Systematics of the Low Energy Action

- Constraints on M theory
  - SUSY non-ren. theorems
- Derivative expansion
  - non-perturbative
    e.g. SL(2,Z) duality and IIB
- Relation between Yang-Mills and gravity
  open ⟷ closed strings
  (viz. Dixon)
- Detailed info about D-instantons
  AdS/CFT Y.M. instantons

I) 11 dim. SUGRA on T^2 – Multi-loop
(with Vanhove) 2

- Comparison with string theory "data"

II) Features of AdS/CFT and N=4 Yang-Mills

SUSY protected processes in type IIB

e.g. four graviton interaction

$$\frac{1}{(\alpha')^4}\left\{e^{-2\phi}R+(\alpha')^3 e^{-\phi/2}R^4 f_1^{(0,0)}(\tau,\bar\tau) + (\alpha')^5 e^{\phi/2}\partial^4 R^4 f_2^{(0,0)}(\tau,\bar\tau)+\dots\right\}$$

$\tau=\tau_1+i\tau_2$

$\tau_2=e^{-\phi}$

Higher derivatives

Related by SUSY

(MBG + Sethi)

$$f_1^{(0,0)}(\tau,\bar\tau) = 2E_{3/2} = \sum \frac{\tau_2^{3/2}}{|m+n\tau|^3}$$

Non-holomorphic Eisenstein series

- $SL(2,\mathbb{Z})$ duality invariance
- Naively, SUSY "protects" terms up to $O(\partial^6 R^4)$ (dimensional analysis)

What is $f_2^{(0,0)}$?

Eisenstein series: $E_s \quad (s > \tfrac{1}{2})$

$$\underline{\tau_2^2 \, \partial_\tau \partial_{\bar\tau} E_s = s(s-1) E_s}$$ eigenfns. of Laplace

$$E_s(\tau,\bar\tau) = \sum_{m,n} \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

$\tau_2 = e^{-\phi} = 1/g$

$$= 2\zeta(2s)\,\tau_2^s + (\ldots\ldots)\,\zeta(2s-1)\,\tau_2^{1-2s}$$

"perturbative"

"non-perturbative" $+\sum c_K \left(e^{2\pi i K\tau} + c.c.\right)\left(1 + a\tau_2^{-1} + \ldots\right)$

∞ no. of fluctuations

$$c_K \approx \sum_{m|K} \frac{1}{m^{s-\frac{1}{2}}}$$

Charge-K D-instantons

zero-dimensional matrix model partition fn.

Similarly for U(1)-violating processes

e.g. $(\alpha')^3 \Lambda^{16} e^{-\phi/2} f_1^{(12,-12)}(\tau,\bar\tau)$ $\Delta_{U(1)} = 16 \times \frac{3}{2} = 24$

dilatino $(q_{U(1)} = \frac{3}{2})$

$$f_1^{(12,-12)} \xrightarrow[SL(2,Z)]{} \left(\frac{c\tau+d}{c\bar\tau+d}\right)^{12} f^{(12,-12)}$$

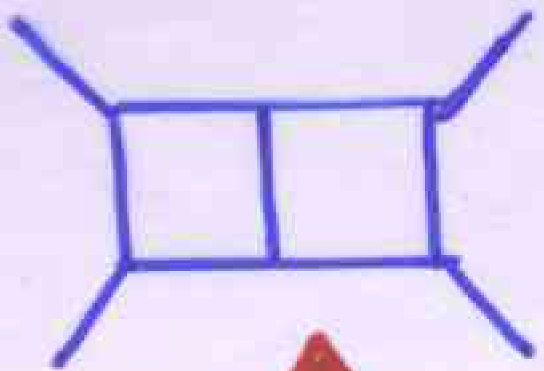
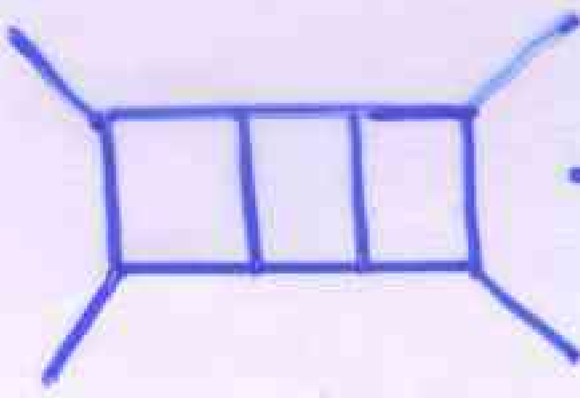
(12,-12) form (MBG, Gutperle + Kwon)

