String Theory and Noncommutative Geometry

N.S. + E. Witten, hep-th/990????

Review of Part I

Study strings in flat space, metric $= g$
constant $B \neq 0$.

In R^n B affects only open strings

$$g_{ij}\,\partial_n x^j + \alpha' B_{ij}\,\partial_t x^j = 0$$ ← boundary condition

$$\langle x^i(t)\, x^j(0)\rangle = \alpha' G^{ij} \log|t| + \frac{i}{2}\,\theta^{ij}\,\epsilon(t)$$

$$G^{ij} = \left(\frac{1}{g+\alpha' B}\right)^{ij}_S$$ ← open string metric $\neq g^{ij} =$ closed string metric

$$\theta^{ij} = \alpha' \left(\frac{1}{g+\alpha' B}\right)^{ij}_A$$

$$[x^i, x^j] = i\theta^{ij}$$

Space is noncommutative in the sense of Connes (Schomerus)

Gauge fields (Connes, Douglas, Schwarz)

$$\hat{\delta}\hat{A} = \partial\hat{\lambda} + i\,\hat{\lambda}*\hat{A} - i\hat{A}*\hat{\lambda}$$

$$f(x)*g(x) = e^{i\frac{\theta^{ij}}{2}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)\,g(y)\Big|_{x=y}$$

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i\hat{A}_i*\hat{A}_j + i\,\hat{A}_j*\hat{A}_i$$

Using vertex operators

$$\mathcal{L}_{eff} \sim \hat{F}*\hat{F} + \alpha'^2\,\hat{F}*\hat{F}*\hat{F}*\hat{F} + \dots$$

metric G

θ only in $*$ product

$$\mathcal{L}_{eff} \sim \hat{F}^2 + \alpha'^2 \hat{F} * \hat{F} * \hat{F} * \hat{F} \ldots = F^2 + \alpha'^2 F^4 + \alpha'^2 B F^{3+} \ldots$$

$F = dA - i[A, A]$ ordinary Yang-Mills field

$\Rightarrow$ change of variables

$$\hat{A}_i(A) = A_i - \frac{1}{4}\theta^{kl}\{A_k, \partial_l A_i + F_{li}\} + O(\theta^2)$$

$$\hat{\lambda}(\lambda, A) = \lambda + \frac{1}{4}\theta^{kl}\{\partial_k \lambda, A_l\} + O(\theta^2)$$

$$\hat{A}(A + \delta_\lambda A) = \hat{A}(A) + \hat{\delta}_{\hat{\lambda}} \hat{A}(A)$$

Noncommutative gauge theory = Commutative gauge theory.

Which of them is more useful depends on parameters (vacuum).

For slowly varying fields (more below)

$$\mathcal{L} = \frac{(\alpha')^{-\frac{d}{2}}}{g_s}\sqrt{\det(g + \alpha' B + \alpha' F)} = \frac{(\alpha')^{-\frac{d}{2}}}{\hat{g}_s}\sqrt{\det(G + \alpha' \hat{F})}$$

+ total derivatives

Agreement for $F = \hat{F} = 0$ with

$$\hat{g}_s = g_s \left[\frac{\det(g + \alpha' B)}{\det g}\right]^{\frac{1}{2}}$$

conclude:

Can use $\quad g_{ij},\ g_s,\ B_{ij},\ A_i(x)$

or $\quad G_{ij},\ \hat{g}_s,\ \theta^{ij},\ \hat{A}_i(x)$

Simplification in zero slope limit

$\alpha' \to 0$

with G_{ij} and θ^{ij} fixed

$(g \sim \alpha'^2 \to 0, \quad G \sim B^2, \quad \theta = \frac{1}{B})$

As for $\theta = 0$:

1. No massive string modes
2. $: f(x) : : g(x) : = : f(x) * g(x) :$

   no singularity, associativity
3. $\mathcal{L}_{eff} \sim \hat{F}^2$ is exact
4. For g_{YM} finite, $\hat{g}_s \sim (\alpha')^{\frac{4-d}{2}}$

In this limit the noncommutative description is more useful

In worldsheet σ-model add $\int A_i \partial_t x^i$ to action.

