Holography & the Future Tube

Hologram :

"information about 3-dim "stored"
on a 2-dim surface

Holographic Principle "A spatial region Σ of volume V and bdy area $A = \text{vol}(\partial\Sigma)$ can contain no more than

$$\exp\left(\frac{1}{4}\frac{A}{l_p^2}\right)$$ "degrees of freedom"

- Surely false if $\partial\Sigma = \phi$
- Cosmological status unclear
- when valid seems to be
  $\approxeq$ Generalized 2^{nd} Law

- If "degrees of freedom" = # (vectors in Hilbert Space) does this mean

$$\dim \mathcal{H}_{qm} < \infty \; ?$$

As yet attempts to extend the "holographic principle" beyond the AdS/CFT correspondence to a general principle comparable in generality & scope to the 2ND Law of Thermodynamics have not been entirely successful.

Isoperimetric Inequality

$$\sqrt{\frac{A}{16\pi}} \leq M$$

$$\Rightarrow \quad A \leq 16\pi M^2$$

$$S \leq \frac{16\pi}{4} M^2$$

$$\boxed{S \leq 4\pi M^2}$$

This probably is true for all physical systems.

What is information & where is it stored?

In Q.Mech Hilbert Space $\mathcal{H}_{qm}$ & s.a. operator, eg energy or Hamiltonian $H : \mathcal{H}_{qm} \to \mathcal{H}_{qm}$

with spectrum $\{E_i\}$

$$N(E) = \#(\text{e-vectors with energy } E_i < E \mid \text{fixed volume } V \text{ etc})$$

$$\boxed{S(E, V) = k \ln N}$$

How does S depend on E, V etc?