- Tree + 1-loop + D-instantons
- AdS/CFT → correlators of $\mathcal{N}=4$ SUSY Yang-Mills (large N, large $\lambda = g^2_{YM}N$)
- "Tested" in charge-K instanton sector (Bianchi et al, Dorey et al)
  - exact agreement for leading $\frac{1}{N}$ term uses semi-classical approximation! (Dorey, Hollowood, Khoze, Mattis, Vandoren)
  - Non-renormalization "theorem"

$\frac{1}{N^\ell}$ term recieves corrections up to $O(\lambda^\ell)$

(MBG + Gopakumar)

b) $\mathcal{N}=8$ Supergravity

R^4  $+\, s^2 R^4$  $+\, s^4 R^4$?  $+ \cdots$

scalar fields

Deduced by : i) Relation between open and closed strings (Kawai, Llewellyn + Tye)

ii) Unitarity (also valid for $D=11$)

- No correction to R^4 term from higher loops
- $s^2 R^4$ gets contributions from 1 and 2 loops?
- Expect SUSY protects terms up to $s^3 R^4$ (dimensional analysis)

11-dimensional supergravity on T^2

$\equiv$ IIA, IIB on S^1

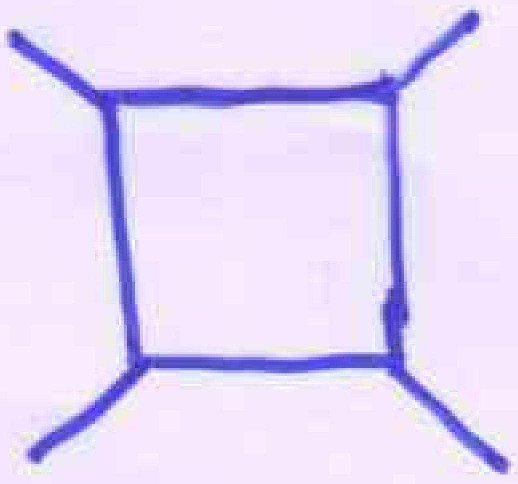
Vol. $V = R_{10} R_{11}$

complex structure $\tau = \tau_1 + i\tau_2$

Recall One Loop - four gravitons

(MBG, Gutperle, Vanhove)

(Russo + Tseytlin)



$$\Longrightarrow R^4 \Big(cV + \frac{E_{3/2}(\tau,\bar\tau)}{V^{1/2}} + A_{\text{threshold}} + O(s^2) \Big)$$

$(-s)^{1/2}$

UV divergence, Λ^3

- Requiring IIA = IIB determines $c = \frac{2\pi}{3}$

** true up to (at least) 2 loops

- No divergence in $V \to 0$ limit (IIB limit).

Type IIB $(V \to 0)$

$$\int d^{10}x \sqrt{g}\, \underbrace{e^{-\phi^B/2} E_{3/2}(\tau,\bar{\tau})}_{} R^4$$ **No** **$O(s^2)$ terms**

$$2\zeta(3)\tau_2^2 + \zeta(2)(\cdots) + O(e^{-2\pi\tau_2})$$

- Tree + 1-loop + D-instantons

Type IIA $(R_{10} \to \infty)$

$$\int d^{10}x \sqrt{g} \Big\{ R^4 e^{-\phi^A/2} \left(2\zeta(3)\tau_2^{3/2} + \zeta(2)(\cdots)\tau_2^{-1/2}\right.$$
$$\left. + \sum s^n R^4 e^{2\phi^A(n-1)} (\cdots) \right\}$$

Not in IIB ?

power series in $g_A^2 = R_{11}^3$

e.g. Two loops in IIA string theory

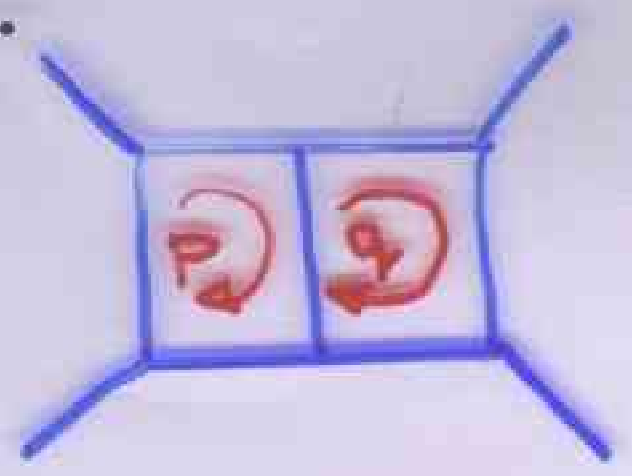
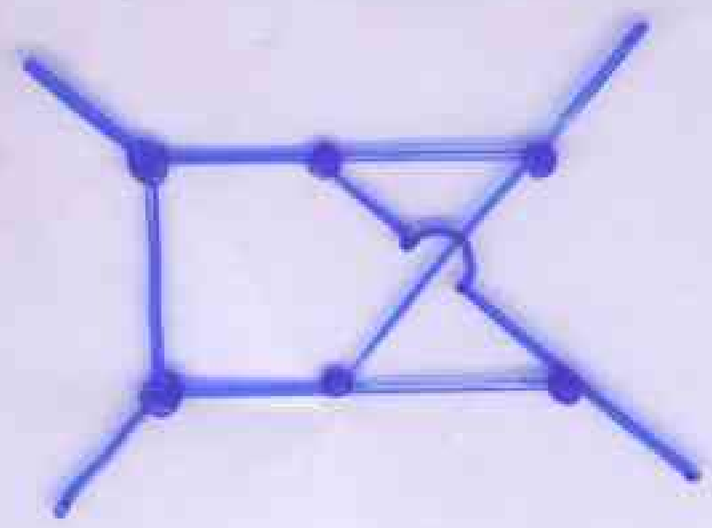
$$\cancel{e^{2\phi^A}}\, \zeta(4)\, s^2 e^{2\phi^A} R^4$$