Under $\delta A_i = \partial_i \lambda$ $\quad \delta \int A_i \partial_t x^i = \int \partial_t \lambda = 0$

$\Rightarrow$ gauge invariance

Same argument with Pauli-Villars.
But with point splitting (no two operators at same point):

$$\int A_i \partial_t x^i(t) \int \delta A_j \partial_{t'} x^j(t') = \int A_i \partial_t x^i(t) \int \partial_{t'} \lambda(t')$$

$$= \int A_i \partial_t x^i \left(\lambda(t^-) - \lambda(t^+) \right) =$$

$$= \int \left(A_i * \lambda - \lambda * A_i \right) \partial_t x^i$$

(complete justification in $\alpha' \to 0$ limit)

Invariance under $\delta A = \partial \lambda + i \lambda * A - i A * \lambda$

Difference between regularizations in contact terms. Hence, $\hat{A}(A)$ exists.

Slowly varying fields on a single D-brane

$$S = \int \frac{(\alpha')^{-\frac{d}{2}}}{g_s} \sqrt{\det(g + \alpha' B + \alpha' F)} + O(\partial F)$$

$$= \int \frac{(\alpha')^{-\frac{d}{2}}}{\hat{g}_s} \sqrt{\det(G + \alpha' \hat{F})} + O(\partial \hat{F})$$

- ordinary gauge fields
- depends on B only through B + F

- noncommutative gauge fields
- Θ dependence only in $*$

Using $\hat{A} = A + \Theta A \partial A + O(\Theta^2)$ can check explicitly when Θ is small

Study in zero slope limit

$$g_{ij} = \epsilon \delta_{ij} \quad , \alpha' \sim \epsilon^{1/2}$$

$$\sqrt{\det\left[g + \alpha'(B+F)\right]} \approx \left| Pf(\alpha' B + \alpha' F) \right| +$$

$$+ \epsilon^2 \left| Pf(\alpha' B + \alpha' F) \right| \mathrm{Tr} \frac{1}{(\alpha' B + \alpha' F)^2}$$

constant + total derivative

non polynomial

Using $\hat{A}(A)$, $\Theta = \frac{1}{B}$, $G \sim B^2$, $\hat{g}_s(g, B, g_s)$

$$\frac{(\alpha')^{-d/2}}{g_s} \epsilon^2 \left| Pf(\alpha' B + \alpha' F) \right| \mathrm{Tr} \frac{1}{(\alpha' B + \alpha' F)^2} =$$

$$= \frac{(\alpha')^{\frac{4-d}{2}}}{\hat{g}_s} \sqrt{|G|} \, G^{ij} G^{i'j'} \hat{F}_{ii'} \hat{F}_{jj'} + \text{total der.}$$

- Consistency check
- Simplification manifest

BPS states

From $\mathcal{L}_{NC} \sim \hat{F}^2$

BPS: $\hat{F}_G^+ = 0$

$\uparrow$

self dual with respect to $G \sim B^2$

Examine for slowly varying fields on one D-brane in terms of an ordinary gauge field. Use DBI

DBI with $N=1$ SUSY

$$\delta\lambda = (B+F)^+$$

It also has nonlinearly realized susy (Bagger and Galperin)

$$\delta^*\lambda = \sqrt{g} - pf[\alpha'(B+F)] + \sqrt{\det(g+\alpha'(B+F))}$$

For $B \neq 0$ both SUSYs are broken (non-linear), but one linear combination is unbroken.