Extensivity $\mathcal{H}_{qm} = \mathcal{H}_1 \otimes \mathcal{H}_2$

non-interacting systems

$$H = H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2$$

$$S \geq S_1 + S_2$$

suggests S increases as V increases

radiation $S \propto V^{\frac{1}{d+1}} E^{\frac{d}{d+1}}$

$d = \dim$ space

Some Old ideas from

Black Hole Physics are suggestive

but are

1) insufficiently detailed

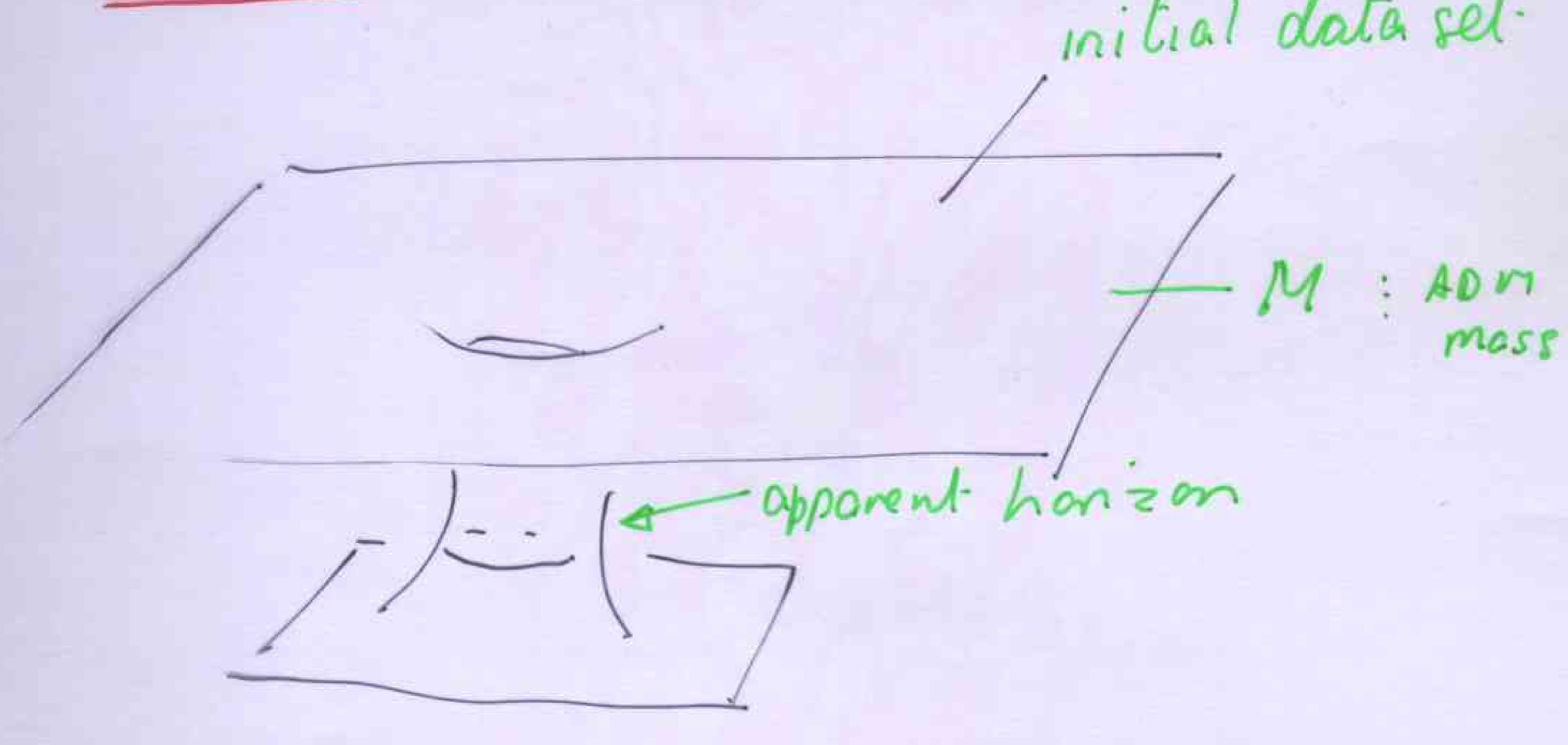
2) at best semi-classical

Example

- Generalized 2^{nd} Law :

$$\sum_i \frac{A_i}{4} + S_{matter} \text{ is non decreasing}$$

- "Isoperimetric Inequalities



$$\sqrt{\frac{A}{16\pi}} \leq M$$

equality $\Leftrightarrow$ Schwarzschild

$$2M \geq \left(\frac{A}{4\pi}\right)^{1/2} + \left(\frac{4\pi}{A}\right)^{1/2} Z^2$$

$$\boxed{A \geq 4\pi Z^2}$$

$$Z = Z(Q, P)$$

Z : central charge.

That is: extreme black holes have least entropy for fixed central charges

or

$$\boxed{|Z| \leq \left(\frac{A}{4\pi}\right)^{1/2}}$$

However these (rigorous) bounds refer to the area of apparent horizons & not an arbitrary surface enclosing an arbitrary system.

Other (putative) bounds (such as the "Bekenstein Bound") seem to be just plain wrong.

This prompts an examination of the question

What is information & where is it stored ?

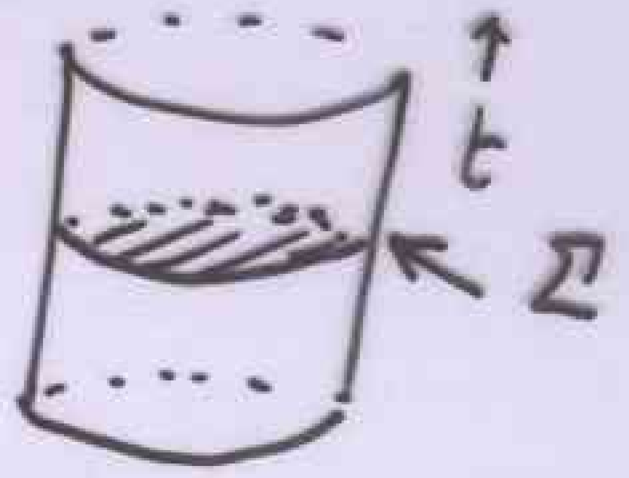
- A few remarks about "the holographic principle"

- A Geometrical description of the future tube & its relation to AdS via Shilov boundaries

- ~~A~~ possible generalizations to non-trivial spacetimes.

Quantum Field Theory (no self-gravity)

$$M = \underset{t}{\mathbb{R}} \times \underset{\substack{\uparrow \\ \text{space}}}{\Sigma} \Leftarrow \text{Globally Hyperbolic}$$



e.g. Free Fields :

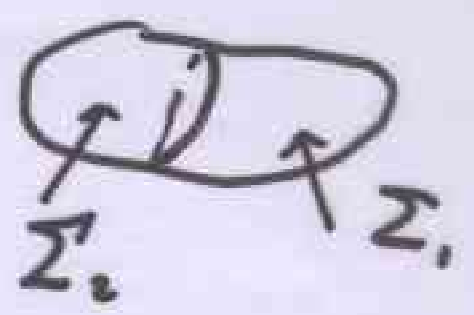
$$\mathcal{H}_{qm} = \mathbb{C} \oplus \mathcal{H}^1 \oplus \mathcal{H}^1 \otimes_{\pm} \mathcal{H}^1 \oplus \ldots$$

$$\mathcal{H}^1 = \{\text{space of Cauchy data}\}$$

$$= \{\phi, \frac{\partial \phi}{\partial t}\} \; : \; \text{maps } \Sigma \to \mathbb{R}^2$$

