ON GAUGE THEORY/SIGMA MODEL CORRESPONDENCE

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Strings'99

based on the joint work
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A tiny bit of philosophy :

Gauge theory studies the gauge fields A in the G-bundles $\mathcal{P}$ over the space-time manifold $\mathcal{X}$.

Sigma models deals with the maps $\varphi : \mathcal{X} \to V$ of the space-time into the target space V.

There is a similarity between two subjects, noted by topologists a long time ago.

Example : if over the target space V one has a G-bundle P with a fixed connection A_0, then to every map $\varphi : \mathcal{X} \to V$ one associates the induced connection

$$A = \varphi^* A_0$$

on the induced G-bundle $\varphi^* \mathcal{P}$. Moreover, by taking topologically non-trivial maps φ ("instantons") one induces topologically non-trivial gauge configurations.

In our today's discussion we will study more specific examples of the interplay between the sigma models and the gauge theories.

- Gauge theory on X with gauge group G $\approx$

  sigma model on LX with target G (Polyakov's book)

- Two dimensional Yang-Mills theory on X G

  $\approx$ sigma model on X with target BG

  (large N limit of Witten's "Verlinde algebra and Cohomology of Grassmanian")

LNS "Issues..." 97

Our original motivation is the desire to understand the exact (instanton corrected) mapping between the **observables** in the *ultraviolet* non-abelian $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory and the

observables of its *low-energy effective abelian theory.*

σ is the complex adjoint scalar in the $SU(N)$ vector multiplet.
Find the polynomials P_R, labelled by the irreps R of $SU(N)$ such that:

$$\text{UV} \longrightarrow \quad \mathrm{Tr}_R \sigma \Longrightarrow P_R(u_1, \ldots, u_{N-1}; \Lambda) \longleftarrow \text{IR}$$

with $u_1, \ldots, u_{N-1}$ – coordinates on the moduli space of vacua:

$$z + \frac{\Lambda^{2N}}{z} = x^N - u_1 x^{N-2} - \ldots u_{N-1}$$

$$u_k = \mathrm{Tr}_{\Lambda^{k+1}\mathbf{C}^N} \sigma$$

and Λ counts instanton corrections.

We found it useful to translate the problem into the language of more familiar two dimensional sigma models.

On the way we found several interesting results which by themselves are worth talking about. !

Plan

1. $\mathcal{N} = 2$ supersymmetric 2d (4d) sigma models (gauge theories).

2. Donaldson theory as gauged linear sigma model.

2. $\mathbf{CP}^{N-1}$ model vs. gauged linear sigma model.

3. Instantons in gauged linear sigma model, 4d gauge theory and non-linear sigma model. $2 \to 4, 6, 8$

4. $\mathcal{N} = 2$ sigma models with disconnected target space arising as effective field theories. Solitons.

5. Conclusions.

1. $\mathcal{N}=2$ supersymmetric 2d sigma models and $\mathcal{N}=2$ 4d gauge theories.

$\mathcal{N}=2$ susy gauged linear sigma model :

Witten? 93

Matter chiral multiplets Φ^i take values in the complex vector space W, with Kähler form ω.

$$\Phi^i = (\phi^i, \psi^i_\pm, \bar{\psi}^i_\pm)$$

Vector multiplets V take values in the Lie algebra of the group G which acts in W preserving Kähler structure.

$$V = (A_\mu, \sigma, \bar{\sigma}, \lambda_\pm, \bar{\lambda}_\pm)$$

The superfield Σ containing the field strength is the twisted chiral superfield with the quantized component F (for abelian G)

The bosonic part of the Lagrangian is (in the absence of superpotential):

$$\int ||D_\mu \phi^i||^2 + \frac{1}{2e^2} F^2_{\mu\nu} + \frac{e^2}{2} ||\mu||^2 + ||[\sigma, \bar{\sigma}]||^2$$

∘ $\mu \sim T^a_{i\bar{j}} \phi^i \bar{\phi}^{\bar{j}}$ – the moment map for the G action in W.

G contains $U(1)$ factors $\Rightarrow$ deform the model by adding a constant per each $U(1)$ to μ, (Fayet-Illiopoulos terms).

$$\mu = T^a_{i\bar{j}} \phi^i \bar{\phi}^{\bar{j}} - r_i \mathbf{1}_i$$

For each $U(1)$ factor $\Rightarrow$ θ term: $\theta_i \int F_i$

- Altogether: a complex parameter $t_i = ir_i + \frac{\theta_i}{2\pi}$ per $U(1)$.
- Correlation functions of chiral observables are **holomorphic** in t_i – important constraint.
- e^2 is the gauge coupling (for several $U(1)$ factors one may have different couplings e_i^2).
- In the infrared $e_i^2 \to \infty$ and the gauged linear model flows to the non-linear sigma model with the target

$$W//G = \mu^{-1}(0)/G$$

2. Donaldson theory ~
$\mathcal{N} = 2$ susy Yang – Mills theory

Bershadsky Johansen Sadov Vafa '95

can also be formulated in terms of the gauged linear (affine) sigma model!

Baulieu Losev Nekrassov '97

Hofman Park '99

Take **W** = the space $\mathcal{A}_\Sigma$ of the gauge fields on a two dimensional Riemann surface Σ

I will skip the details concerning 't Hooft fluxes

Take as **G** the infinite dimensional group of two dimensional gauge transformations.

Then the gauged linear sigma model with matter in **W** and the gauge group **G** is precisely the (partially twisted) $\mathcal{N} = 2$ susy gauge theory formulated on the space-time

$$\Sigma \times \mathbf{R}^{1,1}$$

One may hope to use the knowledge of the 2d sigma models for deducing properties of the 4d gauge theories

Let us see how far can we get

<< Lexicon Cosri >>

Attempt of the Dictionary:

- Both (2d and 4d) models have non-local observables of special kind. There are local observables $\mathcal{O}_R = \mathrm{Tr}_R \sigma$ associated to every irreducible representation R of the gauge group G.
- There are their descendants $\mathcal{O}_R^{(i)} = \{G, \{G, \ldots, \mathcal{O}_R\}\}$ which are i-forms on the space-time.

