Topics in D-Geometry

Strings '99 — M.R. Douglas

based on

- Brunner, Douglas, Lawrence + Römelsberger 9906200
- Diaconescu + Gomis, 9906242
- Douglas + Fiol, 9903031
- Douglas, Greene + Morrison 9704151
- Douglas + Greene, 9707214
- Douglas 9901146
- Govindarajan, Jayaraman + Sarkar 9907131
- Gutperle + Satoh, 9808080.
- Recknagel + Schomerus, 9712186.

See also talk at UPenn, May 99, available at http://www.physics.upenn.edu.

D-geometry is the study of how (one or a few) D-branes see the geometry of space-time.

To zeroth order, this is the study of
calibrated submanifolds
+
vector bundles.

- K3, T^4 - holomorphic curves + bundles.
- CY_3 -

"A" branes = special Lagrangian manifolds

$$\omega|_\Sigma = \mathrm{Re}\, e^{i\theta}\Omega|_\Sigma = 0$$

ex: in $\sum_{i=1}^{5} z_i^5 = 0$, take $\mathrm{Im}\, e^{2\pi i n_i/5} z_i = 0$

ex: in $T^*\mathbb{R}^3$, take $p_i = \partial_i f(x)$

$$\det \partial_i \partial_j f = 1.$$

"B" branes = holomorphic curves, surfaces, vector bundles.

Basic problems:
- count + classify
- find moduli space
- determine $\mathcal{L}_{eff}$ (eg. $\mathcal{N}=1$ $d=4$)

In string/M theory, this zeroth order problem is modified by

1. curvature corrections

$$0 = R_{ij}[g] + l_s^6 R^4_{ij}[g] + \dots$$

ex: what metric does a D0 see on a Calabi-Yau?

2. new light degrees of freedom near singularities make full theory non-singular.

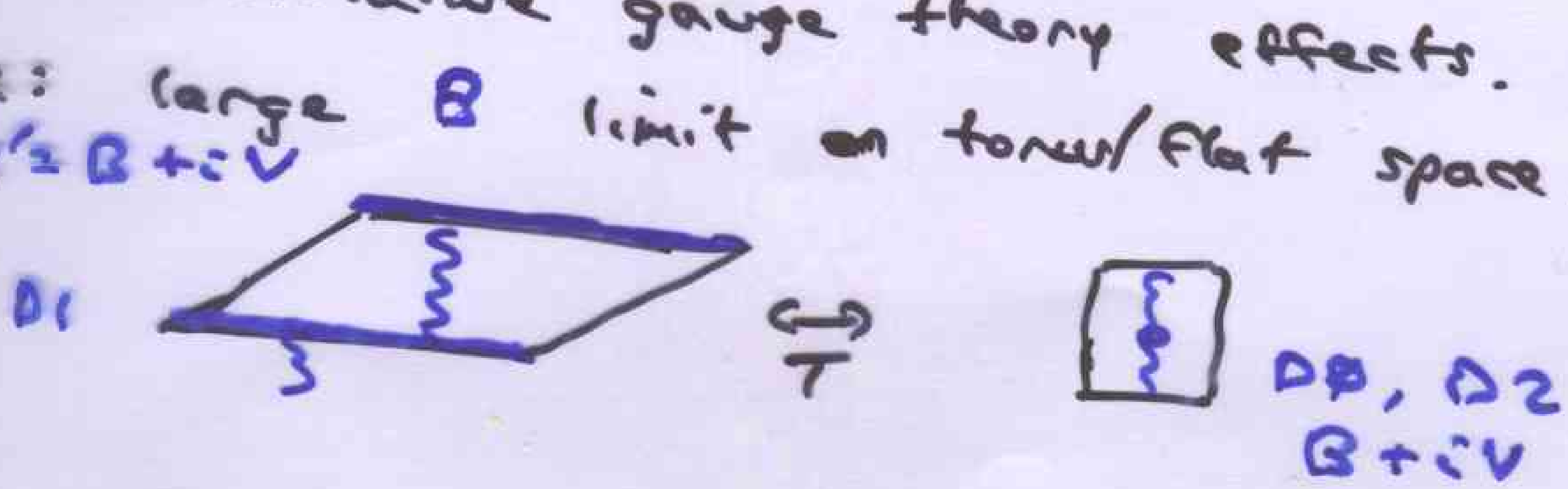
Exact results including effects above can be obtained from gauge theory.

ex: v^{4+2k}/r^{7+7k}, v^4 on ALE, $\mathbb{C}^3/\mathbb{Z}_3$ metric (below).