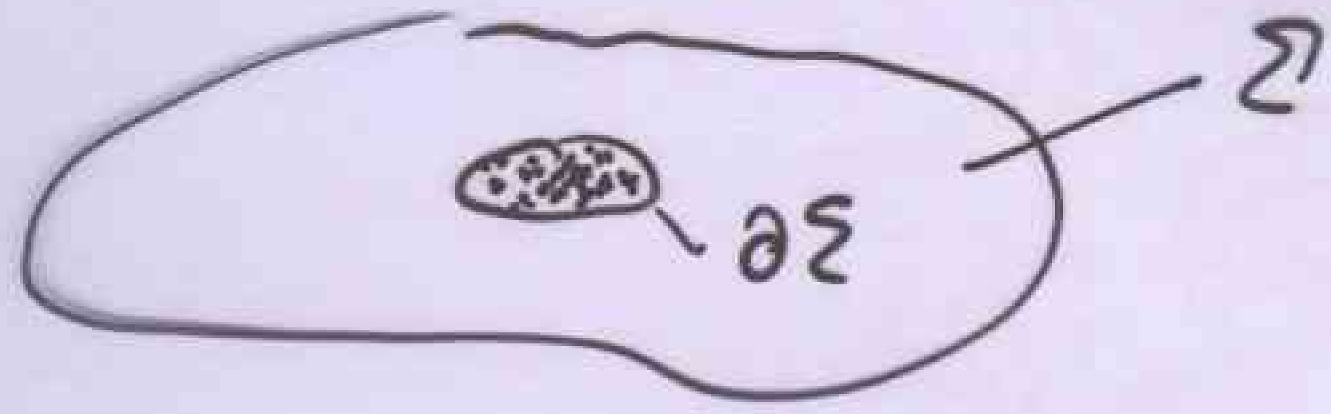
$$\Sigma = \Sigma_1 \sqcup \Sigma_2 \quad ; \quad V = V_1 + V_2$$

$$\mathcal{H}^1 = \mathcal{H}^1_1 \otimes \mathcal{H}^1_2 \quad :$$



- entropy increases as $V = \mathrm{vol}(\Sigma)$ increases
- The information resides in the Cauchy data
- A Cauchy surface of volume V can carry arbitrarilly large amounts of infomation for sufficiently large E

Self-Gravity & Black Holes



Σ acquires (internal) boundary of area $A = \mathrm{vol}(\partial\Sigma)$

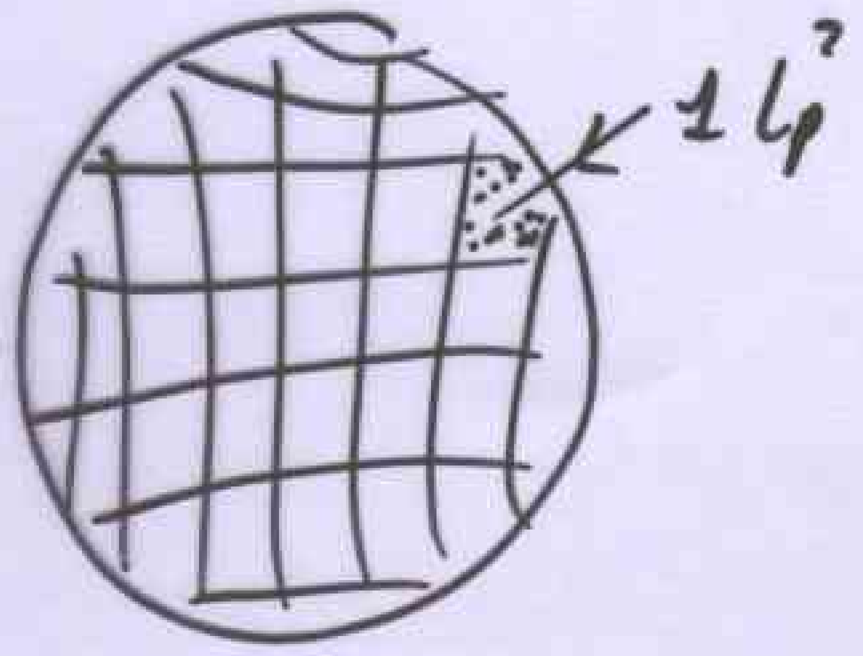
Black Hole Physics $\Rightarrow$

dominant contribution to entropy is

$$\boxed{S_{BH} = \frac{1}{4}\frac{A}{l_p^2}}$$

l_p = Planck length

"1 BIT per Planck area"



Holographic Principle "A spatial region Σ of volume V and bdy area $A = \mathrm{vol}(\partial\Sigma)$ can contain no more than

$$\exp\left(\frac{1}{4}\frac{A}{L_p^2}\right) \text{ "degrees of freedom"}$$

- Surely false if $\partial\Sigma = \phi$
- Cosmological status unclear
- when valid $\cong$ Bekenstein Bound
- If "degrees of freedom" $\equiv$ # (vectors in $\mathcal{H}_{qm}$)