Configurations preserving half of unbroken SUSY (BPS) for $\alpha' \to 0$

$$F^+ - \frac{1}{\mathrm{Pf}B}\left(B\tilde{F} + F\tilde{F}\right) B^+ = 0$$

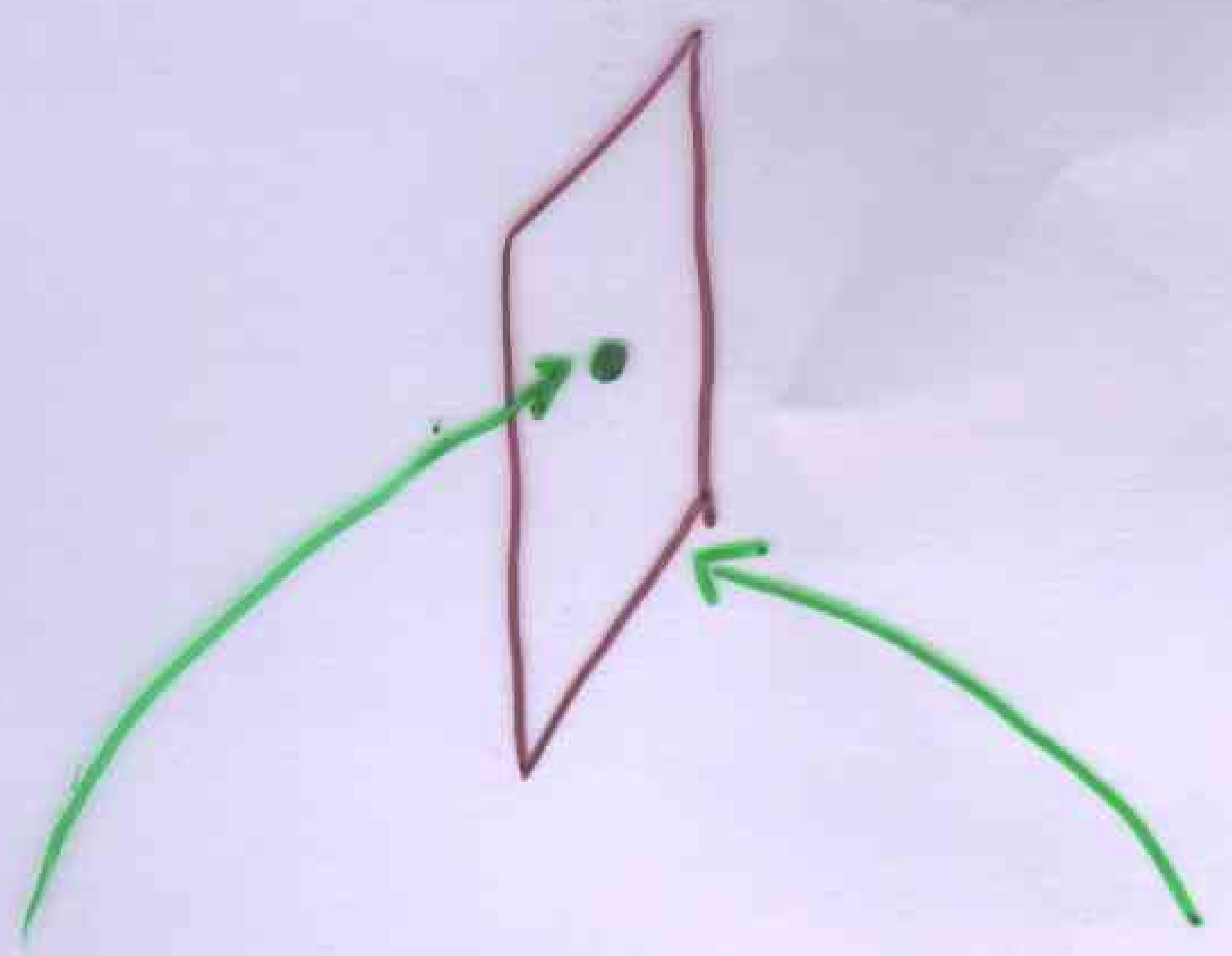
$$F_G^+ - F\tilde{F}\left(\frac{1}{B}\right)_G^+ = 0$$

selfdual with respect to $G \sim B^2$

using the change of variables

$$F_G^+ - F\tilde{F}\,\theta_G^+ = \hat{F}_G^+ = 0$$

D0/D4 with $B \neq 0$



D0 is instanton in D4

$B = 0$

Higgs branch of D0 = moduli space of instantons = ADHM construction

$B^+ = 0$ but $B^- \neq 0$

Theory is noncommutative but the D0s are not affected at low energy $\Longrightarrow$ Higgs branch is unchanged $\Longrightarrow$ moduli space of instantons is unchanged

$B^+ \neq 0$

Higgs branch is deformed by FI term $\Rightarrow$

- No small instanton singularity
- smooth instantons in "$U(1)$"

(Nekrasov, Schwarz)

can analyze using string theory by studying 0-4 strings at origin of moduli space - point instanton:

$B^+ = 0$ susy unbroken.

