Wilson Loops in Large-N Theories

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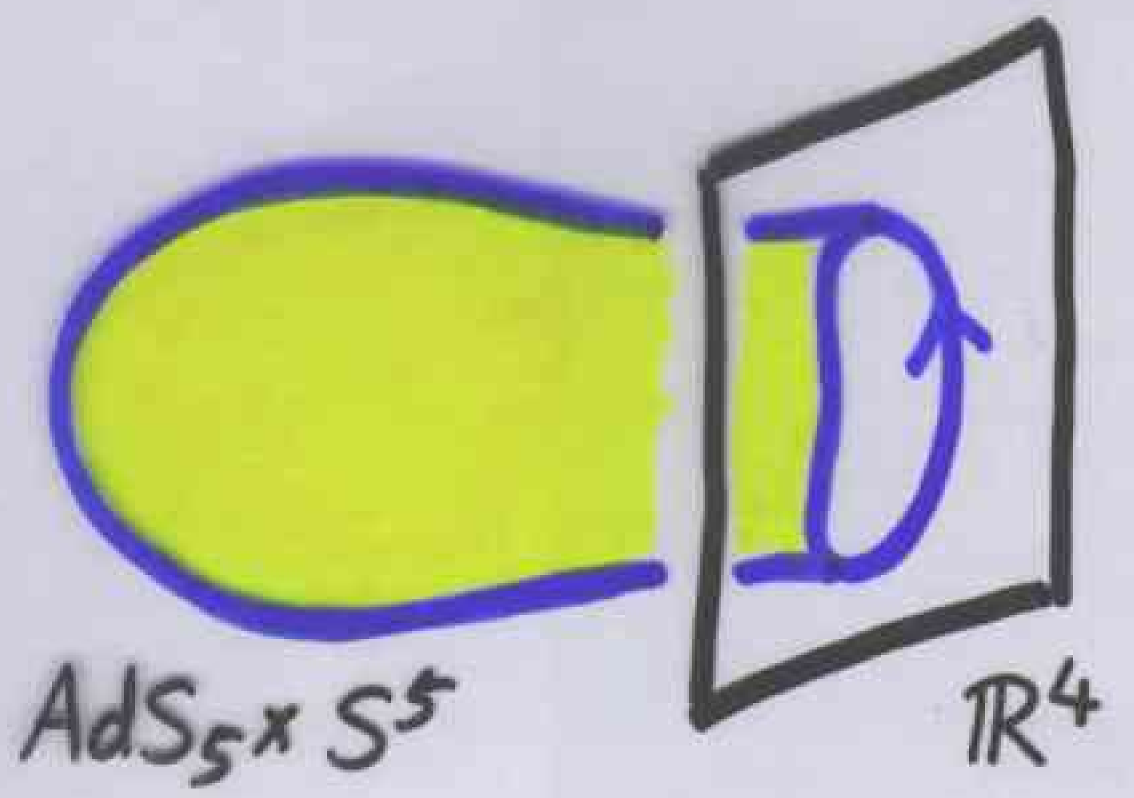
PLAN OF THIS TALK

1. WILSON LOOP IN $\mathcal{N}=4$ SYM_4

$$W = \mathrm{tr}\, P\exp\left[\oint ds\,(i A_\mu \dot{x}^\mu + \phi_i \dot{y}^i + \cdots)\right]$$

2. MINIMUM SURFACE IN $AdS_5 \times S^5$



BASED ON

DRUKKER, GROSS + H.O.

hep-th / 9904191

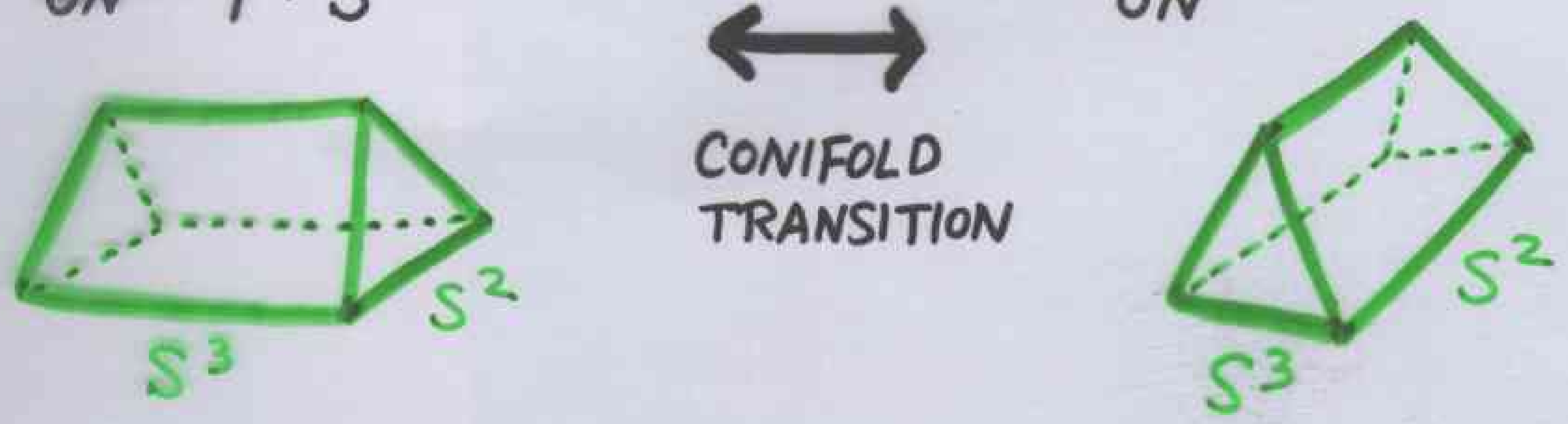
3. WILSON LOOPS IN THE CHERN-SIMONS THEORY

(~ JONES POLYNOMIALS)

$$S_{CS} = \frac{k}{4\pi}\int \mathrm{tr}\left(A\,dA + \frac{2}{3}A^3\right)$$

OPEN STRING ON T^*S^3

CLOSED STRING ON



VAFA + H.O. IN PROGRESS

1. Wilson Loops in $\mathcal{N}=4$ SYM$_4$

A_μ, ϕ^i ← 6 of $SU(4)$, λ^a_α ← 4 of $SU(4)$, ← 4d spinor

[Adjoint of $U(N)$]

To define the Wilson loop operator,

consider the W-boson for

$$\phi^i_{U(N+1)} \rightarrow \left(\begin{array}{c|c} \phi^i_{U(N)} & \\ \hline & u\,\theta^i \end{array}\right)$$

$\theta \in S^5$

$\sum_{i=1}^{6} \theta^{i\,2} = 1$

For $u \to \infty$,

the W-boson becomes infinitely massive.

Phase factor for a trajectory of the W-boson:

$$W = \mathrm{tr}\, P\exp\left[\oint ds \left(i A_\mu \dot{x}^\mu + i \phi_i \theta^i |\dot{x}|\right)\right]$$

in Minkowski space

$$= \mathrm{tr}\, P\exp\left[\oint ds \left(i A_\mu \dot{x}^\mu + \phi_i \theta^i |\dot{x}|\right)\right]$$

in Euclid space

($W \neq$ pure phase)

One may consider a more general type of loop:

$$W = \mathrm{tr}\, P\exp\left[\oint ds \left(i A_\mu \dot{x}^\mu + \phi_i \dot{y}^i\right)\right]$$

The W-boson trajectory gives $\dot{x}^2 = \dot{y}^2$.