  maybe $\dim \mathcal{H}_{qm} < \infty$

Further progress requires more concrete models

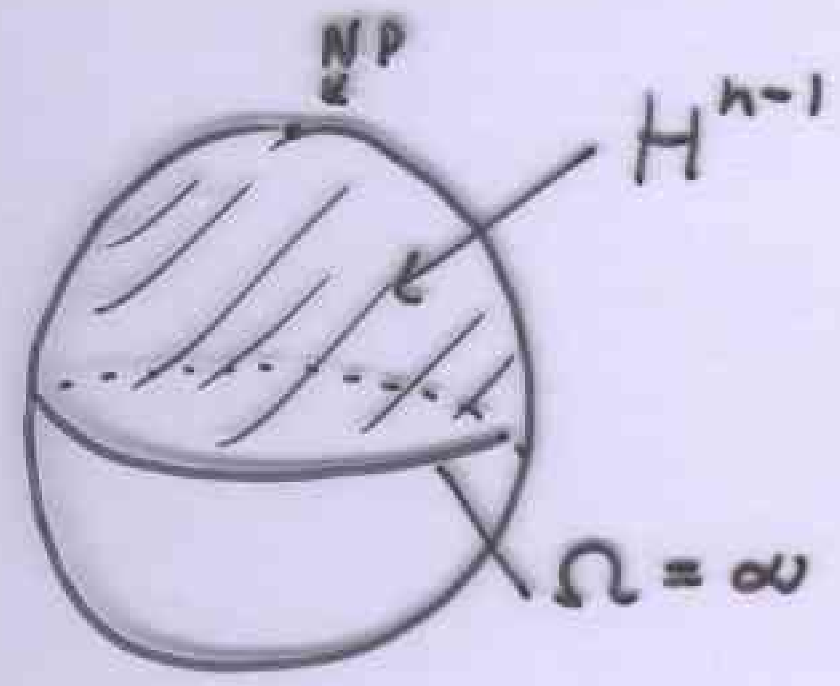
AdS/CFT Correspondence

$$AdS_n = SO(n-1,2)/SO(n-1,1)$$
$$\cong S^1 \times \mathbb{R}^{n-1}$$

with warped product metric $\quad g = -\Omega^2 dt^2 + g_{H^{n-1}}$

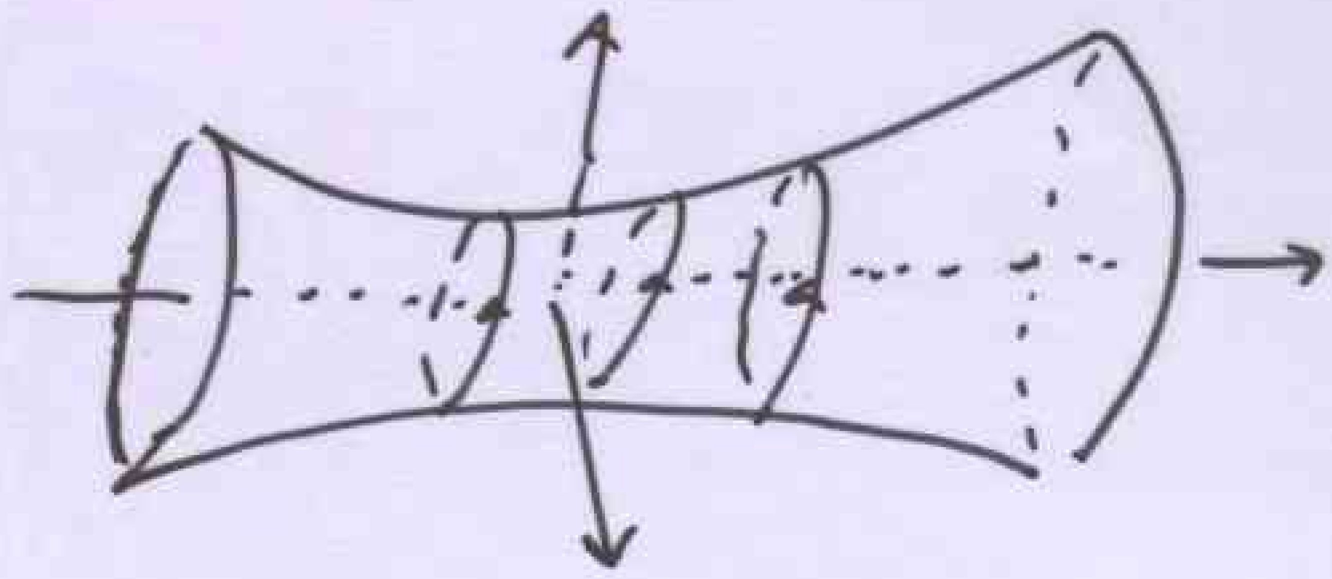
$g_{H^{n-1}}$: standard metric on H^{n-1} s.t.

$$\Omega^{-2} g_{H^{n-1}} = g_{S^{n-1}} \quad \text{on} \quad \tfrac{1}{2} S^{n-1}$$



$\Omega = 1$ at N.P.
$\Omega = \infty$ on equator

or as quadric



$$Q_n \subset \mathbb{E}^{n-1,2}$$

$$(X^1)^2 + \dots (X^{n-1})^2 - (X^n)^2 - (X^{n+1})^2 = -1$$

e.g. $n = 3$: $AdS_3 \equiv SL(2,\mathbb{R})$ with bi-invariant metric

AdS_n is not Globally Hyperbolic :

It has a timelike conformal bdy

$$\partial(AdS_n) = S^1 \times S^{n-2}$$

↑ 2-fold cover of cpt.fied Minkowski spacetime $\mathbb{E}^{n-2,1}$

$$g = \Omega^2 \bar{g}$$

$\bar{g}$ is product metric on $S^1 \times \frac{1}{2} S^{n-1}$

$\Omega^{-1} = 0$, $d(\Omega^{-1}) \neq 0$ on g

The isometry gp of AdS_n is $SO(n-1,2)$
it acts by conformal isometries on $\partial(AdS_n)$