  2d (4d) theory can be deformed by adding the two- (four-) observable to the action.
- The space of deformations preserving (twisted) $\mathcal{N} = 2$ susy has a special structure (special coordinates) which enter the story of mirror symmetry.
- These coordinates are known for a large class of 2d sigma models and are **not** known in 4d case.

  But! 4d case = 2d gauged linear sigma model. The corresponding low-energy effective target space $W//G$ is nothing but the moduli space $\mathcal{M}_\Sigma$ of flat G-connections on Σ. (Considerations involving w_2 can even make it smooth). This is manifold with $c_1 > 0$ for which it was shown (Kontsevich-Manin) that WDVV equations allow to fix the special coordinates uniquely.

Q: *Do we have a solution?*
A: *NO!*
Q: *But what about KM and WDVV?*
A: *One has to be careful comparing linear sigma model and its effective description: instantons are different. Let us look at the familiar example:* $\mathbb{CP}^{N-1}$ *model.*

3. $\mathbb{CP}^{N-1}$ model vs. linear sigma model

This is the model with $W = \mathbf{C}^N$, $G = U(1)$.

BPS field configurations:

$$D_{\bar{A}}\phi^i = 0, F_A = -e^2\left(r - \sum_i |\phi^i|^2\right)$$

$$d\sigma = 0, \qquad \sigma\phi^i = 0$$

In the instanton sector with

$$-\frac{1}{2\pi}\int F = d$$

the action of the BPS configuration is $2\pi i t d$ – it is the minimum in the given topological class.

The moduli space $\mathcal{M}_d$ of BPS configurations is the space for the solutions to these equations modulo gauge transformations. For definiteness let us assume that the worldsheet is the sphere $\mathbf{S}^2$

Let $\beta = e^2 r \mathrm{Area}_{\mathbf{S}^2}$.

$\beta > 2\pi d$: $\mathcal{M}_d$ is compact and $\approx \mathbf{C}\mathbb{P}^{Nd+N-1}$.

$$\langle \sigma(x_1) \ldots \sigma(x_{Nd+N-1}) \rangle = e^{-2\pi i t d}$$

$\beta < 2\pi d$: the moduli space $\mathcal{M}_d$ is empty.

$$\langle \sigma(x_1) \ldots \sigma(x_{Nd+N-1}) \rangle = 0$$

Q: *We get a contradiction with the holomorphicity of the correlators of chiral observables: the dependence on β clashes with the holomorphicity in t ?!*

A: *This phenomenon is well-known in the context of the 4d gauge theory – Donaldson jumps which occur on manifolds with $b_2^+ = 1$ – an example of the failure of the argument which states that the Q-exact terms in the action decouple. Here: 2d analogue of the Donaldson jumps. Correlation functions are piece-wise holomorphic functions. But they jump at certain walls in the Kähler moduli space.*

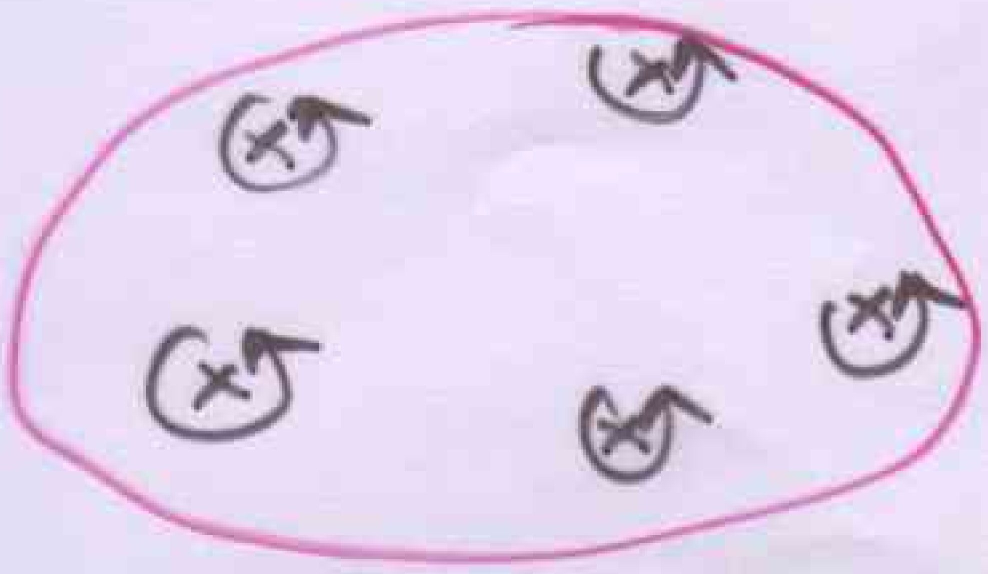
- *not seen in the deep infrared ($\beta \to \infty$)*