Where is IIA term??

Two loops in eleven dimensions

(MBG, Kwon, Vanhove) $(\text{on } T^2)$

Evaluate:

$s^2 R^4$ ( + )

$+ s \to t + s \to u$

$$\sim (s^2+t^2+u^2) R^4 \int_0^\infty d\rho\, d\tau\, d\lambda\, \Delta^{1/2} \sum_{\{m^I, k^I\}} e^{-E_W}$$

$$\Delta = \rho\lambda + \tau\lambda + \rho\tau$$

$$E_W = G_{IJ}\left(\lambda m^I m^J + \tau k^I k^J + \rho (m+k)^I (m+k)^J\right)$$

metric on T^2 winding nos. $m^I = (m^1, m^2)$ $k^I = (k^1, k^2)$

Reparameterize Schwinger parameters:

$\lambda, \rho, \tau \to u, \ \Omega = \Omega_1 + i\Omega_2$ two-torus $\hat{T}^2$

Vol., Complex structure

$$E_W = \frac{u V}{\Omega_2 \tau_2} \left| (1, \tau) A (\Omega, 1) \right|^2 - uV \det A$$

$$A = \begin{pmatrix} m^1 & k^1 \\ m^2 & k^2 \end{pmatrix}$$

$$\Rightarrow (s^2+t^2+u^2) \sum_{\{m^I, k^I\}} \int du\, u^3 \frac{d^2\Omega}{\Omega_2^2} e^{-E_W}$$

$k_r^r = 0$

Map $\underset{(u,\Omega)}{\hat{T}_2}$ into $\underset{(V,\tau)}{T_2}$

Map $\hat{T}_2$ into T_2:

- Degenerate orbits - sub divergences
  $(\det A = 0)$

- Non-degenerate orbits - finite

IIB limit $(V \to 0)$

cut off $\qquad f_2^{(0,0)}$

$$\Rightarrow \int d^{10}x \sqrt{g}\, s^2 R^4 \Lambda^3 \underbrace{e^{\phi^B/2} E_{5/2}(\tau, \bar{\tau})}_{2\zeta(5) e^{-2\phi^B} + (\cdots)\zeta(4) e^{2\phi^B} + O(e^{-2\pi\tau_2})}$$

Two (string) loop behaviour matches with IIA term (fixes Λ) from one loop in 11dim.

IIA limit $(R_{10} \to \infty)$

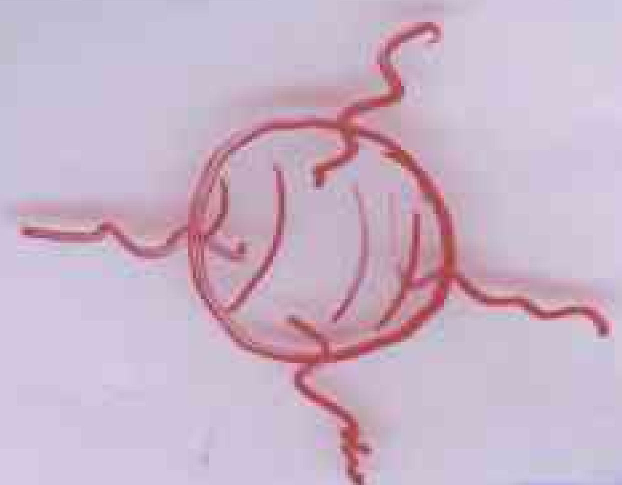
$$\int d^{10}x \sqrt{g}\, s^2 R^4 \Lambda^3 e^{-2\phi^A}\, 2\zeta(5)$$

- No string one-loop contbn. to $s^2 R^4$

- Explicit expression for string two-loop contribution.