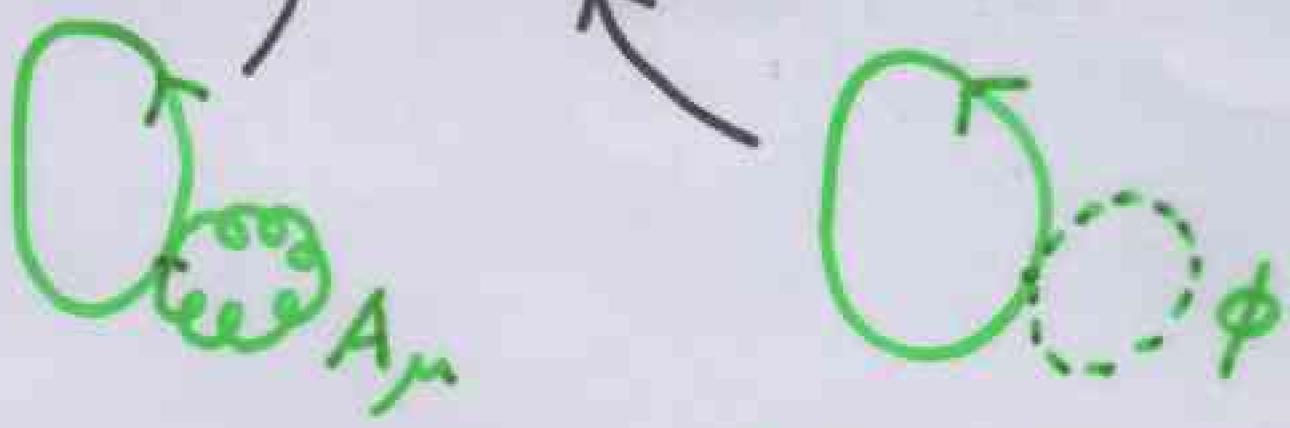
Ultraviolet Divergence

$$W = \mathrm{tr}\, P \exp \left[\oint ds \left(i A_\mu \dot{x}^\mu + \phi_i \dot{y}^i \right) \right]$$

When $g^2 N \ll 1$,

$$\langle W \rangle = 1 + \frac{g^2 N}{(2\pi)^2 \varepsilon} \oint ds\, |\dot{x}| \left(1 - \frac{\dot{y}^2}{\dot{x}^2} \right) + \cdots$$

ε : UV cutoff

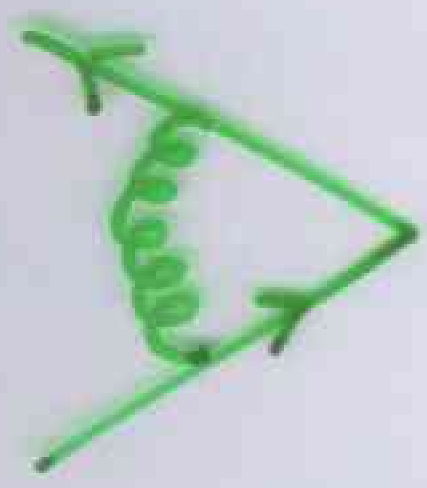


The linear divergence is canceled for $\dot{x}^2 = \dot{y}^2$.

(We will see that this is also the case for $g^2 N \gg 1$, according to AdS/CFT.)

For a loop with a cusp,

a logarithmic divergence remains.

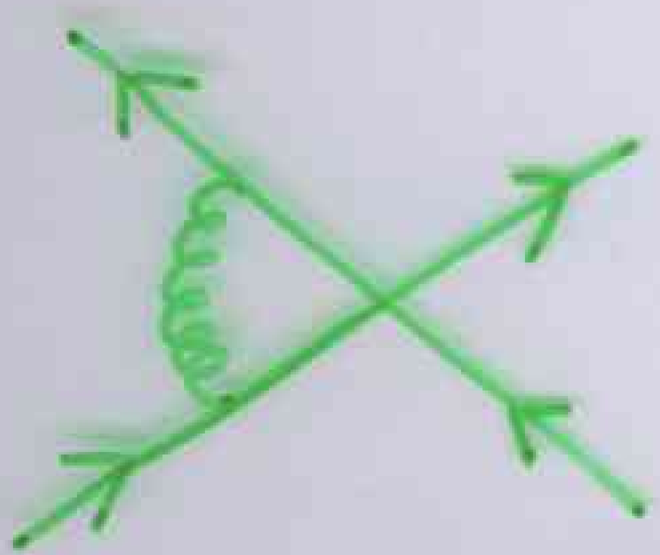


$$\frac{g^2 N}{(2\pi)^2} \cdot \frac{\pi - \Omega}{\sin \Omega} \cdot \left(\cos \Omega + \cos \Theta \right) \cdot \log \left(\frac{1}{\varepsilon} \right)$$

Ω: angle on $\mathbb{R}^4$

Θ: jump of $\theta^i \in S^5$

Also at an intersection,



$$\frac{g^2 N}{2\pi} \cdot \frac{1}{\sin \Omega} \cdot \left(\cos \Omega + \cos \Theta \right) \cdot \log \left(\frac{1}{\varepsilon} \right)$$

LARGE-N LOOP EQUATION

LOOP VARIABLES : $\left(x^{\mu}(s),\ y^{i}(s),\ \underbrace{\xi^{\alpha}{}_{a}(s)}_{\text{FERMIONIC}} \right)$ ← 4 OF SU(4)

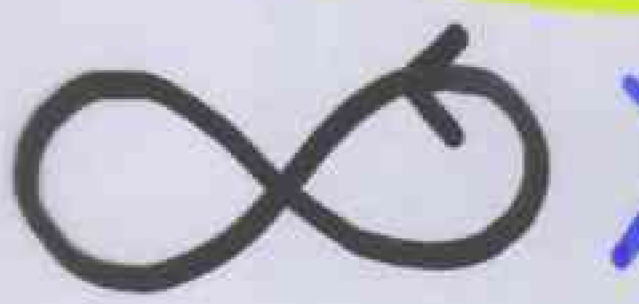
$$W = \mathrm{tr}\, P\exp\left[\oint ds \left(i A_{\mu} \dot{x}^{\mu} + \phi_i \dot{y}^{i} + \frac{i}{2} \xi \underbrace{\left(i \gamma_{\mu} \dot{x}^{\mu} + \Gamma_i \dot{y}^{i} \right)}_{\text{NILPOTENT WHEN } \dot{x}^2 = \dot{y}^2} \lambda + \cdots \right) \right]$$

