\text{can remove by symmetrizing vertices}} - \frac{2}{D-2}\,\eta_{\mu\nu}\,\eta_{\alpha\beta}\right)\frac{i}{k^2}$$

(b) $\mathrm{tr}[h^{2m+1}]$ interaction vertices, e.g. $h_{\underset{LR}{\mu\nu}}\,h_{\underset{LR}{\mu\alpha}}\,h_{\underset{\times}{\nu\alpha}}$

- Solution: • Introduce "auxiliary" scalar (dilaton)

$$\mathcal{L}^{EH} = \frac{2}{\kappa^2}\sqrt{-g}\,R + \sqrt{-g}\,\partial^\mu\phi\,\partial_\mu\phi$$

• Mix it with graviton via field redef:

$$g_{\mu\nu} = \exp\left(\sqrt{\tfrac{2}{D-2}}\,\kappa\phi\right)\exp(\kappa h_{\mu\nu})$$

$$\phi \to -\sqrt{\tfrac{2}{D-2}}\left(\phi + \tfrac{1}{2}h_{\mu\mu}\right)$$

$$\Rightarrow \quad \mathcal{L}_2 = -\tfrac{1}{2}h_{\underset{LR}{\mu\nu}}\,\partial^2 h_{\underset{LR}{\mu\nu}} + \phi\,\partial^2\phi$$

$$\mathcal{L}_3 = \kappa\left[\tfrac{1}{2}h_{\mu\nu}\,h_{\rho\sigma,\mu\nu}\,h_{\rho\sigma} + h_{\nu\mu}\,h_{\rho\mu,\sigma}\,h_{\rho\sigma,\nu} - \tfrac{1}{2}h_{\mu\nu,\kappa}\,h_{\mu\nu,\kappa}\,\phi\right]$$

...

- For n-graviton amp's, integrate out ϕ ⇒ new $\mathcal{L}$ agrees with one constructed directly from KLT, through $\mathcal{L}_5$

Conclusions

- gravity $\sim$ (gauge theory)^2 is a useful way to think about gravity, at least perturbatively.

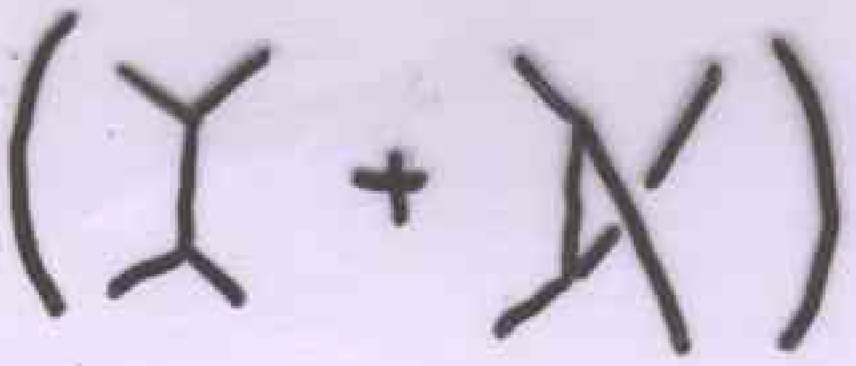
- Unitarity $\Rightarrow$ bootstrap KLT relations to loop level; recycle gauge theory calculations.

- Maximal supersymmetry permits explicit computations of multi-loop scattering amplitudes. For gauge theory applications at least, must extend to non-SUSY cases.

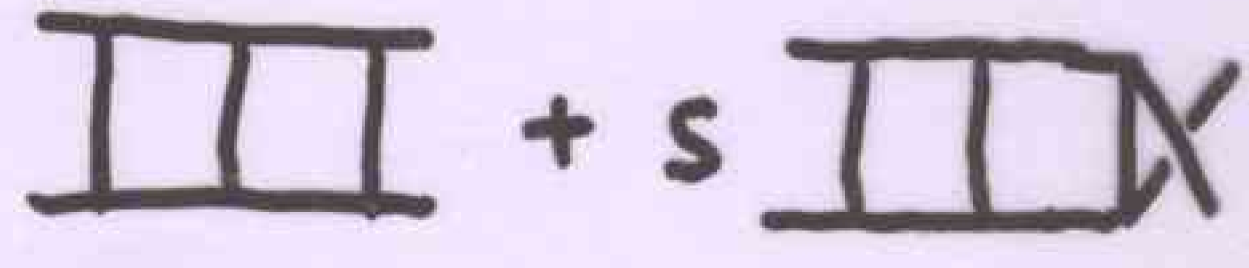
- $N = 8$ supergravity has better UV properties than previously suspected. (But probably still divergent in $D = 4$ by 5 loops.)