3. non-commutative gauge theory effects.

ex: large B limit on torus/flat space.

$\tau' = B + iV$



4. ?

The primary question regarding such modifications is:

To what extent do these effects qualitatively change the zeroth order picture: especially, can they change the spectrum of branes or dimension of moduli spaces?

A (null) hypothesis is the "geometric hypothesis" which asserts that they do not: every brane + effective Lagrangian is continuation of some large volume brane.

True for torus + K3 compactification (16, 32 bulk susys) by susy + duality.

But could fail for CY_3 compactification:

- $\mathcal{N}=2$, $d=4$ bulk theories have lines of marginal stability on which BPS spectrum can change.
- $\mathcal{N}=1$, $d=4$ brane theories can have fairly general superpotentials with many branches of moduli space.

A solvable example: $\mathcal{O}_{\mathbb{P}^2}(-3)$.

Perhaps the simplest non-trivial Calabi-Yau metric is obtained by considering a line bundle over $\mathbb{P}^2$: if $L \cong K^*$ the canonical bundle of the total space will be trivial. This Ricci-flat metric was written down by Calabi (see below) and depends on one real parameter, the volume $J = \int \omega$ of the two-cycle in $\mathbb{P}^2$. As $J \to 0$ the $\mathbb{P}^2$ shrinks to $\mathbb{C}^3/\mathbb{Z}_3$ singularity.

In string theory, this becomes a complex modulus "$B + iJ$" and the exact dependence of periods "$\pi_2 = \int \omega$, $\pi_4 = \int \omega^2$" with all stringy corrections can be found using "local mirror symmetry."

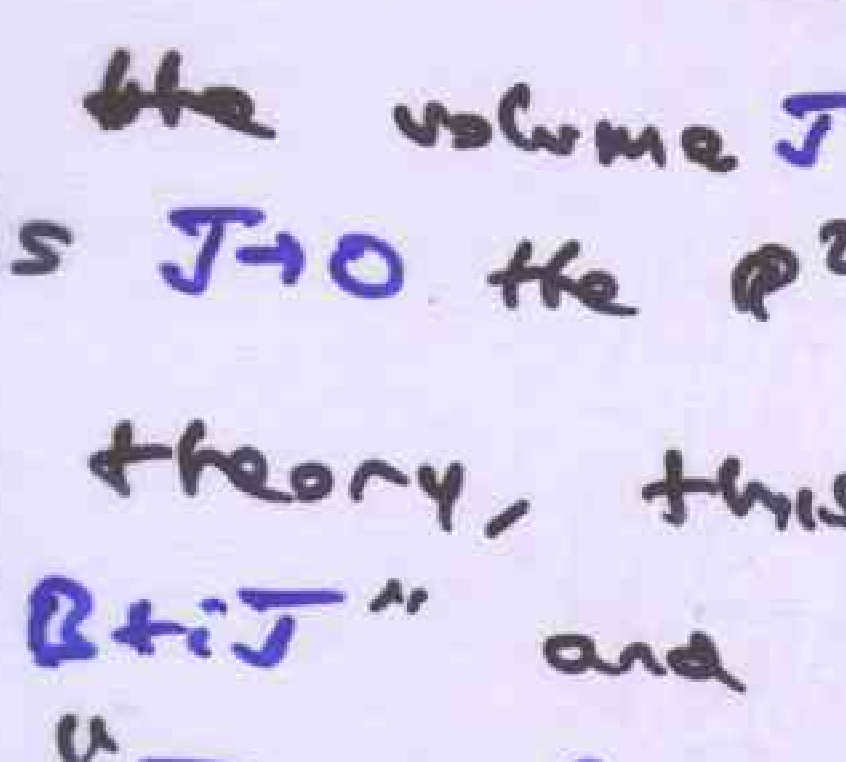
Aspinwall
Katz Klemm Mayr Vafa
Diaconescu + Gomis

