Eisenstein Series and String Thresholds

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Strings 99, Potsdam, July 20, 1999

based on
NO and B. Pioline
hep-th/9903113
hep-th/9812139
hep-th/9809039 (to appear in Phys Rept)
NO, B. Pioline and **E. Rabinovici**,
hep-th/9712084 (NPB)

transparencies available from
http://www.nordita.dk/~obers/transp.html

Introduction

◂ early string days: world-sheet modular invariance has played major role in context of perturbative string theory

▹ partition function invariant under $Sl(2,\mathbb{Z})$ transformations

$$Z(\tau) = Z(\tau+1) = Z(-1/\tau)$$

- $\tau =$ modular parameter of world-sheet torus

◂ advent of target space and non-perturbative dualities: $\Rightarrow$ yet another branch of mathematics of automorphic forms invariant under infinite discrete groups $G(\mathbb{Z})$

▹ physical amplitudes

$$A(\{\Phi\}) \qquad , \qquad \Phi \in H\backslash G(\mathbb{R})$$

- scalar manifold is symmetric space (for theories with many SUSY's)
- duality symmetries identify points related by duality group $G(\mathbb{Z})$

$$G = \begin{cases} Sl(d) & \text{mapping class} \quad \text{diffeomorphism invariant}/T^d \\ SO(d,d) & \text{T-duality} \quad \text{pert. type II string}/T^d \\ E_{d+1(d+1)} & \text{U-duality} \quad \text{non-pert. type II string}/T^d \end{cases}$$

◂ supersymmetry constraints induce 2nd order differential equations on certain BPS-saturated amplitudes $\Rightarrow$ harmonic analysis on scalar moduli spaces can provide us with exact amplitudes, manifestly invariant under duality group $G(\mathbb{Z})$

Example and motivations

▷ prototypical example: type IIB R^4 coupling in 10D

$\tau = a + \frac{i}{g_s}$

$$f_{R^4} = \mathcal{E}^{Sl(2,\mathbb{Z})}_{2;s=3/2} = \sum_{(m,n)\neq(0,0)} \left[\frac{\tau_2}{|m+n\tau|^2}\right]^s$$

$$= \text{tree} + \text{1-loop} + \text{D-instantons}$$

$(\tau_2 \to \infty)$

$$\Delta\,\mathcal{E}_s = s(s-1)\mathcal{E}_s\ , \qquad \Delta_{U(1)\backslash Sl(2)} = \tau_2^2(\partial^2_{\tau_1} + \partial^2_{\tau_2})$$

Green, Gutperle/Berkovits/Pioline/Green, Sethi

cf. talk Green

◂ motivation to study exact amplitudes:

- hints for rules of semi-classical calculus in string theory
  Bachas/Green, Gutperle
  ...

- applications to Matrix models
  Moore, Nekrasov, Shatashvili/Kostov, Vanhove
  Gava, Hammou, Morales, Narain
  ...

- applications in AdS/CFT correspondence
  Banks, Green
  ...

◂ Instead of using symmetries: other way to obtain exact results is via string-string duality

▷ For example for vacua with 16 SUSY

Harvey, Moore/Antoniadis, Pioline, Taylor
Bachas, Fabre, Kiritsis, NO, Vanhove/Kiritsis, NO
Gregori, Kiritsis, Kounnas, NO, Petropoulos, Pioline
Lerche, Stieberger/Lerche, Stieberger, Warner/Foerger, Stieberger

- less general, since need to be able to control the result on one side of the duality map.

Symmetries and exact amplitudes

SUSY	theory	symmetry
16	Het/T^6	$SO(6,22,\mathbb{Z}) \times Sl(2,\mathbb{Z})$
16	IIB/K_3	$SO(5,21,\mathbb{Z})$
32	IIB	$Sl(2,\mathbb{Z})$
⇒ 32	M/T^{d+1} ($\sim$ II/T^d)	$E_{d+1(d+1)}(\mathbb{Z})$

◂ amplitudes are given by automorphic forms of duality group: selection criteria
• leading behavior (first perturbative terms)
• SUSY constraints (partial differential equations)

▹ generalize 10D R^4 coupling to arbitrary toroidal compactification of type II string theory using U-duality
Kiritsis, Pioline/NO, Pioline

• perturbative amplitudes of II/T^d: T-duality invariant
• non-perturbative amplitudes of II/T^d: U-duality inv.