Despite its lack of Global Hyperbolicity one may develop QFT on AdS_n. The quantum fields carry unirreps of $SO(n-1,2)$ as long as the fields obey certain boundary conditions on $\partial(AdS_n)$ (Dirichlet / Neumann)

- In some sense the information is stored on the boundary

Maldacena conjecture

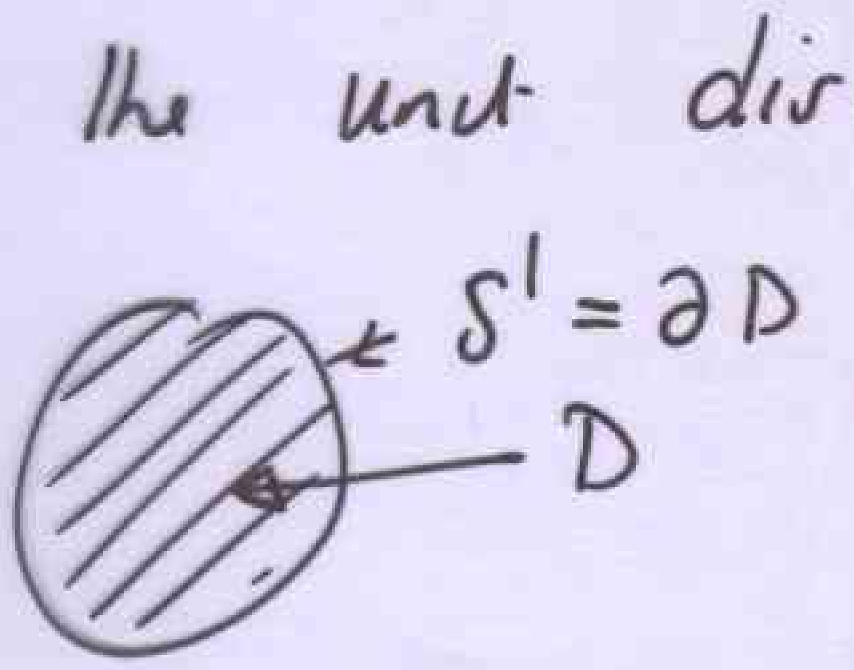
Classical Type IIB String theory on $S^5 \times AdS_5$

$\cong$ large N limit of $SU(N)$ $\left(\begin{smallmatrix}\text{susy}\\ N=4\end{smallmatrix}\right)$ Yang-Mills

on its conformal boundary

Crucial to this correspondence is the relationship between a field theory in the "bulk" & field theory on the "boundary"

The analogy is with holomorphic functions on the unit disc D & their bdy values on $S^1 = \partial D$



$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

restriction to bdy $\Rightarrow f(\theta)$; Fourier coeffs.

boundary values

$$f = g + ih \; : \; \nabla^2 g = 0 = \nabla^2 f$$

harmonic fields in the bulk

In this setting

$$\boxed{\text{Holography} \cong \text{Holomorphy}}$$

Future Tube : T^+_{n-1} of $\mathbb{E}^{n-2,1}$

$= \{ x^\mu + i y^\mu \in \mathbb{C}^{n-1},\ y^\mu \in \text{past light cone} \}$