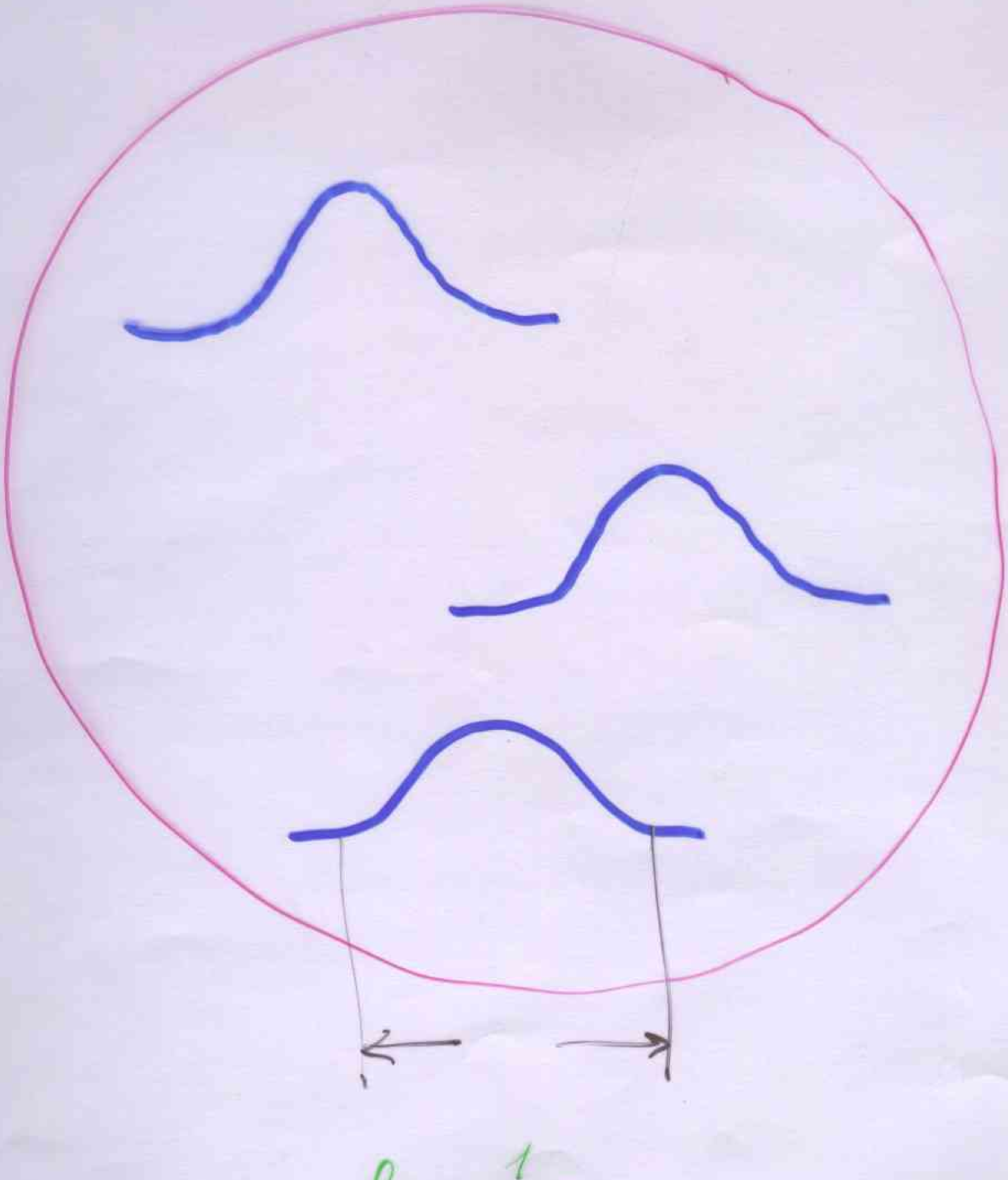
Q: *What is the relation of $\mathcal{M}_d$ to the moduli space M_d of the instantons in the $\mathbb{CP}^{N-1}$ model?*
A: *In the limit $e^2 \to \infty$ for any fixed d the space $\mathcal{M}_d$ can be decomposed as follows:*

$$\mathcal{M}_d = M_d \amalg M_{d-1} \times \mathbb{P}^1 \amalg \ldots \amalg M_{d-k} \times \mathbb{P}^k \amalg \ldots M_0 \times \mathbb{P}^d$$