Kähler moduli space

$\psi \to \infty$ large volume

$\pi_4 = \frac{1}{2}\pi_2^2 + \frac{1}{8} + \dots$

$\pi_2 \equiv B + iJ$



$\psi \to 1$ singularity

$\pi_4 \to 0$

$\pi_2 \sim (\psi-1)\log(\psi-1)$

$\pi_0 \equiv 1.$

$\psi \to 0$ point of enhanced $\mathbb{Z}_3$ symmetry

$\pi_4 = \frac{1}{3} \quad \pi_2 = -\frac{1}{2}$

Stringy corrections modify the naive relation $\pi_4 = \frac{1}{2}(B+iJ)^2$ and "split" $J \to 0$ singularity into "orbifold" and "conifold" singularities.

D∅ probing the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold

$$z^i \cong \omega z^i \qquad 1 \leq i \leq 3 \qquad \omega \equiv e^{2\pi i/3}.$$

$$\gamma^{-1} z \gamma = \omega z$$

leads to quiver theory

$U(N)^3$
$3(N, \bar{N}, 1)$ + cyclic.
$N=1$.

Various applications - for present purposes consider D∅ metric = metric on moduli space of 0+1 reduction of gauge theory.

Computable by symplectic reduction:

$$K = 3v - \sum_{i=1,2} \xi_i \log (v + \xi_i) + (\xi_1 + \xi_2) \log A$$

$$A = 1 + |z_1|^2 + |z_2|^2 \qquad B = |w|^2$$

$$A^3 B = v(v+\xi_1)(v+\xi_2)$$

where ξ_1, ξ_2 are two real (FI) parameters.

$$\xi_1 \sim \mathrm{Re}\, e^{2\pi i/3} \varphi \qquad \xi_2 \sim \mathrm{Re}\, e^{-2\pi i/3} \varphi$$

A non-Ricci flat deformation of Calabi's metric.

So far the picture agrees qualitatively with conventional geometry, with main difference being appearance of new light states at the two singularities.

Qualitative questions:

- Why does D∅ metric depend on both "B" and "J"?
  Must be world-sheet instantons
  But - is not relevant instanton a map from disk to target with Dirichlet b.c. if holomorphic => constant.
  'almost instantons'?
- does D∅ exist near conifold pt? is metric singular?
- do other branes on $\mathbb{P}^2$ exist?
  wrapped 2B + 4B; bundles with c_1, c_2.
  presumably fractional branes (non-regular representations) with computable RR charges:

  (a) → (0 1)
  (b) → (1 -1) + $\frac{1}{3}$ D∅
  (c) → (-1 0)

  in some basis (D4, D2) with some normalization. How to pin this down?

Douglas
Fiol

Intersection form => normalization

H_2 generator Σ_2

H_4 generator Σ_4 $\Rightarrow$ $\Sigma_2 \cap \Sigma_4 = 1$
$\Sigma_4 \cap \Sigma_2 = 1$

compare A branes

$$\Sigma_{3A} \cap \Sigma_{3B} = 1 = -\Sigma_{3B} \cap \Sigma_{3A}.$$

Mirror symmetry equates these, so one of these expressions must be "wrong" (unphysical). Physically this form controls DSZ charge quantization, i.e. D3's wrapped on a, b cycles are electrons + monopoles w. unit charge.
Poincaré duality => DSZ condition saturated.