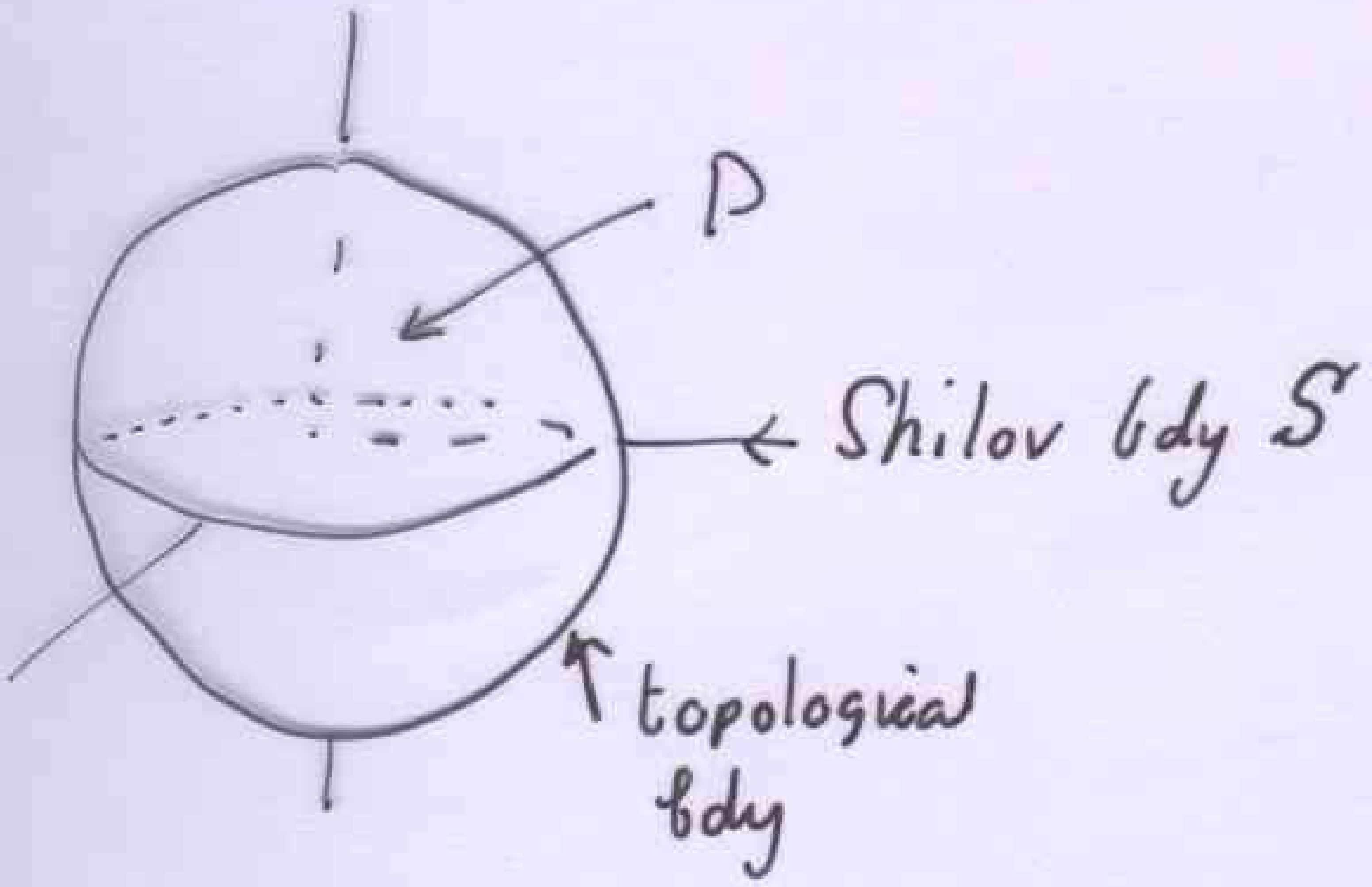
In Q.F.T. a positive $\left\{\begin{matrix}\text{frequency}\\ \text{energy}\end{matrix}\right\}$ function is bdy value of holomorphic function on future tube

$$f^+(x) = \int e^{i p_\mu x^\mu} \mu(p) = \int e^{-iEt + i\underline{p}\cdot\underline{x}} \mu(p)$$

$\mu(p)$ has support in future light cone $\Rightarrow$

$E > 0 \Rightarrow$ holomorphic if $\operatorname{Im} t < 0$.

Claim T^+_{n-1} is homogeneous bounded domain in $\mathbb{C}^{n-1}$; $T^+_{n-1} \cong SO(n-1,2)/\underbrace{SO(n-1)\times SO(2)}_{\text{max cpt. subgroup}}$



$\dim D = 2n-2$

$\dim \partial D = 2n-3$

$\dim S = n-1$

Holomorphic functions on domain $D \in \mathbb{C}^{n-1}$ attain maximum modulus on Shilov bdy

Claim Shilov bdy $\cong (S^1 \times S^{n-1})/\mathbb{Z}_2$

$\cong$ conformally compactified Minkowski Spacetime

- In some sense the information about holomorphic functions is "stored" on the Shilov Boundary

Complex Light Cone in $\mathbb{C}^{n-1,2}$

$$(Z^n)^2 + (Z^{n+1})^2 = (Z^i)^2$$

$$Z^i = w^i/u$$

$$Z^n - i Z^{n+1} = 1/u$$ cplx horospheric coords

$$Z^n + i Z^{n+1} = \frac{w^i w_i}{u}$$

$$w^2 = w^t w \quad ; \quad |w|^2 = w^+ w$$

$$D \; : \; 1 - |w|^2 > \sqrt{|w|^4 - |w^2|^2}$$

$$\partial D \; : \; 1 - 2|w|^2 + |w^2|^2 = 0$$

$$S \subset \partial D \; : \quad w^i = (\exp i\tau)\, n^i \quad \in S^1 \times S^{n-1}/\mathbb{Z}_2$$

$$n^i n_i = 1 \quad ; \; n^i \in S^{n-1}$$

Quantization of a Relativistic particle moving in AdS_n

(i.e. constructing $\mathcal{H}^1$)