◂ need T and U-duality invariant building blocks:
→ mass formulae of T and U-duality multiplets of 1/2 BPS states

▹ use these invariant functions to construct Eisenstein series that describe (non)-perturbative string thresholds

• Eisenstein series for various Lie groups (math)
• similar techniques in context of gauge theories and little string theories
Ganor

Plan

1. T- and U-duality invariant 1/2-BPS mass formulae in toroidally compactified Type II/M-theory

2. Eisenstein series and 1/2 BPS states

3. Exact R^4 couplings in type II string theory

4. Higher genus terms and $R^4 H^{4g-4}$ couplings

5. Conclusions and outlook

T-duality invariant 1/2-BPS mass formulae

▷ Perturbative type II on T^d

- NSNS scalars: dilaton ϕ , g_{ij} , B_{ij} on torus

  moduli space: $\mathbb{R}^+ \times \frac{SO(d,d,\mathbb{R})}{SO(d) \times SO(d)}$

  T-duality symmetry: $SO(d,d,\mathbb{Z}) \supset Sl(d,\mathbb{Z})$

- 1/2-BPS states form representations of T-duality group

$$
\begin{array}{cccccc}
 & & o_0 \ \overline{\boldsymbol{spinor}} & & & \\
 & & | & & & \\
\boldsymbol{spinor}\ o_1 & - & o_2 & - \quad o_3 & - \cdots - & o_{d-1}\ \boldsymbol{vector}
\end{array}
$$

$\boldsymbol{vector}$: KK,winding (m_i, n^i)

$\boldsymbol{spinor}$: D-strings $=$ D2/T^1, D4/T^3,.. $m^1, m^3, m^5, ..$

$\overline{\boldsymbol{spinor}}$: D-particles $=$ D0, D2/T^2, D4/T^4,.. $m, m^2, m^4, \ldots$

▷ T-duality invariant mass formulae

$$\mathcal{M}^2(\boldsymbol{vector}) = (m_i + B_{ij} m^j) g^{ik} (m_k + B_{kl} m^l) + m^i g_{ij} m^j$$

1/2 BPS condition: $k \equiv \|m\|^2 = 2 m_i m^i = 0$

- similar mass formulae + BPS condition for spinor

  Dijkgraaf, Verlinde, Verlinde/Pioline, Kiritsis/NO, Pioline

U-duality of toroidally compactified M-theory

◂ M-theory on T^{d+1} $\sim$ non-perturbative IIA on T^d

Witten/Townsend

• scalars: g_{IJ}, C_3 (and duals)

▹ take values in coset space: $H_{d+1} \backslash E_{d+1(d+1)}(\mathbb{R})$

D	$d+1$	$E_{d+1(d+1)}(\mathbb{R})$	H_{d+1}
10	1	$\mathbb{R}^+$	1
9	2	$Sl(2,\mathbb{R}) \times \mathbb{R}^+$	$U(1)$
8	3	$Sl(3,\mathbb{R}) \times Sl(2,\mathbb{R})$	$SO(3) \times U(1)$
7	4	$Sl(5,\mathbb{R})$	$SO(5)$
6	5	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$
5	6	$E_{6(6)}$	$USp(8)$
4	7	$E_{7(7)}$	$SU(8)$
3	8	$E_{8(8)}$	$SO(16)$