LOOP DIFFERENTIAL OPERATOR

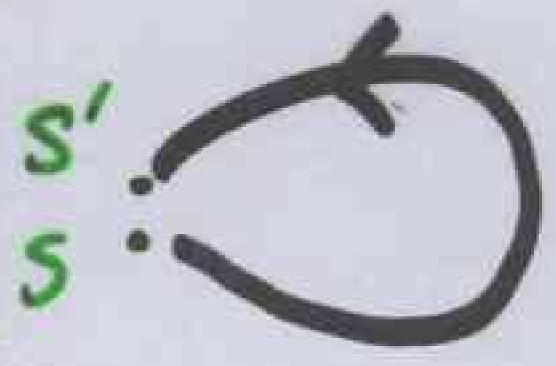
$$\mathcal{L} \equiv \lim_{\eta \ll \varepsilon} \oint ds \int_{|s'-s|<\eta} ds' \left[\frac{\delta^2}{\delta x(s)\,\delta x(s')} - \frac{\delta^2}{\delta y(s)\,\delta y(s')} + \frac{\delta^2}{\delta \xi(s)\,\delta \bar{\xi}(s')} \right]$$

↑ UV CUTOFF

$N \to \infty$, $g^2 N$: FIXED

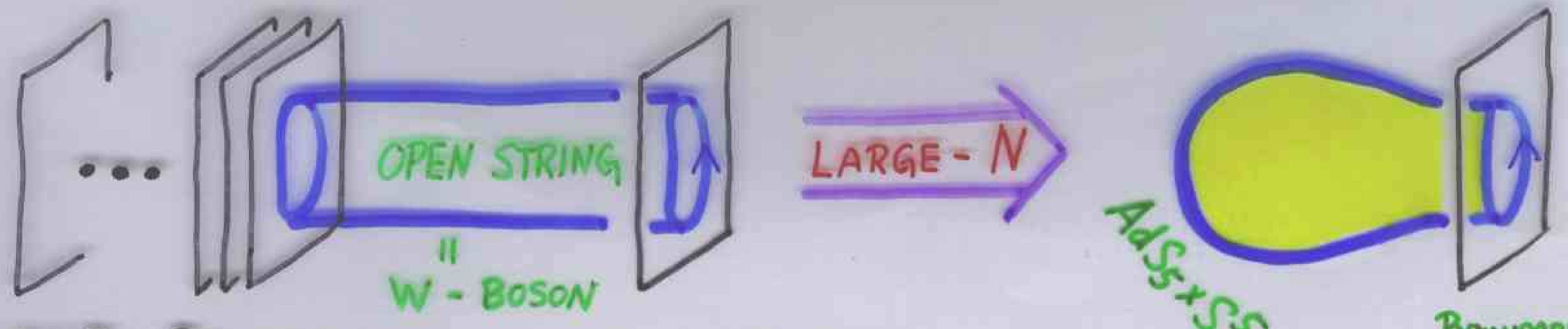
$\mathcal{L} \langle$  $\rangle \big|_{\xi = 0,\ \dot{x}^2 = \dot{y}^2}$

$$= g^2 N \oint ds \oint ds'\, \delta^{(4)}\left(x(s) - x(s') \right) \times \left(\dot{x}(s) \cdot \dot{x}(s') - \dot{y}(s) \cdot \dot{y}(s') \right)$$

$\times \langle$  $\rangle \langle$  $\rangle$

LOOP EQUATION, REGULARIZED AND UNRENORMALIZED.

2. Minimum Surface in $AdS_5 \times S^5$



$\Rightarrow U(N)\ SYM_4$

MALDACENA
REY + YEE

$$ds^2 = \sqrt{g^2 N}\,\frac{1}{y^2}\left(dy^2 + \sum_{\mu=1}^{4} dx^{\mu 2}\right) + \sqrt{g^2 N}\, d\theta^2$$

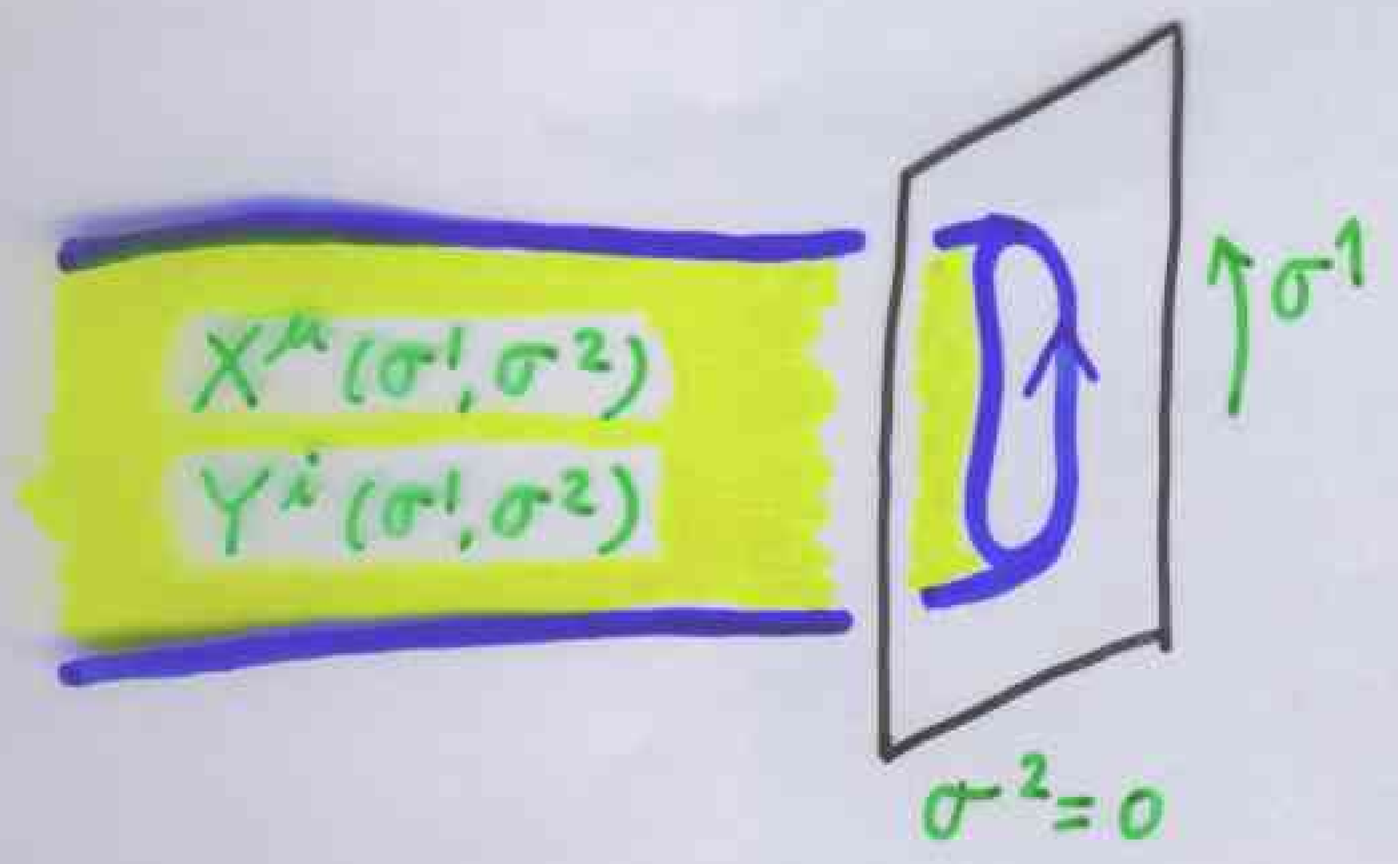
AdS_5 $\qquad$ S^5

$$\boxed{= \sqrt{g^2 N}\,\frac{1}{y^2}\left[\sum_{\mu=1}^{4} dx^{\mu 2} + \sum_{i=1}^{6} dy^{i\,2}\right]}$$

$$y^i = y\,\theta^i \quad \left(\theta \in S^5;\ \sum_i \theta^{i\,2} = 1\right)$$