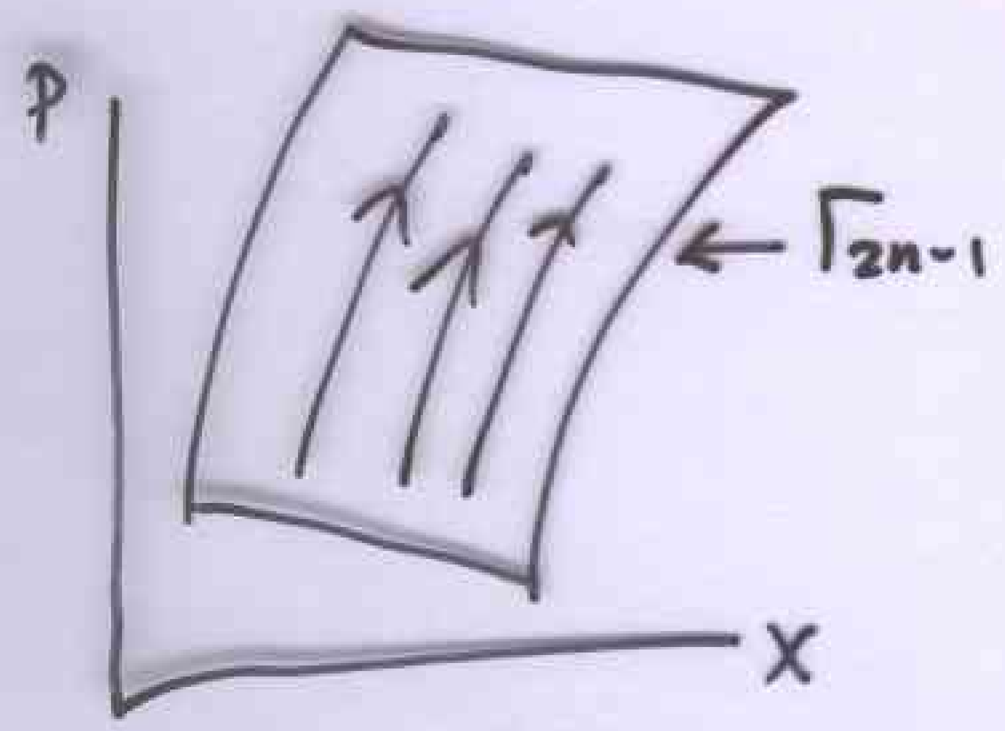
Look at Hamiltonian flow on $T^*(AdS_n)$ with classical Hamiltonian function

$$H^{class} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$$

subject to constraint

$$H^{class} = -\frac{1}{2} m^2 \qquad (\text{timelike geodesics})$$

level set of $H^{class^{-1}}(-\frac{1}{2}m^2) = \Gamma_{2n-1}$



orbits are circles

Marsden-Weinstein quotient or reduced phase space

$\Gamma_{2n-1} / S^1 \cong P_{2n-2}$ is

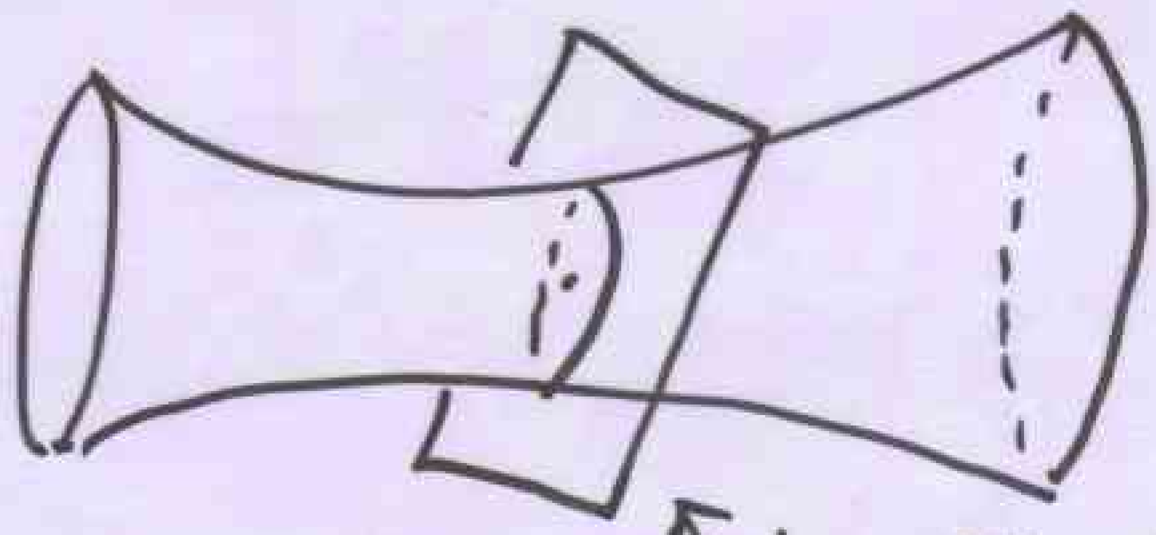
$(2n-2)$-diml symplectic mfd.