◂ U-duality symmetry of M-theory on T^{d+1} *Hull, Townsend*

$$E_{d+1(d+1)}(\mathbb{Z}) = Sl(d+1, \mathbb{Z}) \bowtie SO(d,d,\mathbb{Z})$$

• discrete versions of continuous (Cremmer-Julia) hidden symmetry groups above

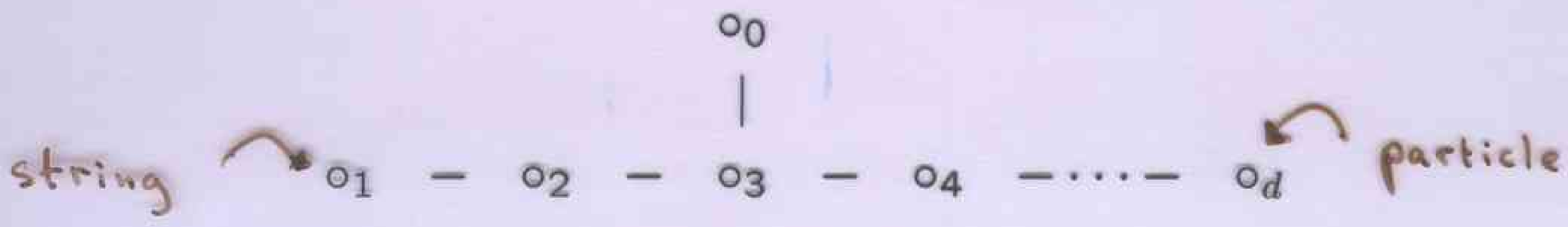
• includes perturbative T-duality

• includes also $R_{11} \leftrightarrow R_i$ (with $R_{11} = g_s l_s$)

$\Rightarrow$ non-perturbative (mix type II NSNS and RR scalars)

Representations of U-duality and BPS states

- 1/2-BPS states: reps. of U-duality group $E_{d+1(d+1)}$



$$\begin{array}{lll} \textit{particle}: & \text{KK,M2}/T^2\text{,M5}/T^5.. & (m_1, m^2, m^5, ..) \\ \textit{string}: & \text{M2}/T^1\text{,M5}/T^4.. & (n^1, n^4, ..) \end{array}$$

- ▷ U-duality invariant mass formulae

$$\mathcal{M}^2(\textit{particle}) = (\tilde{m}_1)^2 + l_p^{-6} \left(\tilde{m}^2\right)^2 + l_p^{-12} \left(\tilde{m}^5\right)^2 + \dots$$

$$\begin{aligned} \tilde{m}_1 &= m_1 + \mathcal{C}_3 m^2 + \mathcal{C}_3 \mathcal{C}_3 m^5 + \dots \\ \tilde{m}^2 &= m^2 + \mathcal{C}_3 m^5 + \dots \quad , \quad \tilde{m}^5 = m^5 + \dots \end{aligned}$$

NO, Pioline, Rabinovici

- 1/2-BPS condition on the particle multiplet = string multiplet constructed out of the particle charges

$$k^1 = m_1 m^2 \equiv 0 \quad , \quad k^4 = m_1 m^5 + m^2 m^2 \equiv 0$$

Ferrara, Maldacena/Hofman, Verlinde, Zwart/NO, Pioline

- ▷ branching under perturbative $SO(d,d)$ subgroup

$$\begin{array}{rllll} \textit{particle} = & \textit{vector} & \oplus & \textit{spinor} & \oplus \dots \\ & (KK, w) & & \text{D-particles} & \\ \textit{string} = & \textit{singlet} & \oplus & \textit{spinor} & \oplus \dots \\ & \text{F-string} & & \text{D-strings} & \end{array}$$

General structure for toroidal compactified type II and M-theory

▷ moduli space spanned by scalars is symmetric space of non-compact type

$$H\backslash G(\mathbb{R}) \quad , \quad H = \text{maximal compact subgroup}$$

- T-duality of II/T^d → perturbative: $G = SO(d,d)$
- U-duality of $\mathrm{M}/T^{d+1} \sim \mathrm{II}/T^d$ → non-perturbative: $G = E_{d+1(d+1)}$