Comparison with string pert. theory

Tree Level

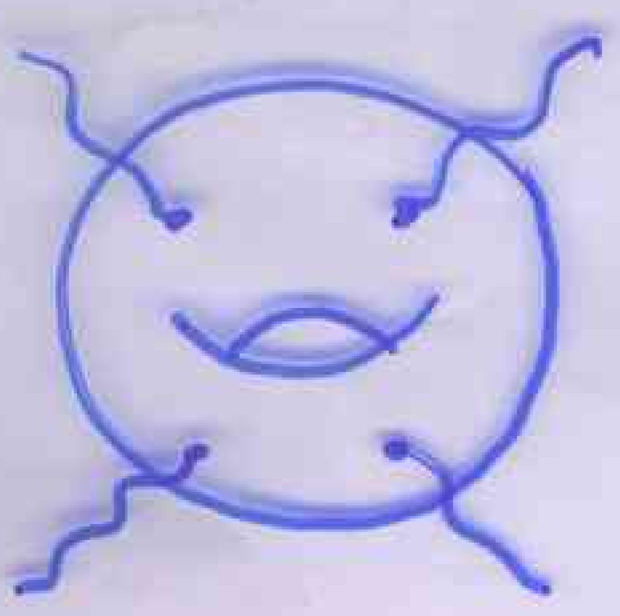


$$A^{tree} = e^{-2\phi} R^4 \frac{1}{(\alpha')^4 stu} \frac{\Gamma(1-\alpha' s)\,\Gamma(1-\alpha' t)\,\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\,\Gamma(1+\alpha' t)\,\Gamma(1+\alpha' u)}$$

$$= \frac{e^{-2\phi} R^4}{(\alpha')^4 stu} \exp\left\{ \sum_{\ell=1}^{\infty} \frac{2\zeta(2\ell+1)}{2\ell+1} (\alpha')^{2\ell+1} \left(s^{2\ell+1} + t^{2\ell+1} + u^{2\ell+1}\right) \right\}$$

$$= \frac{e^{-2\phi} R^4}{(\alpha')^4} \left\{ \frac{1}{stu} + (\alpha')^3 2\zeta(3) + (\alpha')^5 \zeta(5)(s^2+t^2+u^2) \right.$$
$$\left. + (\alpha')^6 \zeta^2(3)(s^3+t^3+u^3) + (\alpha')^7 \zeta(7)(s^5+t^5+u^5) + \ldots \right\}$$

One String Loop



$$A^{loop} \equiv R^4\, I$$

$$I = \int \frac{d^2\tau}{\tau_2^2} \int \prod_{1}^{4} \frac{d^2\nu^{(i)}}{\tau_2}\, e^{D}$$

$$D \equiv 4\alpha' s \ln \frac{\chi_{12}\chi_{34}}{\chi_{13}\chi_{24}} + 4\alpha' t \ln \frac{\chi_{14}\chi_{23}}{\chi_{13}\chi_{24}}$$

Scalar propagator: $P(\nu^{ij}|\tau) = \ln \chi_{ij}(\nu_{ij}|\tau)$

$\nu^{ij} = \nu^i - \nu^j$

- Need to evaluate integrals of powers of P over the torus

- $$A^{loop} = \frac{\pi}{3} + \alpha' A_{threshold} + (\alpha')^2 c\, s^2 + (\alpha')^3 d\, s^3 + \cdots$$

  $\sim s \ln s + t \ln t + u \ln u$

- Find $c = 0$ ✓

  $d = \frac{4}{3\pi} \zeta(2)\zeta(3)$

All the known coefficients in the low energy expansion of the IIB four graviton amplitude agrees with the $SL(2Z)$ invariant

"SILLY" formula: (Russo / MBG + Moore)

$$A = \frac{e^{-\phi} R^4}{(\alpha')^4 stu} \exp\left\{ \sum_{\ell=1}^{\infty} \frac{E_{\ell+\frac{1}{2}}}{2\ell+1} e^{(\ell+\frac{1}{2})\phi} \left(s^{2\ell+1} + t^{2\ell+1} + u^{2\ell+1}\right) \right\}$$

$\left(2\zeta(2\ell+1)\, \tau_2^{\ell+\frac{1}{2}} + \ldots\right)$ tree term

$$= \frac{1}{stu} \prod_{p,q} (stu) A^{tree}(T_{p,q})$$

D-string $\quad \frac{1}{2\pi\alpha'} |p+q\tau|$

Product of D-string amplitudes!

$$\sim \frac{e^{-\phi} R^4}{(\alpha')^4} \left\{ \frac{1}{stu} + E_{3/2} + \frac{1}{2} E_{5/2} (s^2+t^2+u^2) + \frac{2}{3} \left(E_{3/2}\right)^2 (s^3+t^3+u^3) + \ldots \right\}$$

BAD
- No normal thresholds!
- ∞ no. of potentially unstable non-BPS states.