- We need to "quantize" $P_{2n-2} = \{\text{set of timelike geodesics in } AdS_n\}$

Claim $\boxed{T^+_{n-1} \equiv P_{2n-2}}$

Reduced phase space $\equiv$ Future tube

Proof (sketch)



$\nwarrow$ timelike 2-plane through origin

P_{2n-2} = {space of totally timelike 2-planes through origin}

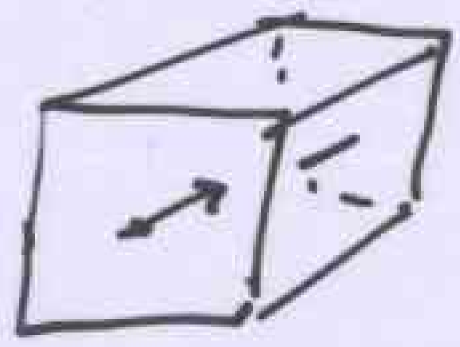
Grassmannian = $G_{-2}(\mathbb{E}^{n-1,2}) \equiv SO(n-1,2)/SO(n-1)\times SO(2)$

- Thus P_{2n-2} is a Kähler mfd & we may adopt the holomorphic polarization & construct $\mathcal{H}^1$ from holomorphic functions (or sections of some bdle) on P_{2n-2} which are determined by their behavior on the Shilov boundary, i.e. on compactified Minkowski spacetime.

Generalization : Adapted Complex Structures

(Szöke, Guillemin, Stenzel)

studied $T^*(M)$, M riemannian, in nbd. of zero section. They gave cplx structure s.t. M is totally real submfd.



"Grauert tube"

cplx structure s.t. map $\mathbb{C} \to T^*M$ given by
$$\sigma + i\tau \to (x^\mu(\sigma), \tau p_\mu(\sigma))$$

is holomorphic $\forall$ geodesics $x^\mu(\sigma)$

Let w^i be adapted cplx coords.

- $H^{class}(w, \bar{w})$ is plurisubharmonic
- $\sqrt{H^{class}}$ satisfies homogeneous Monge-Ampère eqn.

- $\dfrac{\partial^2 H^{class}}{\partial w^i \partial \bar{w}^j} > 0$

- $\det\left(\dfrac{\partial^2}{\partial w^i \partial \bar{w}^j} \left(\sqrt{H^{class}} \right) \right) = 0$

Example $M = S^n$

cplx structure to all T^*S^n &

carries Calabi-Yau metric (Ricci - flat Kähler, $\cong SU(n)$ holonomy) related to $\mathcal{H}^{class}$

- The "Wick rotated" version of this construction is the AdS_n example discussed earlier

- Formally the general construction works when M is Lorentzian & T^*M becomes pseudo Kähler (i.e. 2 -ve signs in metric)

Conclusions ?

- A more precise formulation of holographic ideas requires making precise ideas like "degrees of freedom" & "information".
- In AdS much current activity is involved with relating holomorphic functions to their behavior on the Shilov bdy of $P_{2n-2} = T_{n-1}^{+}$
- There are indications that these ideas extend to the inhomogeneous case
- Complex domains, the Kähler metrics on them & Geometric quantization are coming into play in this Lorentzian context.

Stenzel Example

$T^*S^n \leftrightarrow$ affine quadric in $\mathbb{C}^{n+1}$

$\underline{X}.\underline{X} = 1 \qquad (\underline{X}, \underline{P}) \subset \mathbb{R}^{2n+2}$

$\underline{X}.\underline{P} = 0$

$$\underline{\underline{Z}} = \cosh|P|\,\underline{X} + i\,\frac{\sinh|\underline{P}|}{|\underline{P}|}\,\underline{P}$$

$$\boxed{Z^t Z = 1}$$

let $\tau = Z^\dagger Z$ & seek $F(\tau)$ st.

$$\det\left|\frac{\partial^2 F}{\partial w^i \partial \bar{w}^j}\right| = 1 \qquad \text{inhomogeneous Monge-Ampère}$$

in fact $\tau = 8\cosh^2\left(8 H^{class}(w,\bar{w})\right)$

$$H^{class} = \frac{1}{2} P^2$$

Cheng - Mok - Yau - Anti-de-Sitter Spacetimes

Replace $H^n_{\mathbb{C}}$ by general Einstein-Kähler metric on some inhomogeneous domain D

if $dA = \omega \leftarrow$ Kähler form

we have

$$\begin{array}{ccc} S^1 & \longrightarrow & M \\ & & \downarrow \\ & & D \end{array}$$

&

$$g_L = -(dt + A_\circ dx^\circ)^2 + g_{CMY}$$

will be Einstein - Lorentz metric

Conformal bdy is S^1 bdle over ∂D.

Another appearance of homogeneous complex domains

AdS_{2p+1} : $|Z^1|^2+|Z^2|^2+\dots|Z^p|^2-|Z^{p+1}|^2=-1$

(anti)-Hopf fibration $Z^\alpha \to e^{i\theta} Z^\alpha$

orbits are timelike

$\leftarrow H^p_{\mathbb{C}}$ cplx. hyperbolic space with Bergmann metric

$$ds^2 = -(dt + A_i dx^i)^2 + g_{ij}\,dx^i dx^j$$

Sagnac connection on Canonical Bundle

$H^p_{\mathbb{C}} = SU(p,1)/U(p)$ cplx domain $D \subset \mathbb{C}^p$

& is Einstein-Kähler

($H^p_{\mathbb{C}} = \widehat{\mathbb{CP}}^p$, symmetric space dual)