◂ duality symmetry identifies different points in moduli space: discrete subgroup of $G(\mathbb{R})$

$$\text{duality symmetry :} \qquad G(\mathbb{Z})$$

▷ BPS states fall into representations $\mathcal{R}$ of duality group

◂ 1/2-BPS states have $G(\mathbb{Z})$-invariant masses

$$\mathcal{M}^2(\mathcal{R}) = m \cdot M_{\mathcal{R}}(g) \cdot m$$

- moduli matrix in representation $\mathcal{R}$

$$M_{\mathcal{R}}(g) = \mathcal{V}^t\mathcal{V} \quad , \quad \mathcal{V} = A \cdot N \quad \text{(Iwasawa)}$$

$$\text{1/2-BPS constraints :} \qquad k = m \wedge m = 0$$

▷ physical amplitudes should be invariant under $G(\mathbb{Z})$

→ for type II/T^d

- perturbative amplitudes invariant under $SO(d,d,\mathbb{Z})$
- non-perturbative amplitudes invariant under $E_{d+1(d+1)}(\mathbb{Z})$

Generalized Eisenstein series

▷ Using U-duality and T-duality invariant mass formulae → can construct automorphic forms for T and U-duality group

$$\mathcal{E}^{G(\mathbb{Z})}_{\mathcal{R};s}(g) = \sum_{m \in \Lambda_{\mathcal{R}} \setminus \{0\}} \delta(m \wedge m) \, \frac{1}{[\mathcal{M}^2(\mathcal{R})]^s}$$

NO, Pioline

• depend on $G(\mathbb{Z})$-invariant 1/2-BPS mass formulae

$$\mathcal{M}^2(\mathcal{R}) = m \cdot M_{\mathcal{R}}(g) \cdot m$$

• $g \equiv$ moduli (parametrizing symmetric space $H \backslash G(\mathbb{R})$)
• $M_{\mathcal{R}}(g)$ = moduli matrix in certain rep of $G(\mathbb{Z})$
• generalize standard non-holomorphic Eisenstein series on fundamental domain of Poincaré upper half plane
• $G(\mathbb{Z})$ stands for

	group	moduli
mapping class group	$Sl(d, \mathbb{Z})$	g_{ij}
T-duality group	$SO(d, d, \mathbb{Z})$	g_{ij}, B_{ij}
U-duality group	$E_{d+1(d+1)}$	g_{IJ}, C_{IJK}

◂ the δ insertion imposes the half-BPS condition
▷ required both on physical and mathematical grounds

$$\Delta_{H \backslash G} \, \mathcal{E}^{G(\mathbb{Z})}_{\mathcal{R}_\lambda;s} = s(\lambda, \rho - s\lambda) \, \mathcal{E}^{G(\mathbb{Z})}_{\mathcal{R}_\lambda;s}$$

• λ is highest weight, ρ is Weyl vector

▷ Basis of invariant functions (aside from cusp forms) is expected to correspond to nodes of Dynkin diagram

One-loop amplitude and Eisenstein series

◂ type II R^4 (+ other half-BPS couplings):
one-loop integral is modular integral of lattice partition function

$$I_d = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{d,d}(g,B;\tau)$$

▹ integral I_d is eigenmode of two differential operators

$$\Delta_{SO(d,d)} I_d = \frac{d(2-d)}{4} I_d \quad , \quad \Box_d I_d = \frac{d(2-d)}{8} I_d$$

Kiritsis, Kounnas/NO, Pioline

- order $s=1$ spinor and $s=d/2-1$ vector Eisenstein series have same eigenvalues
- large volume expansion

$$I_d = \frac{2\pi^2}{3} V_d + 2V_d\, \mathcal{E}^{Sl(d,\mathbb{Z})}_{d;s=1}(g_{ij}) + \dots$$