☆ Coupling of the Gauge Field (A_μ, ϕ^i)

to the String Worldsheet $(X^\mu(\sigma), Y^i(\sigma))$.



$$\oint d\sigma^1 \left(A_\mu \frac{\partial X^\mu}{\partial \sigma^1} + \phi_i P^i\right)_{|\sigma^2=0}$$

$$\boxed{P^i = \frac{1}{\sqrt{g}}\, g_{1\alpha}\, \epsilon^{\alpha\beta} \frac{\partial Y^i}{\partial \sigma^\beta}}$$

Momentum conjugate to Y^i

$\dfrac{\partial Y^i}{\partial \sigma^1} \Leftrightarrow P^i$: T - Duality

THIS SUGGESTS THAT THE WILSON LOOP OPERATOR

$$W = \mathrm{tr}\, \mathcal{P} \exp\left[\oint ds \left(i A_\mu \dot{x}^\mu + \phi_i \dot{y}^i \right) \right]$$

COUPLES TO A STRING WORLDSHEET WITH THE MIXED BOUNDARY CONDITION.

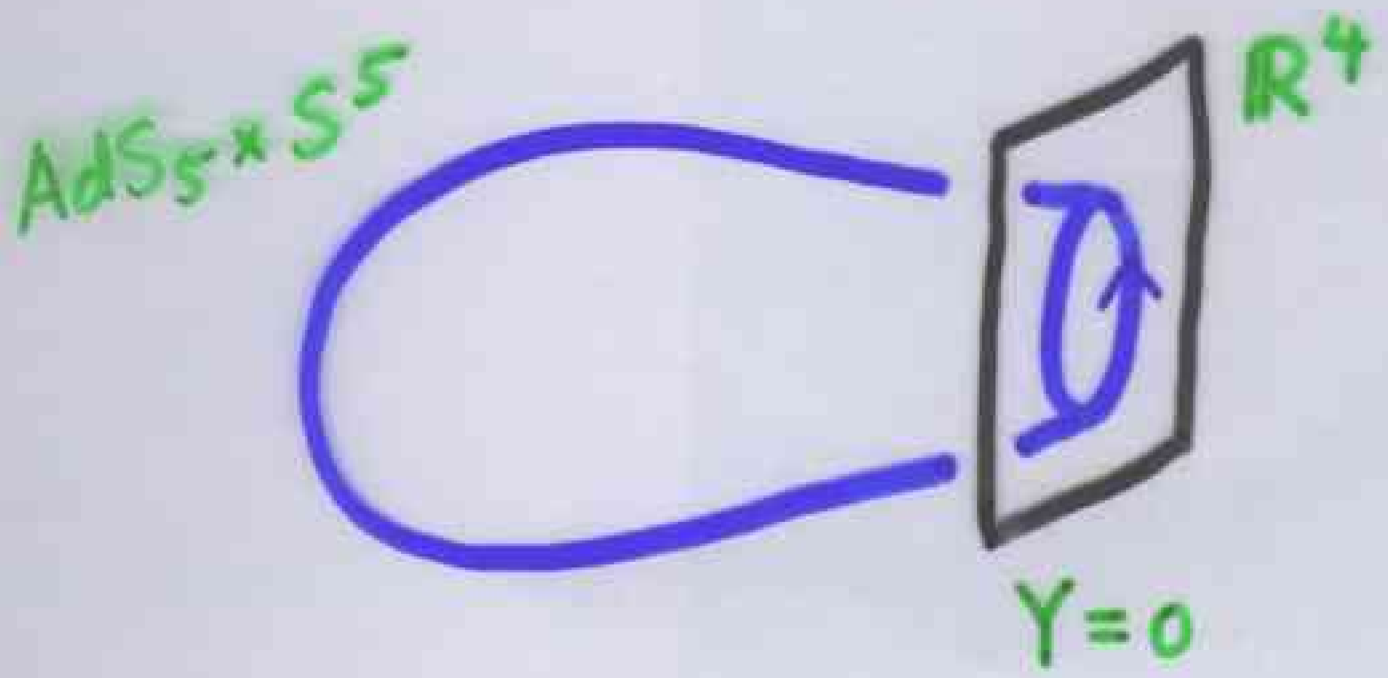
$$X^\mu(\sigma^1, \sigma^2 = 0) = x^\mu(\sigma^1) \quad \text{DIRICHLET}$$
$$P^i(\sigma^1, \sigma^2 = 0) = \dot{y}^i(\sigma^1) \quad \text{NEUMANN}$$

MOMENTUM CONJUGATE TO Y^i

$g^2 N \to \infty \rightarrow$ SEMI-CLASSICAL STRING WORLDSHEET $\rightarrow$ MINIMUM SURFACE

FOR A GIVEN $(x^\mu(s), y^i(s))$,

THERE IS A UNIQUE MINIMUM SURFACE IN $AdS_5 \times S^5$.



THE SURFACE CAN END AT $Y = 0$ ONLY IF $\dot{x}^2 = \dot{y}^2$.

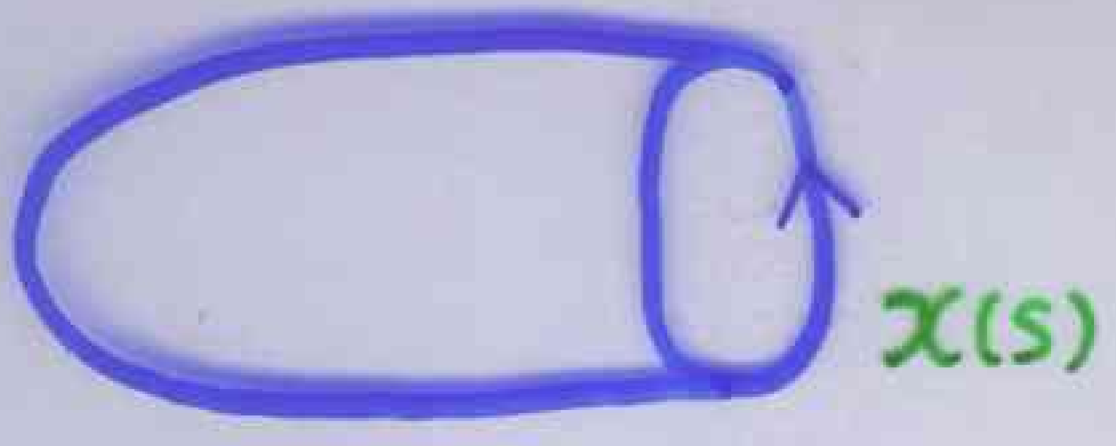
(∵) HAMILTON - JACOBI EQUATION

$$\dot{x}^2 - \dot{y}^2 = P^\mu P^\mu - \frac{\partial Y^i}{\partial \sigma^1} \frac{\partial Y^i}{\partial \sigma^1}$$

- IF $Y^i = 0$ AT $\sigma^2 = 0$, $\frac{\partial Y^i}{\partial \sigma^1} = 0$.
- FOR A SMOOTH LOOP, $P^\mu = 0$; OTHERWISE COSTS ∞ AREA.)