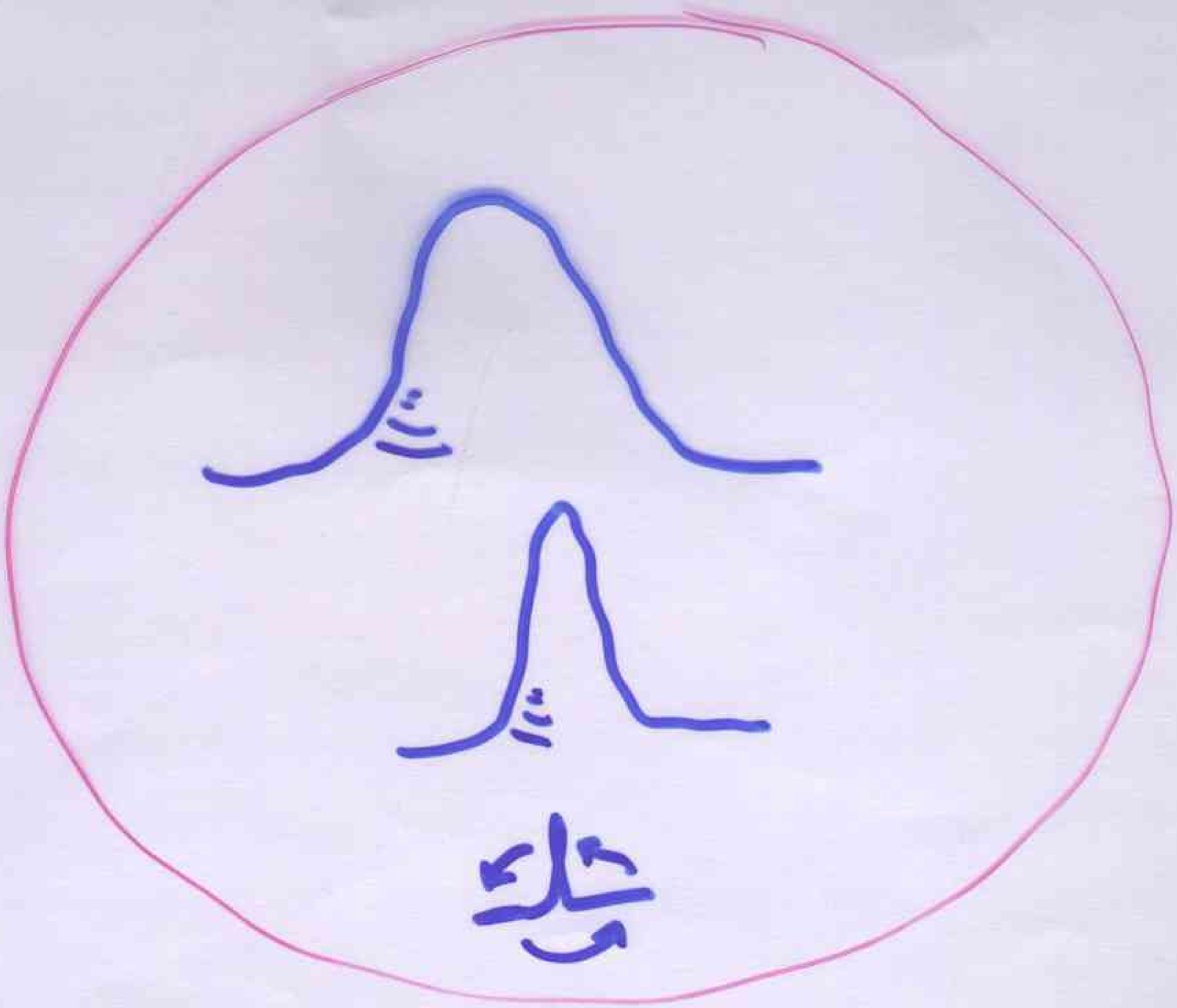
where M_d is the moduli space of the degree d holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^{N-1}$ and $\mathbb{P}^k$ is the **space of charge k Nielsen-Olesen-like vortices** which show up as the point-like instantons of some kind.
If $\sum_i |\phi^i|^2 \neq 0$ everywhere $\Rightarrow$ BPS configuration corresponds to the actual holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^{N-1}$.
Otherwise the zeroes of $\sum_i |\phi^i|^2$'s are the centers of the vortices.

• The unexpectedly non-trivial $\mathbb{CP}^0$ model is the analogue of the $U(1)$ "instantons"
on non-commutative spaces in 4d (NS). In both cases the moduli space is the symmetric product of the space-time manifold (with the blowup in 4d).





$$\ell \sim \frac{1}{e\sqrt{z}}$$



Abrikosov-
Nielsen -
Olesen -

vortex

Q: *What is the relation of* $\mathcal{M}_d$ *to the Kontsevich compactification of the space* M_d *?*

A: It is different (coarser) compactification, not suitable for WDVV formalism

Q: *Then what is good about it?*

Answer 1: *It is the compactification which is suitable for the 4d story*

Indeed, let us have a look at the **BPS field configurations** in the gauge theory.

Using the complex structure on $\Sigma \times \mathbb{CP}^1$ we write BPS eqs. as:

- $F_{\bar{i}\bar{j}} = 0$, $\bar{i}, \bar{j} = 1, 2$ – the analogue of $D_{\bar{A}}\phi^i = 0$ of the $\mathbb{CP}^{N-1}$ model.
- $F_A = -\beta F_\Sigma$, $\beta = \frac{\mathrm{Area}_{\mathbf{S}^2}}{\mathrm{Area}_\Sigma}$ $\leftrightarrow$ $F_A = -e^2\mu$
- The analogue of the condition $\sum_i |\phi^i|^2 \neq 0$ is

$$A_\Sigma(x) \neq A_{\mathrm{YM}}, \quad \text{for} \quad \text{any} \quad x \in \mathbf{S}^2$$

where A_{YM} is the **non-flat** Yang-Mills gauge field on Σ, i.e. the solution to the YM equation:

$$D_A^* F_A = 0$$

with $F_A \neq 0$. In the holomorphic language: the first BPS equation tells that the instanton defines a holomorphic bundle $\mathcal{E}$ on $\Sigma \times \mathbb{P}^1$, the second is the semistability condition.

If the restriction of $\mathcal{E}$ on every fiber Σ is semi-stable as the bundle over a curve Σ then $\mathcal{E}$ defines a holomorphic map

$$\varphi_{\mathcal{E}} : \mathbb{P}^1 \to \mathcal{M}_{\Sigma}$$

But, just as in the $\mathbf{C}\mathbb{P}^{N-1}$ case we have the analogues of the Nielsen-Olesen vortices, which occur whenever one gets the unstable bundle over a fiber, or, in the physical terms, whenever the fiber gauge field is the Yang-Mills connection.

- Even after inclusion of the Nielsen-Olesen vortices the moduli space of the instanton gauge fields is not compact, due to 4d point-like instantons.

- Analogously one can formulate 6 and 8 dimensional theories.

raises questions about equivalence of

D5-D1

D5-D1 on X vs. sigma model on $\mathcal{M}_X$

Answer 2: *It is sufficient for studies of the correlators of zero-observables - quantum cohomology ring*

In the case of $\mathbf{CP}^{N-1}$ this is shown above by computing $\langle \sigma \ldots \sigma \rangle$
if one works in appropriate chamber $r >> 0$.