- explicit evaluation for low d

$$I_1 = \frac{2\pi^2}{3}\left(R + \frac{1}{R}\right)$$

$$I_2 = -2\pi \ln U_2 |\eta(U)|^4 - 2\pi \ln T_2 |\eta(T)|^4$$

◂ can show that

$$I_d = \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbf{vector};s=\frac{d}{2}-1} = \begin{cases} \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbf{spinor};s=1} + \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\overline{\mathbf{spinor}};s=1} & d=1,2 \\ \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbf{spinor};s=1} = \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\overline{\mathbf{spinor}};s=1} & d \geq 3 \end{cases}$$

$$\sum \frac{1}{m^2_{BPS,T}}$$

Exact R^4 couplings in low dimensions

◂ U-duality invariant R^4 coupling has to reproduce tree-level contribution and one-loop term:

$$f_d^{R^4} = 2\zeta(3) V_d + I_d + \text{non-pert.}$$

$$= \frac{V_d}{g_s^2}\, \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbf{singlet};s=3/2} + \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbf{spinor};s=1} + \text{non pert.}$$

▹ use branching rules of $E_{d+1,d+1}$ into $SO(d,d)$:

$$\mathcal{M}^2_{\text{string}} = \mathcal{M}^2_{\text{singlet}} + \frac{V_d}{g_s^2}\mathcal{M}^2_{\text{spinor}} + \frac{V_d^2}{g_s^4}\mathcal{M}^2_{\dots}$$

→ Eisenstein series in the string representation has the correct small coupling expansion

$$f_d^{R^4} = \frac{V_{d+1}}{l_p^9}\, \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\mathbf{string};s=3/2} \qquad \sim \sum \frac{1}{m^3_{BPS,4}}$$

• natural generalization: involves M-theory strings

◂ Beyond tree-level and one-loop terms, this exhibits non-perturbative $\mathcal{O}(e^{-1/g_s})$ effects from instantonic D-branes wrapped on T^d

• in $D \leq 6$ there are also contributions from the extra rep (singlet for $D = 6$), which scale as e^{-1/g_s^2}

▹ also evidence for identities

$$\frac{V_{d+1}}{l_p^9}\, \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\mathbf{string};s=3/2} = \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\mathbf{particle};s=d/2-1} = \frac{V_{d+1}}{l_p^9}\, \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\mathbf{membrane};s=1}$$

Higher genus integrals and higher derivative couplings

▹ can apply similar techniques to genus g integrals

$$I^g_d = \int_{\mathcal{M}} d\mu \; Z^g_{d,d}(g_{ij}, B_{ij}; \tau)$$

- eigenmode of $SO(d,d)$ Laplacian → conjecture

$$I^g_d \propto \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbf{spinor};s=g} + \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\bar{\mathbf{spinor}};s=g}$$

◂ N=4 topological string on T^2
→ higher derivative couplings $R^4 H^{4g-4}$ in IIB on T^2

Berkovits, Vafa/Ooguri, Vafa

▹ $R^4 H^{4g-4}$ couplings in IIB on T^d ($H = dB_{RR}$):

$$I = \int d^{10-d}x\sqrt{-\gamma}\,\hat{\sum_m}\delta(m\wedge m)\; e^{6(g-1)\phi}\; \frac{R^4 (m\cdot H_{RR})^{4g-4}}{(m\cdot M(\boldsymbol{spinor})\cdot m)^{3g-2}}$$

◂ use group theory arguments and large volume expansion to conjecture non-perturbative completion

$$I = \frac{V_{d+1}}{l_M^9}\int d^{10-d}x\sqrt{-\gamma}\,\hat{\sum_m}\delta(m\wedge m)\;\frac{R^4\;(m\cdot H)^{4g-4}}{(m\cdot M(\boldsymbol{string})\cdot m)^{3g-\frac{3}{2}}}$$

NO, Pioline

- appropriate scaling dimension
- correctly gives tree-level NS and g-loop RR result
- generalize the $Sl(2,\mathbb{Z})$ (covariant) modular functions $f^{p,q} = \hat{\sum}\tau_2^{(p+q)/2}/[(m+n\tau)^p(m+n\bar{\tau})^q]$