LEGENDRE TRANSFORMATION

THE AREA (NAMBU-GOTO ACTION) IS
A GOOD ACTION FUNCTIONAL
FOR A FULLY DIRICHLET PROBLEM.

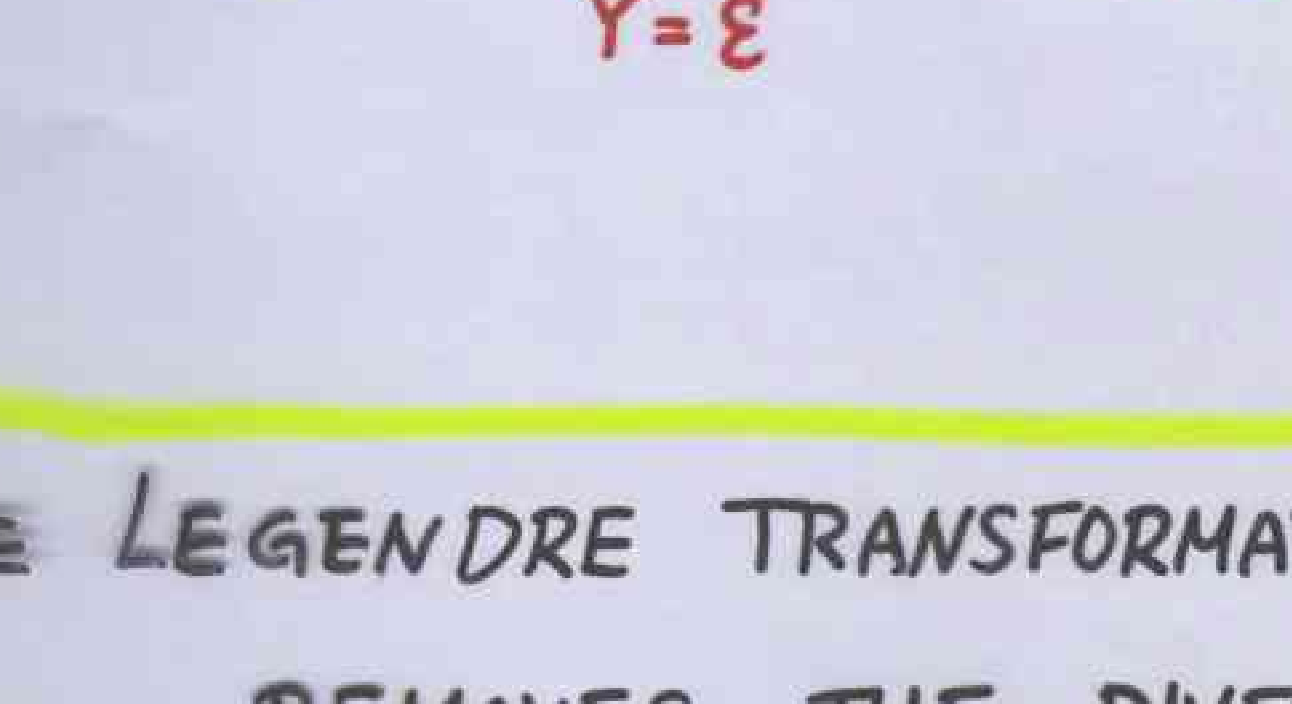


$$\Rightarrow \quad \text{AREA} = \frac{\sqrt{g^2 N}}{2\pi} A[x]$$

A CANONICAL ACTION FOR A NEUMANN PROBLEM
IS GIVEN BY ITS LEGENDRE TRANSFORM:

$$\tilde{A} = A - \oint ds\, P_i Y^i$$

FOR A SMOOTH LOOP WITH $\dot{x}^2 = \dot{y}^2$,
A IS LINEARLY DIVERGENT.



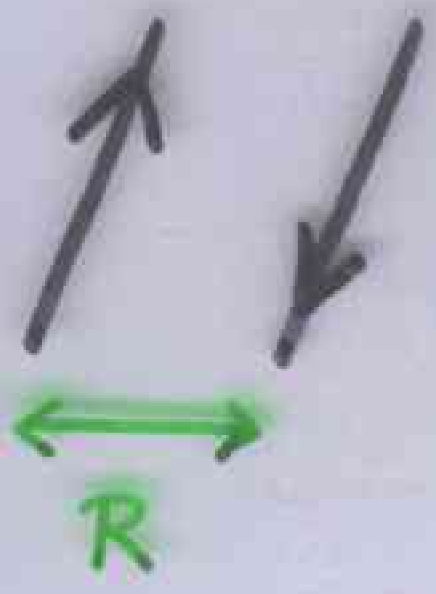
$$A = \frac{1}{\varepsilon} \oint ds\, |\dot{x}| + \text{FINITE}$$

ε : IR REGULARIZATION IN AdS_5
$\Leftrightarrow$ UV REGULARIZATION OF SYM_4

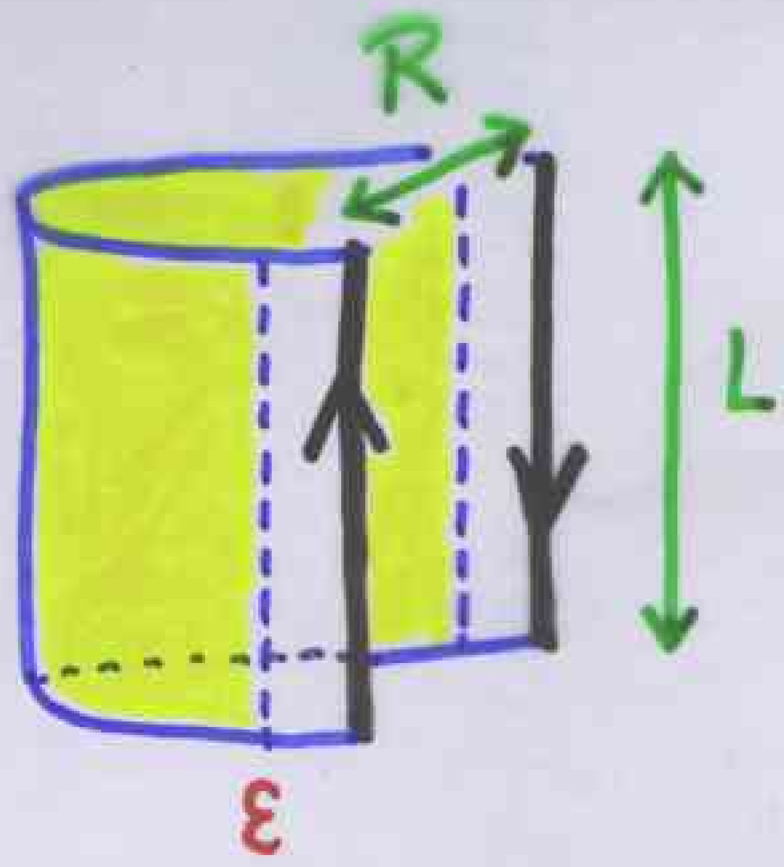
THE LEGENDRE TRANSFORMATION REMOVES THE DIVERGENCE.