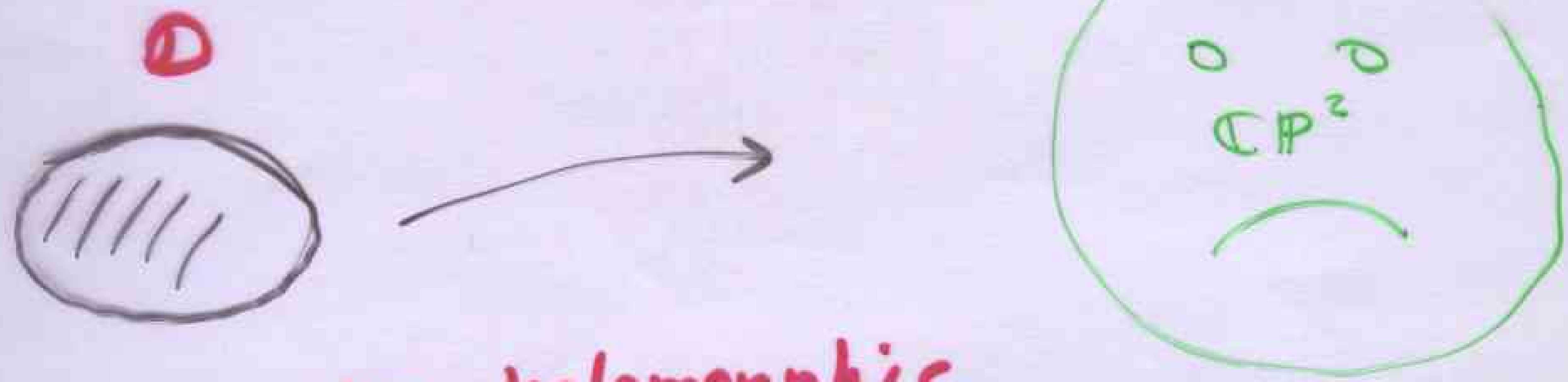
For $\mathcal{M}_\Sigma$ the story is more difficult.

- chambers. One has to take the limit $\beta \to \infty \Leftrightarrow$ $\mathrm{Area}_\Sigma << \mathrm{Area}_{\mathbf{S}^2}$.
- explicit knowledge of the space of BPS configurations is lacking. Got to use Seiberg-Witten solution, taking care of the contact terms and other issues.

Answer 2.5

Comment on world-sheet instantons for open-strings

Douglas' lecture



no holomorphic non-constant maps

$$\phi_1(z) = a_1 z$$
$$\phi_2(z) = a_2 z$$
$$\phi_3(z) = a_3 z$$

$\longrightarrow$ BPS in linear σ-model

$z \neq 0$ $\quad (\phi_1 : \phi_2 : \phi_3) \in \mathbb{P}^2$

$z = 0$ VORTEX

The modular invariant part of the cohomology ring of $\mathcal{M}_\Sigma$ for the group $G = SO(3), w_2 \neq 0$ is generated by three generators a, b, c of degree $2, 4, 6$:

$$a = \int_\Sigma \mathcal{O}^{(2)}_{\mathrm{Tr}\sigma^2}, \qquad b = \mathcal{O}^{(0)}_{\mathrm{Tr}\sigma^2}$$

$$c = \sum_i \int_{A_i} \mathcal{O}^{(1)}_{\mathrm{Tr}\sigma^2} \int_{B^i} \mathcal{O}^{(1)}_{\mathrm{Tr}\sigma^2}$$

Quantum cohomology ring (= relations on a, b, c) – conjectured by BJSV in 1995.

MW + LNS + MM + LNS $\Rightarrow$ explicit formula:

$$\langle e^{\epsilon_1 a + \epsilon_2 b + \epsilon_3 c} \rangle \propto$$

$$\oint \frac{du dz}{(u^2-1)^g z^{g+1}} e^{2\epsilon_2 u + \left(\epsilon_1 u + \epsilon_3 (u^2-1)\right) z} \frac{\sigma_3(\epsilon_1 + z)}{\sigma(\epsilon_1)\sigma_3(z)}$$

where σ_3 is the Weierstraß elliptic function $\sigma_3(z) = 1 + \frac{u}{24} z^2 + \ldots$, associated to the SW curve:

$$y^2 = 4x^3 - \frac{x}{4}\left(\frac{u^2}{3} - \frac{1}{4}\right) - \frac{1}{48}\left(\frac{2u^3}{9} - \frac{u}{4}\right)$$

• Qualitatively the ring defined by this formula looks like the quantum ring of BJSV, but the precise mapping between the generators is different.

Answer 3.: *it has effective description as the theory on the σ-plane with superpotential*

Back to $\mathbb{CP}^{N-1}$ model. Integrate out the matter fields. We are left with the vector multiplet V, but only its gauge invariant part Σ enters the Lagrangian. It is known (BT) that one can replace the gauge field and the gauge fixing ghosts by the field strength without changing the functional measure, with only one subtlety:

- The $F_{\mu\nu}$ component of the twisted chiral superfield Σ is quantized: $\int F \in 2\pi\mathbb{Z}$.

This subtlety leads to non-trivial consequences. The matter induces the effective twisted superpotential:

$$\widetilde{W}(\Sigma) = it\Sigma - \frac{N}{2\pi}\Sigma\log\Sigma$$

which only makes sense (due to the logarithm) when Σ is the superfield with quantized F. Now let us perform a series of the manipulations: fix a point $\star \in \mathbf{S}^2$ and view the field $\log\sigma$ as the univalent function by restricting:

$$0 \leq \arg\sigma(\star) < 2\pi$$

and defining $\log\sigma(x) = \log\sigma(\star) + \int_\star^x d\sigma/\sigma$.