$B^+ \neq 0$ tachyon. Instability toward large instantons

$m^2 \sim -|\theta^+|$ (for $\mathrm{Pf}B<0$) corresponds to FI term in D0 Lagrangian $\sim |\theta^+|$

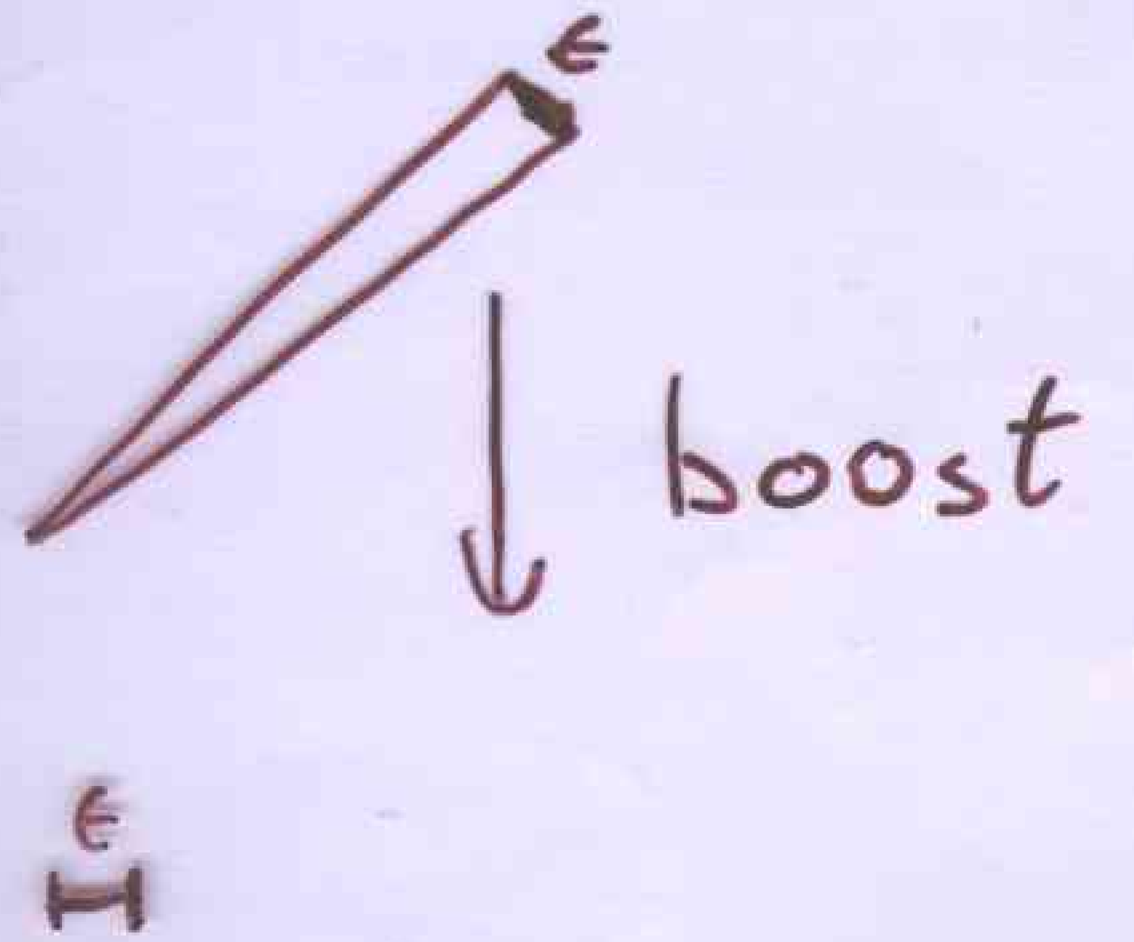
$\Rightarrow$ no point instanton, deformation by θ^+

M theory in DLCQ

M theory in DLCQ $(x^- \sim x^- + R)$ is described by Matrix Model (BFSS).

If $C_{-ij} \neq 0$, it is noncommutative (Connes, Douglas, Schwarz, Brace, Morariu, Zumino).

DLCQ is limit $(\epsilon \to 0)$ of spacelike circle



$\downarrow$ Rescale energy by $\frac{1}{\epsilon}$

Effective string theory at weak coupling with small transverse dimensions $g \sim \epsilon$, $\alpha' \sim \epsilon^{1/2}$, $g_s \sim \epsilon^{3/4}$

$C_{-ij} \longrightarrow B_{ij}$ and N D0-branes

If T^p in transverse space, T duality.

$B \neq 0 \Rightarrow$ transverse dimensions remain small. $D0 \rightarrow Dp$

$g \sim \epsilon$, $\alpha' \sim \epsilon^{1/2}$, $g_s \sim \epsilon^{3/4}$, $B \sim \frac{1}{\epsilon}$

This is our zero slope limit

G = dual of original g, finite

$\theta \sim C_{-ij}$, finite

$g_{YM} \sim$ finite

$\Downarrow$

$$\mathcal{L} = \frac{1}{g_{YM}^2} \int \hat{F}^2$$