EXAMPLES

(1) PARALLEL LINES

$\langle$  $\rangle = \exp\left(\sqrt{g^2 N}\,\frac{4\pi\sqrt{2}}{\Gamma(1/4)^4}\,\frac{L}{R}\right)$

$L \gg R$

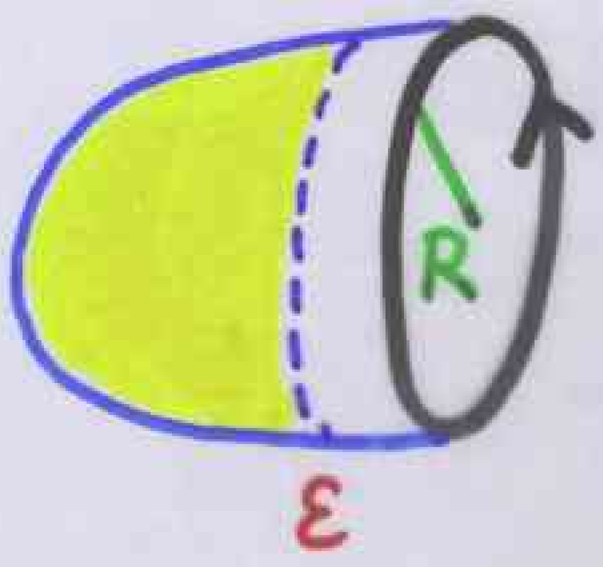


MALDACENA ; REY + YEE

$q\bar{q}$-POTENTIAL $\sim \dfrac{-\sqrt{g^2 N}}{R}$

(2) CIRCLE

$\langle$  $\rangle = \exp\left(\sqrt{g^2 N}\right)$



INDEPENDENT OF R

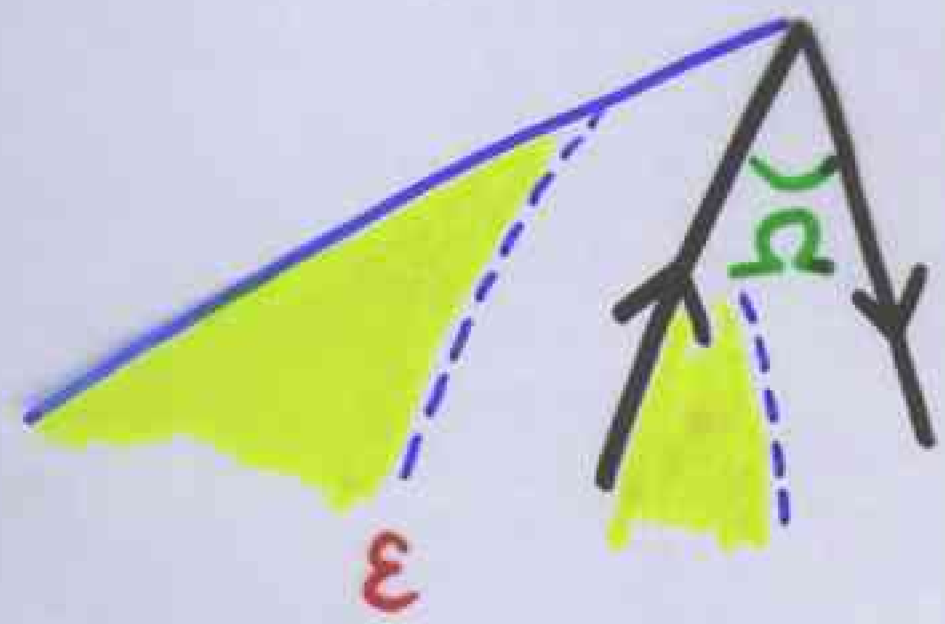
THEY ARE BOTH FINITE.

(3) CUSP

$\langle$  $\rangle = \exp\left(\sqrt{g^2 N}\, F(\Omega, \Theta)\, \log\left(\frac{L}{\varepsilon}\right)\right)$

ANGLE ON $\mathbb{R}^4$

JUMP ON S^5



- LOGARITHMIC DIVERGENCE
- $F(\Omega, \Theta)$ IS GIVEN IN TERMS OF THE ELLIPTIC INTEGRALS.

WHEN $\dot{x}^2 = \dot{y}^2$, $\langle W \rangle \simeq \exp(-\sqrt{g^2 N}\,\tilde{A})$

FOR $g^2 N \gg 1$, $\varepsilon \to 0$.

QUESTION : HOW TO COMPUTE $\langle W \rangle$ WHEN $\dot{x}^2 \neq \dot{y}^2$?

$\langle W \rangle = 0$ SINCE THERE IS NO MINIMUM SURFACE FOR SUCH $(x(s), y(s))$ WHICH ENDS ON THE BOUNDARY OF $AdS_5 \times S^5$. (NOT A GOOD ANSWER)

$\Downarrow$

WE NEED TO UNDERSTAND STRINGY CORRECTIONS BETTER.

IMPORTANT TEST : CONSISTENCY WITH THE LOOP EQUATION.

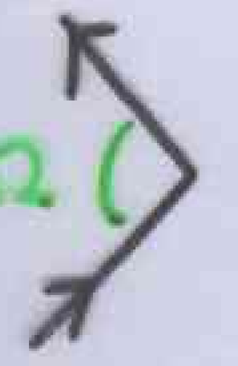
FOR EXAMPLE, IF (AS A PURE SPECULATION)

$$\langle W \rangle \simeq \exp\left(\sqrt{g^2 N}\left(\frac{1}{\varepsilon}\oint ds\,(|\dot{x}| - |\dot{y}|) - \tilde{A}\right)\right)$$

WHEN $\dot{x}^2 \neq \dot{y}^2$,

$\mathcal{L}\langle$  $\rangle \sim \dfrac{g^2 N}{\varepsilon^2}(\cos\Omega + \cos\Theta)\,\langle$  $\rangle$.

THIS CAN BE COMPARED WITH THE GAUGE THEORY RESULT :

$\mathcal{L}\langle$  $\rangle = \dfrac{g^2 N}{\varepsilon^2}(\cos\Omega + \cos\Theta)\cdot\dfrac{\pi - \Omega}{(2\pi)^2 \sin\Omega}$

$\times \langle$  $\rangle$

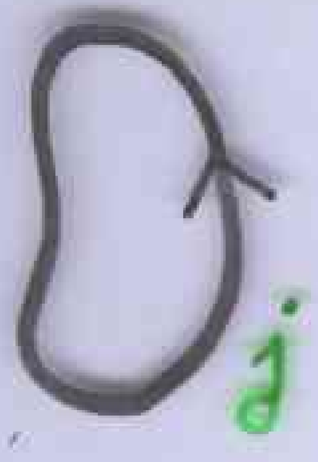
3. Wilson Loops in the Chern-Simons Theory

$$S_{CS} = \frac{k}{4\pi} \int tr \left(A dA + \frac{2}{3} A^3 \right)$$

$G = SU(N)$

$$Z(S^3) = S_{00}$$

Witten / 1989

$\langle$  $\rangle_j = \dfrac{S_{0j}}{S_{00}}$

$\langle$  $\rangle = \dfrac{S_{ij}}{S_{00}}$

S_{ij} IS GIVEN BY THE MODULAR TRANSFORMATION OF THE $\widehat{SU(N)}_k$ CHARACTERS.