Rewrite the functional integral over Σ as follows:

$$\int_{\substack{0 \leq \arg\sigma(\star) < 2\pi \\ F \quad \text{quantized}}} e^{-S_{eff} = \frac{i}{2\pi}\int NF\arg\sigma + \ldots} =$$

$$\sum_{m\in\mathbb{Z}} \int_{\substack{0 \leq \arg\sigma(\star) < 2\pi \\ F \quad \text{unconstrained}}} e^{im\int F + \frac{i}{2\pi}\int NF\arg\sigma + \ldots} =$$

now write $m = Nl + k$, $k = 0, \ldots, N-1$ and shift $\arg\sigma(\star)$:

$$= \sum_{k=0}^{N-1} \sum_{l\in\mathbb{Z}} \int_{\substack{2\pi l \leq \arg\sigma(\star) < 2\pi(l+1) \\ F \quad \text{unconstrained}}} e^{ik\int F + \frac{i}{2\pi}\int NF\arg\sigma}$$

$$\sum_{k=0}^{N-1} \int_{\arg\sigma(\star), F \quad \text{free}} e^{-S_{eff,k}}$$

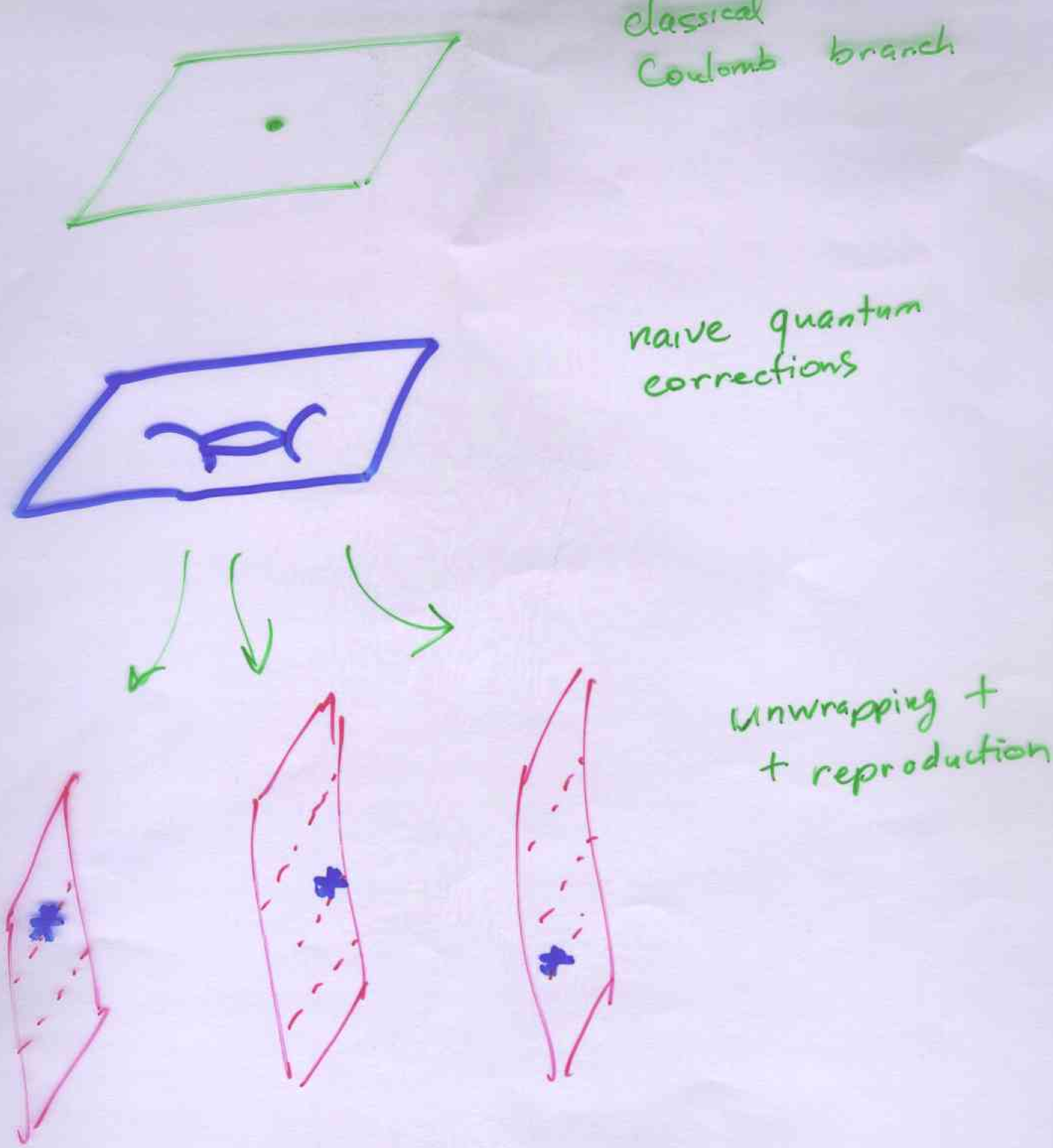
where $S_{eff,k}$ is susy Lagrangian derived from the twisted superpotential

$$\widetilde{W}_k = i(k+t)\sigma - \frac{N}{2\pi}\sigma\log\sigma$$

which is now defined on the universal cover of $\mathbf{C}^*$ where logarithm is well-defined. So, we observe the **mutation** of the Coulomb branch:

$$\mathbf{C}^* \to \mathbf{C} \amalg \ldots \amalg \mathbf{C} \qquad N \quad \text{copies}$$

with $\tilde{W}$ having an isolated vacuum on each component. The story is completely general:



classical Coulomb branch
naive quantum corrections
unwrapping +
+ reproduction

4. Sigma models with disconnected target spaces

Two dimensional sigma models with the twisted chiral multiplets with quantized F's can be "dualized" in a way similar to the above into the sigma model with the unconstrained fields. The target space of the dual model is typically disconnected. We won't need the general description but rather concentrate on examples:

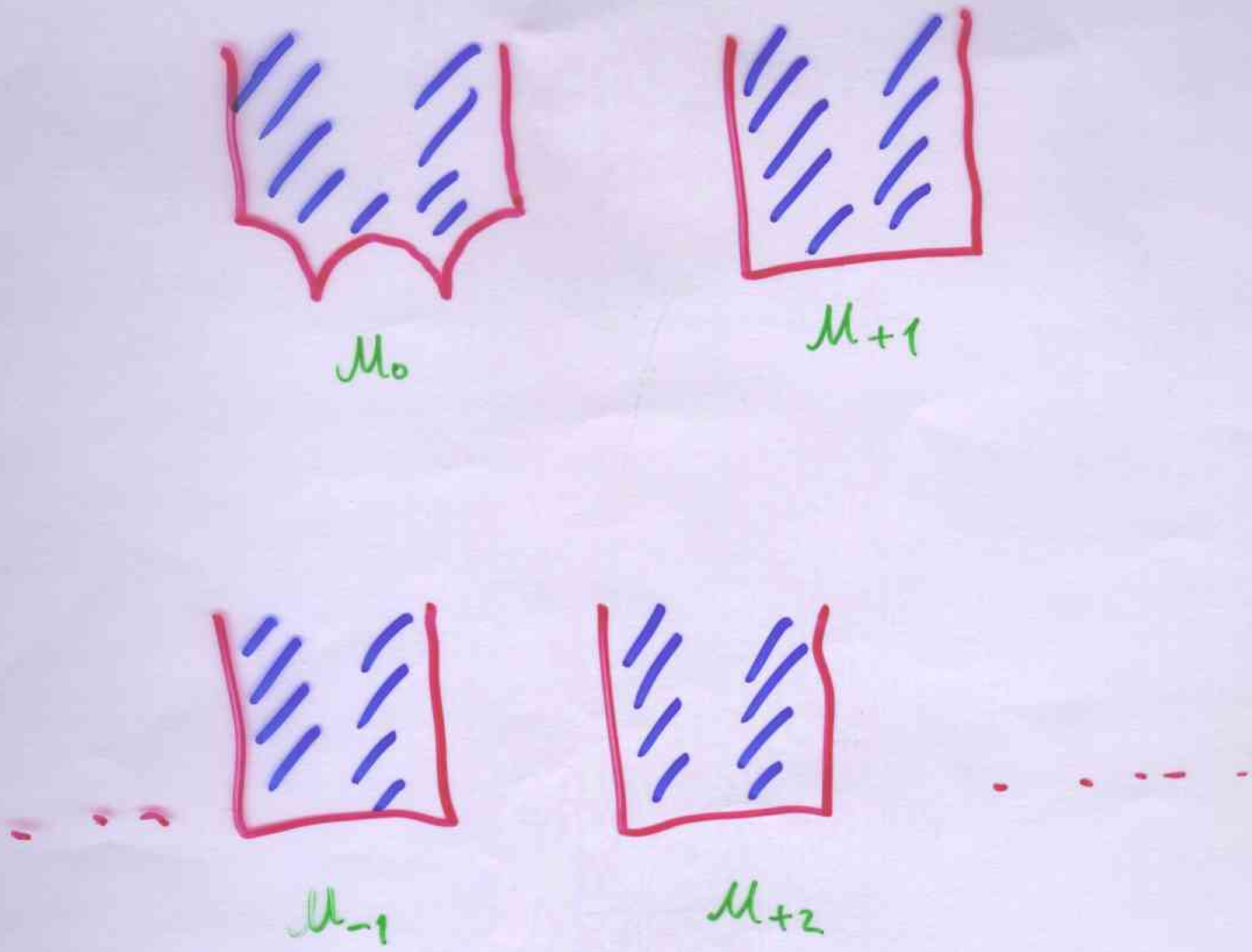
$\mathcal{N} = 2$ super-Yang-Mills theory
on $\Sigma \times \mathbf{R}^{1,1}$ for $\Sigma \approx \mathbb{P}^1$.

If one makes a (partial) twist so as to preserve $\mathcal{N} = 2$ susy in 2d one arrives at the theory with twisted chiral multiplets $\leftarrow$ 4d abelian vector multiplets. The target space of effective sigma model is the space of pairs:

(C_u, γ), where C_u is one of the curves:

$$z + \frac{\Lambda^{2N}}{z} = x^N - u_1 x^{N-2} - \ldots u_{N-1}$$

while $\gamma \in H_1(C_u, \mathbb{Z})$. The target space contains the component $\mathcal{M}_0$ where $\gamma = 0$ which is 4d moduli space of vacua. It also has components with $\gamma \neq 0$ which are (partial) covers of $\mathcal{M}_0$.



$\mathcal{M}_0$
$\mathcal{M}_{+1}$
$\mathcal{M}_{-1}$
$\mathcal{M}_{+2}$

The model also has a twisted superpotential $\tilde{W}$:

$$d\widetilde{W} = \oint_\gamma dx \wedge \frac{dz}{z}$$

which vanishes on $\mathcal{M}_0$. The model can be deformed by adding a "mass" term to the superpotential $\tilde{W} \to \tilde{W} + \epsilon_1 u$. Then one gets a critical point on every component.