A non-geometric def'n follows from computing 'DSZ' interaction in CFT; it is Berry phase produced by massless open fermionic strings.

$$H = \sum_{i=1}^{n_L} \vec{X} \cdot \sigma_i + \sum_{i=1}^{n_R} \vec{X} \cdot (-\sigma_i)$$

$$\Rightarrow \frac{1}{2\pi} \int F = n_L - n_R = Tr_{ab} (-1)^F.$$

intersection form = index.
For B branes,

$$= \int ch(F_1)\, ch(-F_2)\, \hat{A}(R).$$

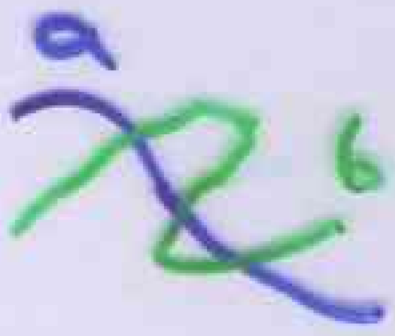
In particular, # generations of matter for GB carrying unbroken gauge group is

$$\mathrm{Tr}(-1)^F = \int ch(F_2)\,\hat{A} = c_3/2$$

(if $c_1 = 0$)

This is exactly the "OB" charge and we are counting intersections of two branes.

In mirror picture, even more intuitive.



Combining this with $N=4$ SCFT representation theory for compactification on K3 leads to simple proof of Mukai's formula for dimension of moduli space: $\mathrm{tr}(-1)^F$ gets contributions from

vector multiplet $\quad \mathrm{tr}(-1)^F = -2$

hypermultiplet $\quad = 2$

so

$$\int ch(F)\,ch(-F)\,\hat{A} = \mathrm{tr}(-1)^F$$

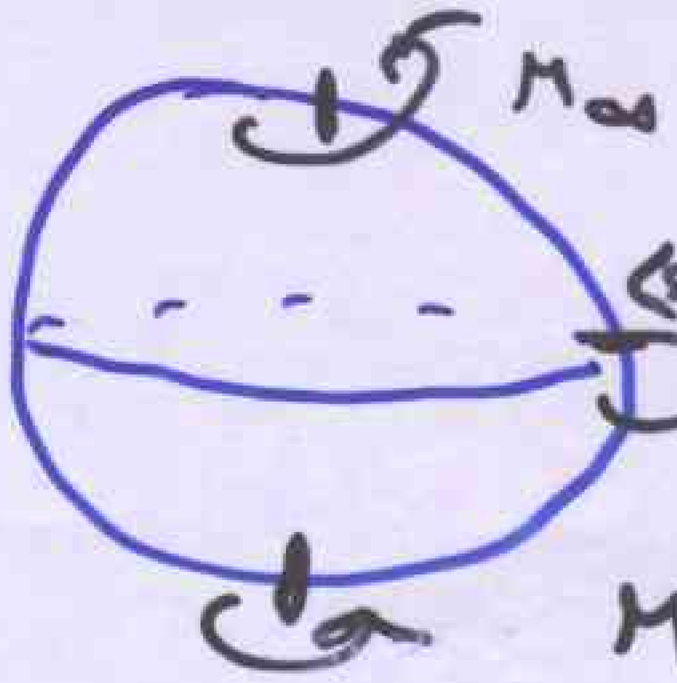
$$= N_{hyper} - N_{vector}$$

$$= \frac{1}{2}\dim_{\mathbb{C}}(\text{moduli space}) - 1$$

(since an overall $U(1)$ vector does not act).

Returning to $\mathbb{C}^3/\mathbb{Z}_3$, the intersection number $\mathrm{tr}(-1)^F$ in a quiver construction will be the total (signed) number of links between the two nodes (each chiral multiplet contributes ± 1), so ± 3 in our example.