$$\chi_i \left(-\frac{1}{\tau}\right) = \sum_j S_{ij} \, \chi_j(\tau)$$

FOR EXAMPLE:

$$S_{00} = \frac{e^{i\frac{\pi}{8}N(N-1)}}{(k+N)^{N/2}} \sqrt{\frac{k+N}{N}} \prod_{l=1}^{N-1} \left[2 \sin\left(\frac{\pi l}{N+k}\right) \right]^{N-l}$$

CONJECTURE

GOPAKUMAR + VAFA / 9811131

THE $SU(N)_k$ CHERN-SIMONS THEORY ON S^3

(= TOPOLOGICAL OPEN STRING THEORY ON T^*S^3

· N D-BRANES ON S^3

· $g_{CS}^2 = \frac{2\pi}{k+N}$ WITTEN / 9207094)

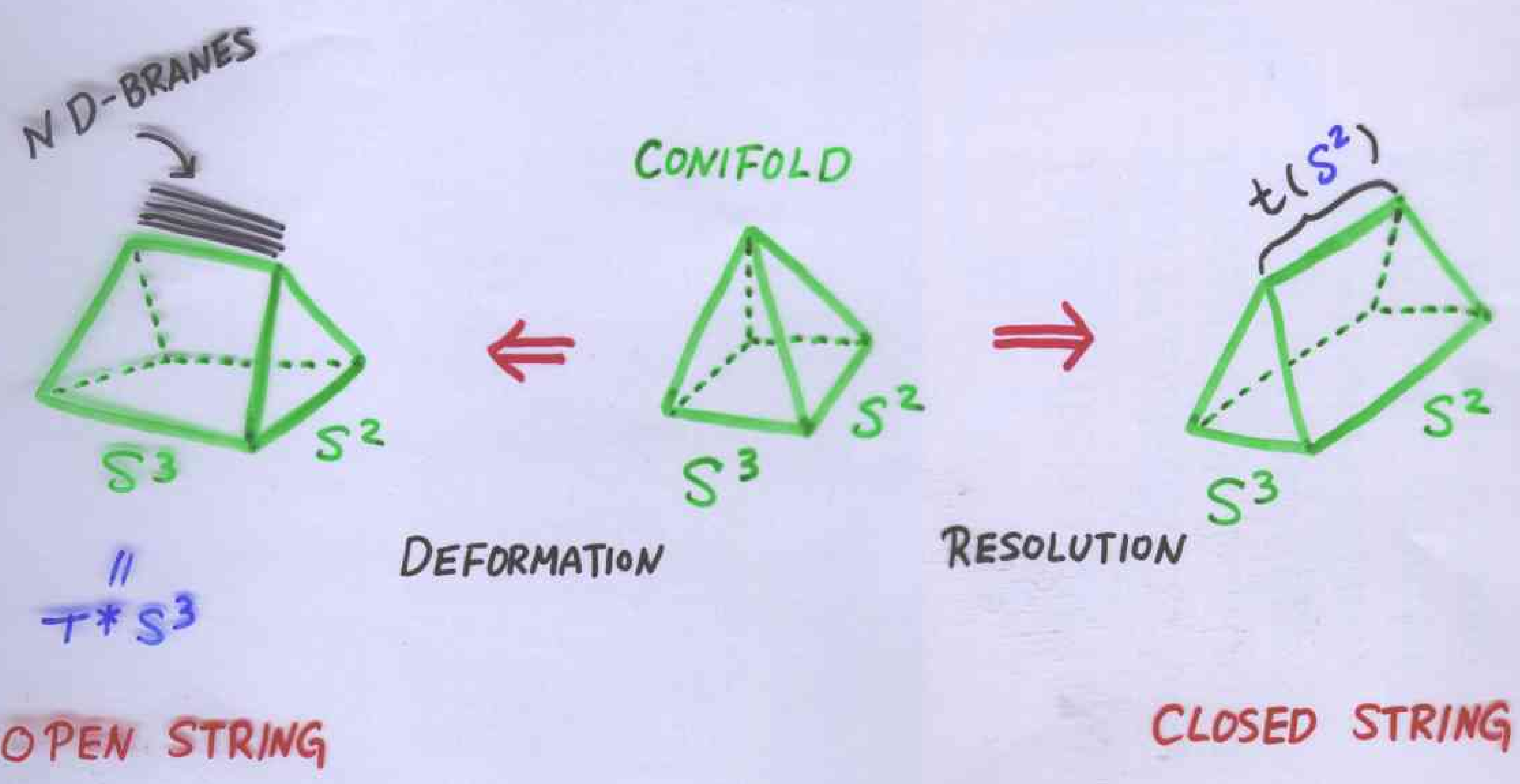
$\Updownarrow$

TOPOLOGICAL CLOSED STRING THEORY (A-MODEL)

ON THE RESOLVED CONIFOLD GEOMETRY, WITHOUT BRANE.

· $g_{STRING} = \frac{1}{N}$

· $t(S^2) = i\, g_{CS}^2 N = \frac{2\pi i N}{k+N}$



EVIDENCE

PERIWAL / 9305115

GOPAKUMAR + VAFA / 9809187

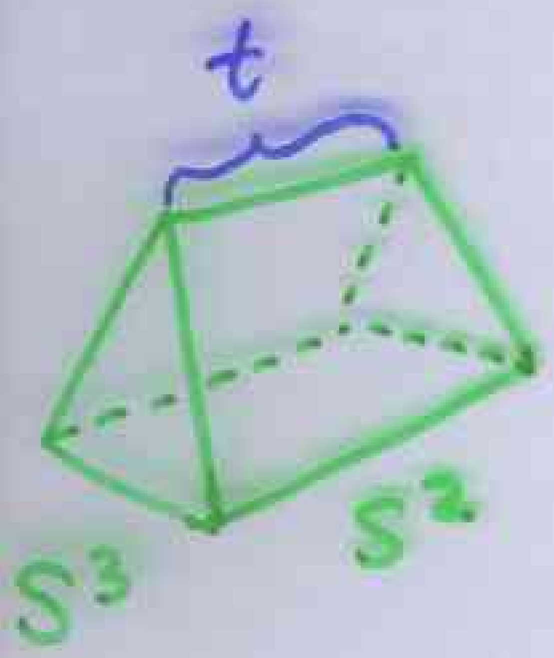
$$Z(S^3) = \frac{e^{i\frac{\pi}{8}(N-1)N}}{(k+N)^{N/2}}\sqrt{\frac{k+N}{N}}\prod_{\ell=1}^{N-1}\left(2\sin\left(\frac{N}{k+N}\right)\right)^{N-\ell}$$

$$= \exp\left[-\sum_{g=0}^{\infty} N^{2-2g} F_g(t)\right]$$

$$t = \frac{2\pi i N}{k+N}$$

F_g CAN BE EXPRESSED AS

$$\boxed{F_g = \int_{\mathcal{M}_g} c_{g-1}^3 - \frac{\chi_g}{(2g-3)!}\sum_{n=1}^{\infty} n^{2g-3} e^{-nt}}$$