For $G = SU(2)$ one gets one copy of the u-plane of SW and an infinite number of strips $-2 < \mathrm{Re}\tau < 2, \quad \mathrm{Im}\tau > 0$ which show up in the "unfolding" method of computing the u-plane integrals of Rankin-Selberg-Borcherds-Moore-Witten et al. $\tilde{W}_N = N \oint_0 x \frac{dz}{z}$ is the superpotential on the N'th component

As an example of the computation of the correlation function using this presentation of the theory consider Donaldson theory on $\mathbb{P}^1 \times \mathbb{P}^1$ (the notations a, b as before):

$$\langle e^{\epsilon_1 a + \epsilon_2 b} \rangle = \sum_N \oint \frac{(du)^2 e^{2\epsilon_2 u}}{N da + \epsilon_1 du} =$$

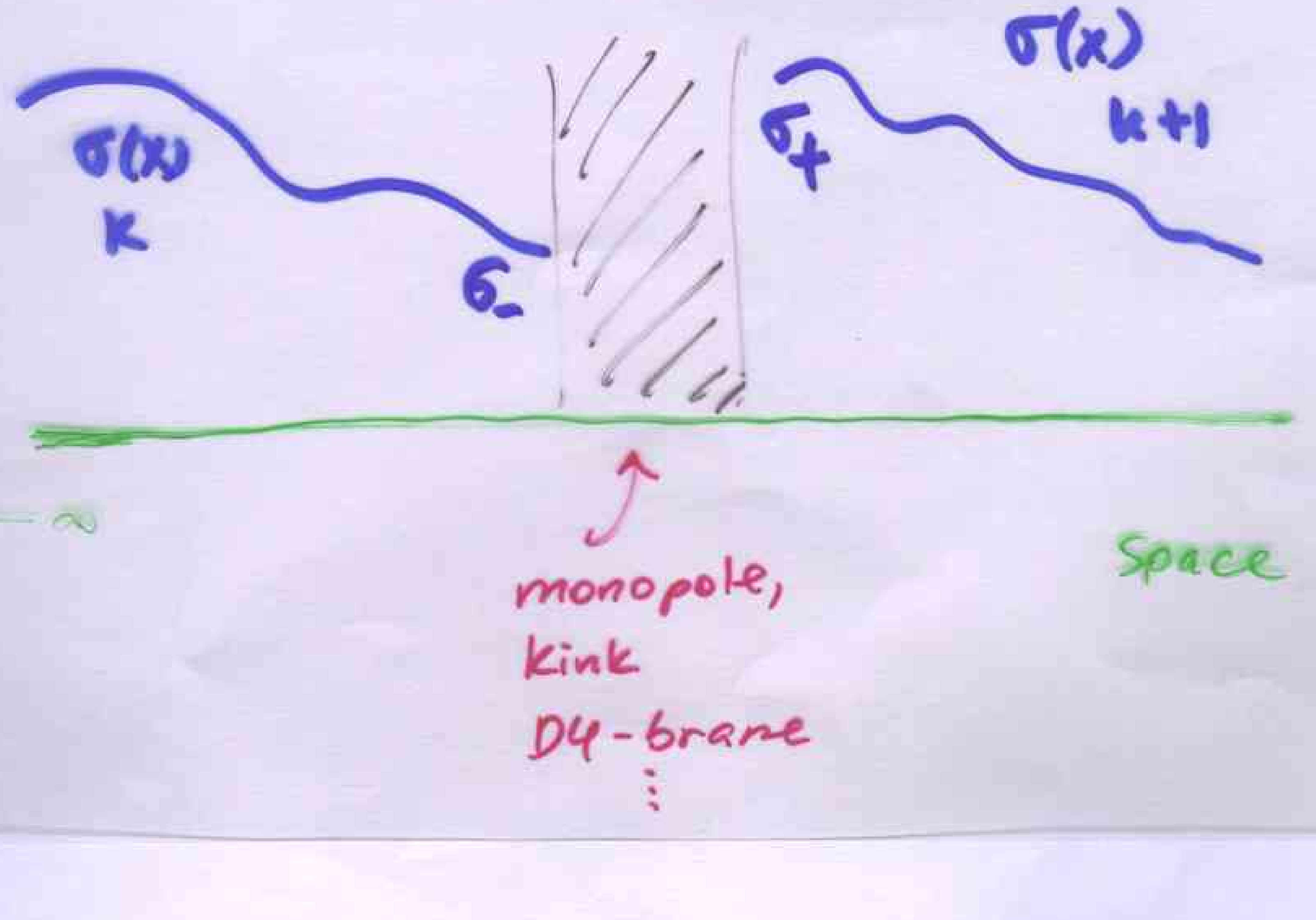
$$\oint \frac{(du)^2 e^{2\epsilon_2 u}}{da \tanh\left(\epsilon_1 \frac{du}{da}\right)}$$

where as usual $a = \oint_0 x \frac{dz}{z}$.

Q : *If the target space of the effective sigma model for* $\mathbb{CP}^{N-1}$ *is disconnected while the vacua sit on different components, then what about the solitons?*

A: *This is an extremely interesting question. If the sigma model on the σ-plane was the fundamental theory then we would conclude that the solitons are absent. But, the σ-theory is an effective one, it can break down at short distances, in particular, the trajectories in the σ-space may jump. Where they jump and how this happens will be analyzed elsewhere. At the moment it seems that they can break at the "attractor" points. The useful examples of the models where this breaking occurs are:*

- Type IIA string compactified on CY_4 with the G -flux. Then the solitons are represented by the D4-branes wrapping 4-cycles
- $\mathcal{N} = 2$ 4d $SU(N)$ on $\mathbb{P}^1$ with the twist – the solitons are the monopoles and dyons.



5. Conclusions

1. There is a deep and fruitful interplay between the 2d sigma models and 4d (and higher?) gauge theories, especially in the context of susy.
2. We found precise relationship between the 4d and 2d instantons.
3. By comparing the quantum rings we see the non-trivial renormalization which relates
   4d observables to the 2d observables
4. We found a non-trivial topological transmutation experienced by the Coulomb branch of the linear sigma model and the universal description of the superpotential.

Q: *Is that all?*

A: *Of course not. For example, we left aside several interesting points*:

1. applications to the compactifications of string theory,
2. further discussion of solitons in the sigma models, related questions concerning domain walls in susy QCD
3. higher dimensional gauge/tensor theories,
   theories on D-branes and the corresponding sigma

models

4. four dimensional mirror symmetry