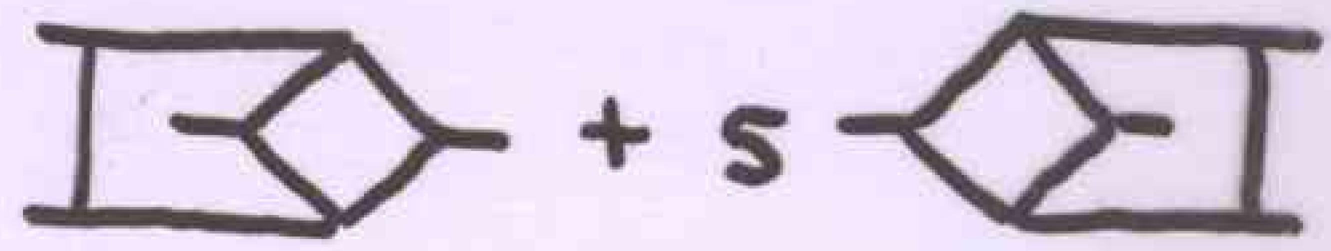
- Recent progress in understanding KLT relations from Einstein-Hilbert (+ dilaton) Lagrangian.

All-loop 4-point Amplitudes in

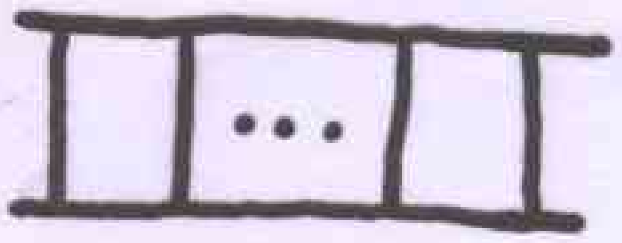
$= \dfrac{st A_4^{tree}}{s}$ 

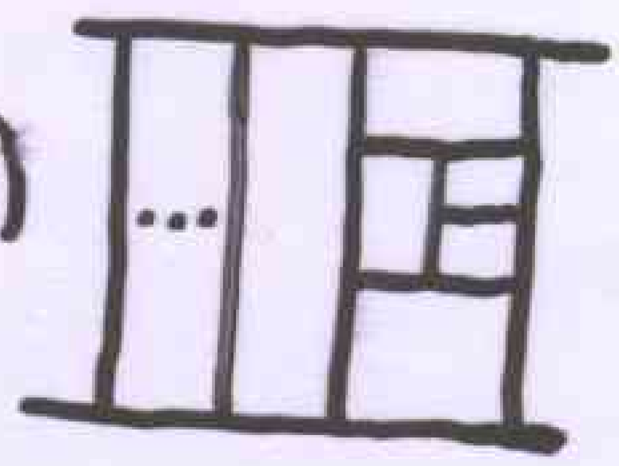
$= st A_4^{tree}$ 

$= st A_4^{tree}$ (s 

+ s 

+ cyclic(2,3,4))

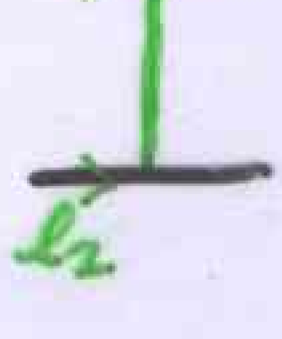
$= st A_4^{tree}$ (s^{l-1} 

+ … + $M(l_i, k_i)$ 

+ ? 

+ …)

$M(l_i, k_i)$ determined recursively by "rung rule"



$\Rightarrow (l_1 + l_2)^2$