$\mathcal{M}_g$: MODULI SPACE OF GENUS $-g$ CURVE

χ_g : EULER CHARACTER OF $\mathcal{M}_g$

$$= (-1)^{g-1} B_g / 2g(2g-2)$$

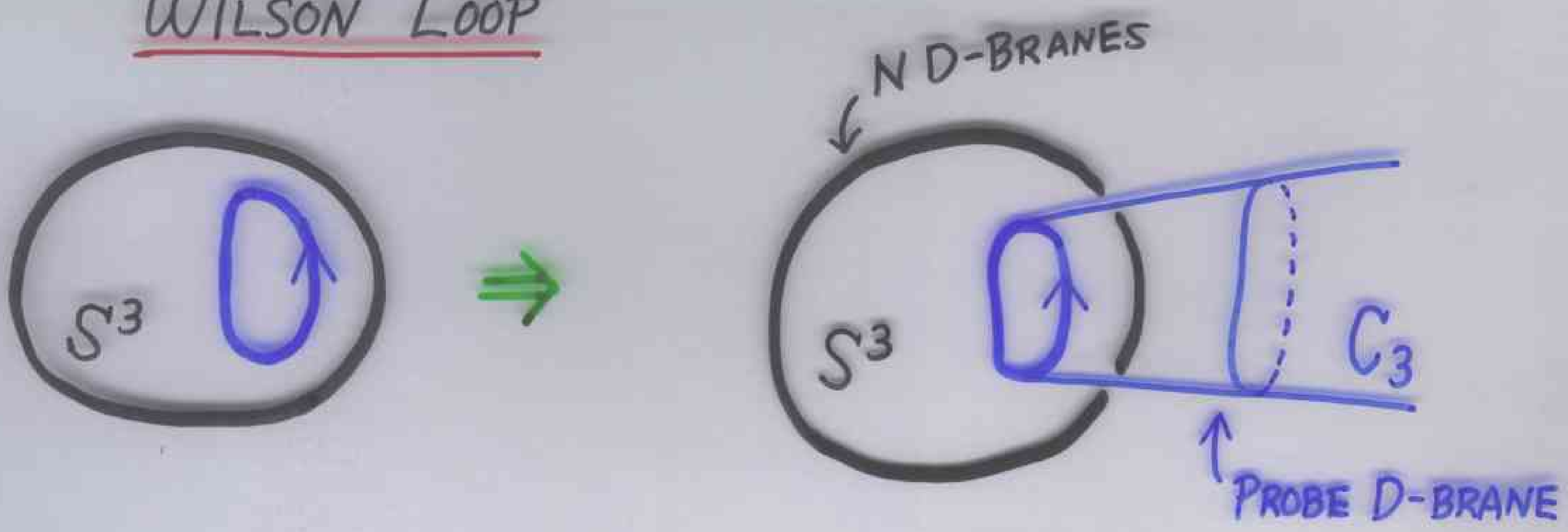
$$\int_{\mathcal{M}_g} c_{g-1}^3 = (-1)^{g-1}\chi_g\, \zeta(2g-2)\cdot 2/(2\pi)^{2g-2}$$

THIS EXPRESSION AGREES WITH THE g-LOOP AMPLITUDE OF TOPOLOGICAL CLOSED STRING ON THE RESOLVED CONIFOLD GEOMETRY.

BERSHADSKY, CECOTTI, VAFA + H.O. / 9309140

$\sum_{n=1}^{\infty} n^{2g-3} e^{-nt}$: A SUM OVER DEGENERATE GENUS $-g$ CURVES ON S^2 OF SIZE t.

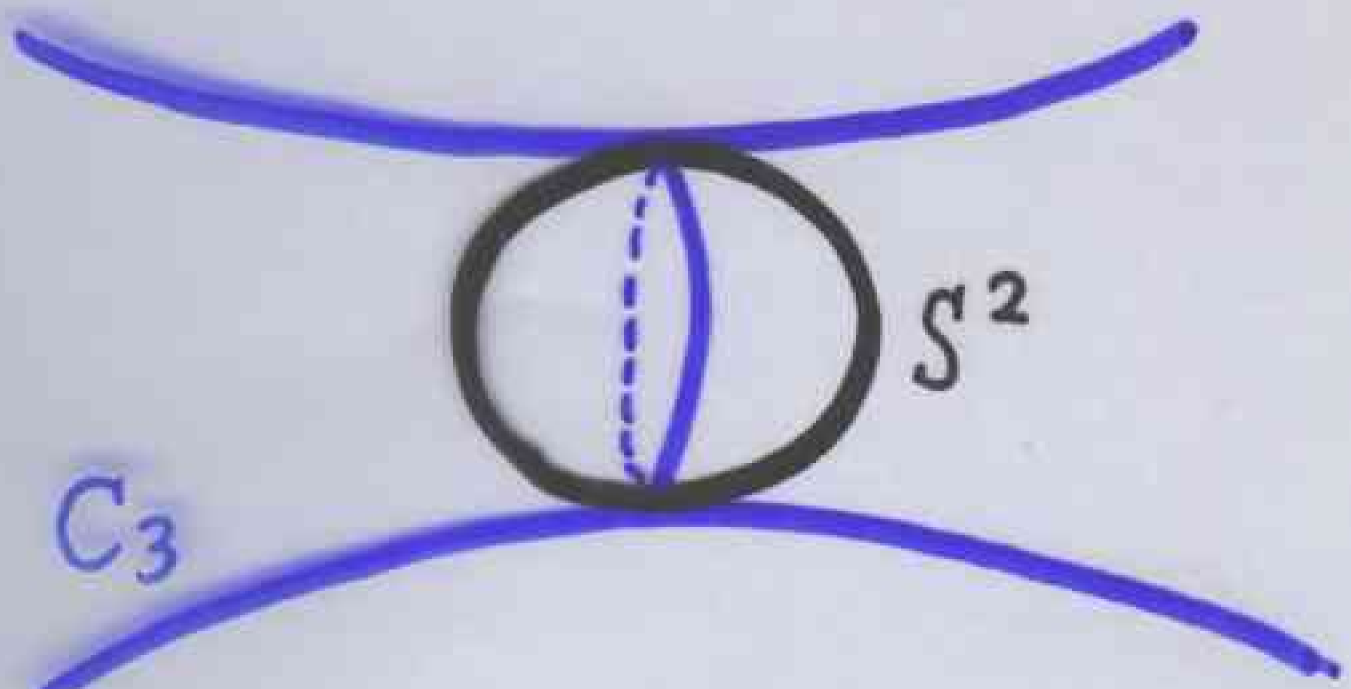
Wilson Loop



3-CYCLE $C_3 = \{ (x, p) \in T^*S^3 : x = x(s),\ p \perp \dot{x}(s) \}$

THIS IS LAGRANGIAN WITH RESPECT TO THE SYMPLECTIC FORM $\omega = dx \wedge dp$.

$\Downarrow$ CONIFOLD TRANSITION



THE 3-CYCLE C_3 TOUCHES THE EQUATOR OF THE BLOWN-UP S^2.

$$\langle \mathrm{tr}\, U^m \rangle_{S^3} = \frac{\sin\left(\frac{Nm\pi}{k+N}\right)}{\sin\left(\frac{m\pi}{k+N}\right)} = \frac{e^{m\frac{t}{2}} - e^{-m\frac{t}{2}}}{2\sinh\left(m\, g_{\text{STRING}}\frac{t}{2}\right)}$$

$\frac{t}{2} = $ AREA ()

MATCHES WITH TOPOLOGICAL STRING COMPUTATION.