Kehagias, Partouche/Green, Gutperle, Kwon

Asymetric thresholds and elliptic genus

▷ one-loop BPS saturated couplings for toroidal compactification of heterotic string

$$I_{d,k} \equiv \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, Z_{d,d+k}(g,B,Y;\tau)\ \mathcal{A}_k(F,R,\tau)$$

• $\mathcal{A}_k =$ almost holomorphic modular form, weight $-k/2$
• BPS condition only constrains left movers to ground state → all right-moving oscillations contribute

$$\mathcal{A}(F,R,\tau) = \sum_{\nu=0}^{\nu_{\max}} \frac{1}{\tau_2^{\nu}} \mathcal{A}^{(\nu)}(F,R,q) \qquad , \qquad q = e^{2\pi i\tau}$$

◂ $\mathcal{A}^{(\nu)}$ have Laurent expansions, with at most simple pole in q (left-moving tachyon)

▷ use analogous differential equations

$$\left[\Delta_{SO(d,d+k)} + \frac{d(d+k-2)}{4} - \tau_2^2 \partial_{\bar\tau} D_\tau\right] Z_{d,d+k} = 0$$

$I_{d,k}$
• not eigenmode of the operators $\Delta_{SO(d,d+k)}$ due to presence in $\mathcal{A}$ of the tachyonic pole in $1/q$

▷ pole term in $\mathcal{A}$ generates harmonic anomaly localized at enhanced symmetry points in the moduli space
→ not captured by any candidate Eisenstein series

◂ the term integrated by parts involves the descendant of the elliptic genus
• automorphic forms of $SO(2,2+k)$

Borcherds/Harvey, Moore/Kiritsis, NO

Conclusions

◂ can construct Eisenstein series for general class of duality groups

$$\mathcal{E}_s \sim \sum \frac{1}{[\mathcal{M}^2_{\text{BPS}}]^s}$$

▹ building blocks are T and U-duality multiplets + invariant 1/2 BPS mass formulae

$$\begin{array}{lcll} 1/2 \text{ BPS} & : & \mathcal{M}^2 & = \mathcal{M}_0^2(m) \\ 1/4 \text{ BPS} & : & \mathcal{M}^2 & = \mathcal{M}_0^2(m) + \sqrt{\mathcal{T}^2(n)} \quad , \quad n \simeq m \wedge m \end{array}$$

◂ Eisenstein series describe various BPS saturated couplings in type II or M-theory on T^d

▹ use T-, U-duality symmetry and SUSY to determine exact couplings:

◂ obtained convenient representations of manifestly T-duality invariant 1-loop and g-loop amplitudes

$$I_d \sim \sum \frac{1}{\mathcal{M}^{2g}_{\text{BPS;T}}}$$

◂ exact non-perturbative $R^4 H^{4g-4}$ couplings in toroidally compactified M-theory in terms of order $s = 3g - 3/2$ Eisenstein series of string multiplet of M-theory

$$f_d \sim \sum \frac{1}{\mathcal{M}^{6g-3}_{\text{BPS;U}}} = \text{tree} + g\text{-loop} + \text{non-perturbative}$$

- e^{-1/g_s} instanton effects: Euclidean D-branes wrapped on various cycles
- e^{-1/g_s^2} instanton effects: NS5-brane wrapped on T^6

◂ Open directions:

▹ have focused on 1/2-BPS saturated couplings in maximal SUSY

• couplings preserving lesser amount of SUSY
E.g. 1/4 BPS states: expect cubic Casimir, generalized Eisenstein series (also: higher derivative (Green))

• 1/2-BPS states in theories with lesser SUSY
(Eisenstein series seem of little use when gauge symmetry enhancement at particular point in moduli space occurs)

▴ Het, IIB on K_3, FHSV model ...

▹ more careful analysis of instanton contributions: e^{-1/g_s^2}-effects from NS5-branes

▹ stable non-BPS states ?

◂ mathematics:

• wealth of explicit examples of modular functions on symmetry spaces of non-compact type

• many conjectured relations between Eisenstein series in various representations

• spectrum also contains discrete cusp forms: from string theory ?