This turns out to be consistent with one of the branes being $a\,\Sigma_{\mathbb{P}_1} + b\,\Sigma_{\mathbb{P}_2}$ with $(a,b)=1$ (here $\Sigma_{\mathbb{P}_1} \cap \Sigma_{\mathbb{P}_2} = 3$) with the charges of the others determined by using the monodromy on periods + charges derived from local mirror symmetry:



$$M_\infty = M_1 M_0$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & -3 \\ 0 & 1 & 1 \end{pmatrix}$$

This is not enough information and we must also use the fact that each state has D0 charge $\frac{1}{3}$. (D0 does not intersect locally), to find branes

$$\overline{D4}, \quad \overline{D4} + D2 \quad \text{and} \quad 2D4 + \overline{D2} + D0.$$

All 3 are sensible - the last is a known rank 2 bundle on $\mathbb{P}^2$ with no moduli.

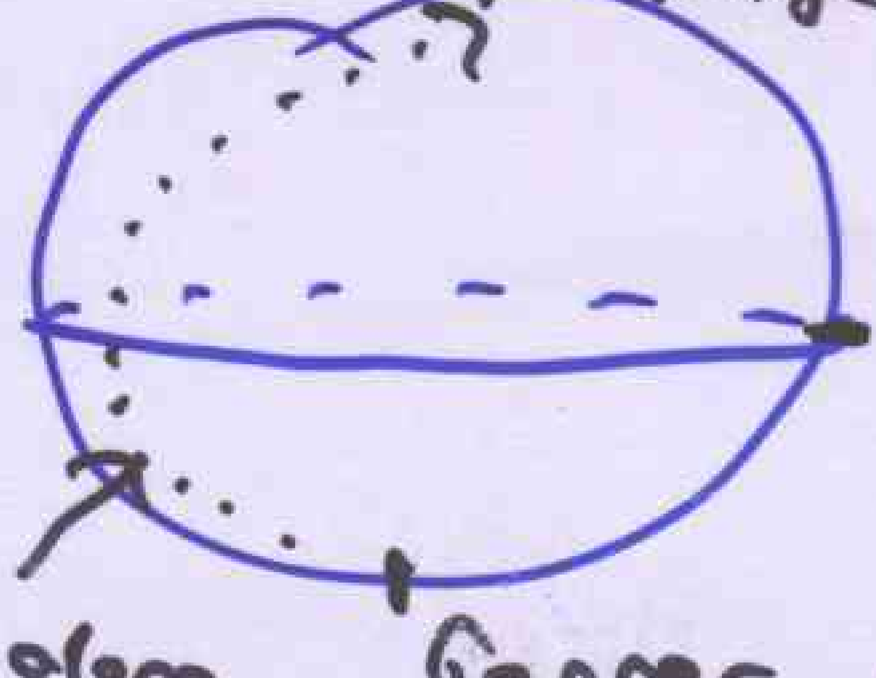
D-geometry of quintic.

In this case fewer large volume results are known. We do know various facts, such as

- $\exists$ 625 supersymmetric 3-cycles
$$\mathrm{Re}\, e^{2\pi i n_i/5} z_i = 0$$
on the Fermat quintic.
They are topologically $\mathbb{RP}^3$'s and as such should have no continuous moduli, but a choice of $\mathbb{Z}_2$ Wilson line.
Their intersection form is calculable.
e.g. $\Sigma_{(0,0,0,0,0)} \cap \Sigma_{(1,1,1,0,0)} = +1$.
It can be written in terms of $\mathbb{Z}_5^4$ generators: $\prod g_i = 1$.
$$I = \prod_{i=1}^{5} (g_i + g_i^2 - g_i^3 - g_i^4)$$

- A state with 6B charge r should correspond to a rank r semistable holomorphic vector bundle (DUY). These can have arbitrary 4B charge (take $B \to B+E$); if this is set to zero we have 2B charge
$$Q_2 = 0 \qquad \text{for } r=1 \quad \text{(line bundle)}$$
$$Q_2 \geq 0 \qquad \text{for } r>1 \quad \text{(Bogomolov inequality)}$$

Explicit boundary states in the corresponding Gepner model 3^5 have been constructed by Recknagel and Schomerus. By computing $Tr(-1)^F$ for these states, and combining with stringy geometry of quintic (Candelas et al), we can derive large volume interpretation for these branes.



By continuing along this trajectory, we

1. express large volume intersection form in Gepner basis:

$$I_{LV} = g(1-g)^3$$

2. express Gepner basis periods in terms of large volume periods.

Recknagel + Schomerus' boundary states are "rational" boundary states: each factor in the 3^5 Gepner model has an independent boundary condition of the type constructed by Cardy; the resulting state then undergoes orbifold / GSO projection à la Gepner.

A states ($J_L = J_R$ in closed string channel) are labelled by a choice of closed string primary field $\phi^{l_1}_{M_1} \phi^{l_2}_{M_2} \phi^{l_3}_{M_3} \phi^{l_4}_{M_4} \phi^{l_5}_{M_5}$.
Orbifolding identifies $\vec{M} \sim \vec{M} + \vec{S}$.
These form multiplets of $\mathbb{Z}_5^4$ (shift M_i).

B states ($J_L = -J_R$) exist only by virtue of orbifolding and are labelled by l_i and overall M. They transform under quantum $\mathbb{Z}_5$.

Open string spectra between branes given by fusion coefficients (Cardy):

$$Z_{ab} = \sum_c N_{ab}^c \chi_c$$

This makes computation of $\mathrm{tr}(-1)^F$ easy:

$$\mathrm{Tr}_{ab}(-1)^F = \sum_{\substack{\text{chiral} \\ \text{primaries} \\ c}} N_{ab}^{c}\, \mathcal{C}(1)_c$$

= operator, depending on L_i, L_i', acting on M or M_i indices:

$$N_{L=0,L=0} = 1 - g^4$$
$$N_{L=0,L=1} = g - g^3$$
$$N_{L=1,L=1} = g + g^2 - g^3 - g^4.$$

There are two candidate multiplets of 625 A-branes to identify with $\mathbb{RP}^3$'s:

$L=0$: $\quad I = \prod_i (1-g_i^4)$.

$L=1$ $\quad I = \prod_i (g_i + g_i^2 - g_i^3 - g_i^4) \Leftarrow$ yes.

However the $L=1$ branes, which seem to work, also have a (naively) marginal operator!

Likely resolution — induced superpotential.

B branes are labelled by L_i, M.

take $M=0$ and map charges:

L_i	6B	4B	2B	0B	(dimension)
0 0 0 0 0	−1	0	0	0	0
0 0 0 0 1	2	0	5	0	4
0 0 0 1 1	1	0	5	0	11
0 0 1 1 1	3	0	10	0	24
0 1 1 1 1	4	0	15	0	50
1 1 1 1 1	7	0	25	0	101

These all look "pretty geometrical" except for 00011. Perhaps this is a "non-commutative U(1) brane"?

Charges for other M follow from action of $\mathbb{Z}_5$ monodromy and do not look very geometrical. Some of them violate stability inequality.

Conclusions + further directions:

- We can study branes in stringy regime using orbifold + Gepner model techniques.
- Orbifold results look fairly geometrical. Quintic results – much less clear.
- Evidence for non-trivial superpotentials, should be computable from CFT.

  "Topological hypothesis":
  superpotential for A (B) brane is computable in A (B) topological open string theory, and has only (nontrivial) dependence on A (B) closed string moduli.
  B superpotential would then be computable in large volume limit.
  Many examples + (formal) world-sheet argument.
- More direct geometric interpretation?
  $L=1$ A state looks like Re z Neumann, Im z Dirichlet – a known LG model boundary condition (Warner?).
  What are others?

- Lines of marginal stability?

Much studied in $d=4$, $N=2$ supersymmetric gauge theory.

e.g. Seiberg - Witten

Spectrum simplifies inside circle (strong coupling) – only monopole, dyon present.

Could the same happen for CY_3 compactification?

- Do marginal stability + decay considerations also operate for branes which extend in 3+1 Minkowski space (eg. wrapped 3,5,7,9 B). Geometric picture of stability

(geodesics on SW curve)

suggests yes.

Combining these two points could lead to strong constraints on allowed brane configurations in string scale type II compactifications