Yang Mills Integrals

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+ ongoing, unpublished work with
Werner Krauth + Jan Plefka (AEI - Potsdam)

What are Yang-Mills integrals ?

Answer: D-dimensional Euclidean gauge field theory reduced to zero dimensions.

$$\mathcal{Z}^{\mathcal{N}}_{D,N} = \int \prod_{A=1}^{N^2-1} \left(\prod_{\mu=1}^{D} \frac{dX_\mu^A}{\sqrt{2\pi}} \right) \left(\prod_{\alpha=1}^{\mathcal{N}} d\Psi_\alpha^A \right) e^{-S[X,\Psi]}$$

gauge group: $SU(N)$

Euclidean action:

$$S[X,\Psi] = -\frac{1}{2} \operatorname{Tr} [X_\mu, X_\nu][X_\mu, X_\nu] + \operatorname{Tr} \Psi_\alpha [\Gamma^\mu_{\alpha\beta} X_\mu, \Psi_\beta]$$

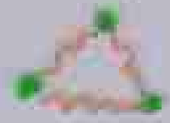
Susy: $\mathcal{N} = 2, 4, 8, 16 \Rightarrow D = 3, 4, 6, 10$

"Bosonic": $\mathcal{N} = 0 \Rightarrow D = 2, 3, 4, 5, \ldots$

- Integrating out fermions gives a Pfaffian: Homogeneous polynomial in X

Why should we study them?

- "D-branes": $p = -1$: D-instantons



- "Partonic attempts" to define M-theory

  "matrix string theory": $10 \to 2$

  "BFSS" or "matrix QM": $10 \to 1$

  "IKKT": $10 \to 0$ $\quad Z_{IKKT} = \sum_{N=2}^{\infty} Z^{16}_{10,N}\, e^{-\beta N}$

- Susy index calculations: $(-1)^F$
- $\mathcal{N}=4$, $D=4$ $SU(\infty)$ multi-instanton calculus

  K-instanton sector related to $Z^{\mathcal{N}=16}_{D=10,\, N=K}$

  (Dorey et.al. hep-th/9901128)

- An old idea: "Eguchi-Kawai reduction" (1982)

  (see Gross + Kitazawa 82)

Do YM integrals make sense ?

- They look very singular ("valleys"!).
- back-of-the-envelope calculation: $D=2$ diverges.

However, for (say) $SU(2)$ it was ("accidentally") discovered (Yi, Sethi-Stern) that convergence is possible:

special gauge : $X_\mu^A = b_\mu\, \delta_\mu^A$ (no sum.) $A=1,2,3$,

using $SU(2)\times SO(D)$ invariance:

$$Z^{\mathcal{N}}_{D,N=2} \sim \int\limits_{0\le b_3\le b_2\le b_1} db_1\, db_2\, db_3\ \Delta(b_i^2)\ (b_1 b_2 b_3)^{D-3+\frac{\mathcal{N}}{2}}\, e^{-\frac{1}{2}b_1^2 b_2^2-\frac{1}{2}b_1^2 b_3^2-\frac{1}{2}b_2^2 b_3^2}$$

Vandermonde: $\Delta(b_i^2) = (b_1^2-b_2^2)(b_1^2-b_3^2)(b_2^2-b_3^2)$

Convergence properties can be seen after a "Nicolai map":

$y_1 = b_2 b_3 \qquad y_2 = b_1 b_3 \qquad y_3 = b_1 b_2$

$$Z^{\mathcal{N}}_{D,N=2} \sim \int\limits_{0\le y_1\le y_2\le y_3} dy_3\, dy_2\, dy_1\ \Delta(y_i^2)\ (y_1 y_2 y_3)^{\underbrace{\frac{D}{2}-3+\frac{\mathcal{N}}{4}}_{=D-4\ (\text{Susy}:\ \mathcal{N}>0)}}\, e^{-\frac{1}{2}y_1^2-\frac{1}{2}y_2^2-\frac{1}{2}y_3^2}$$

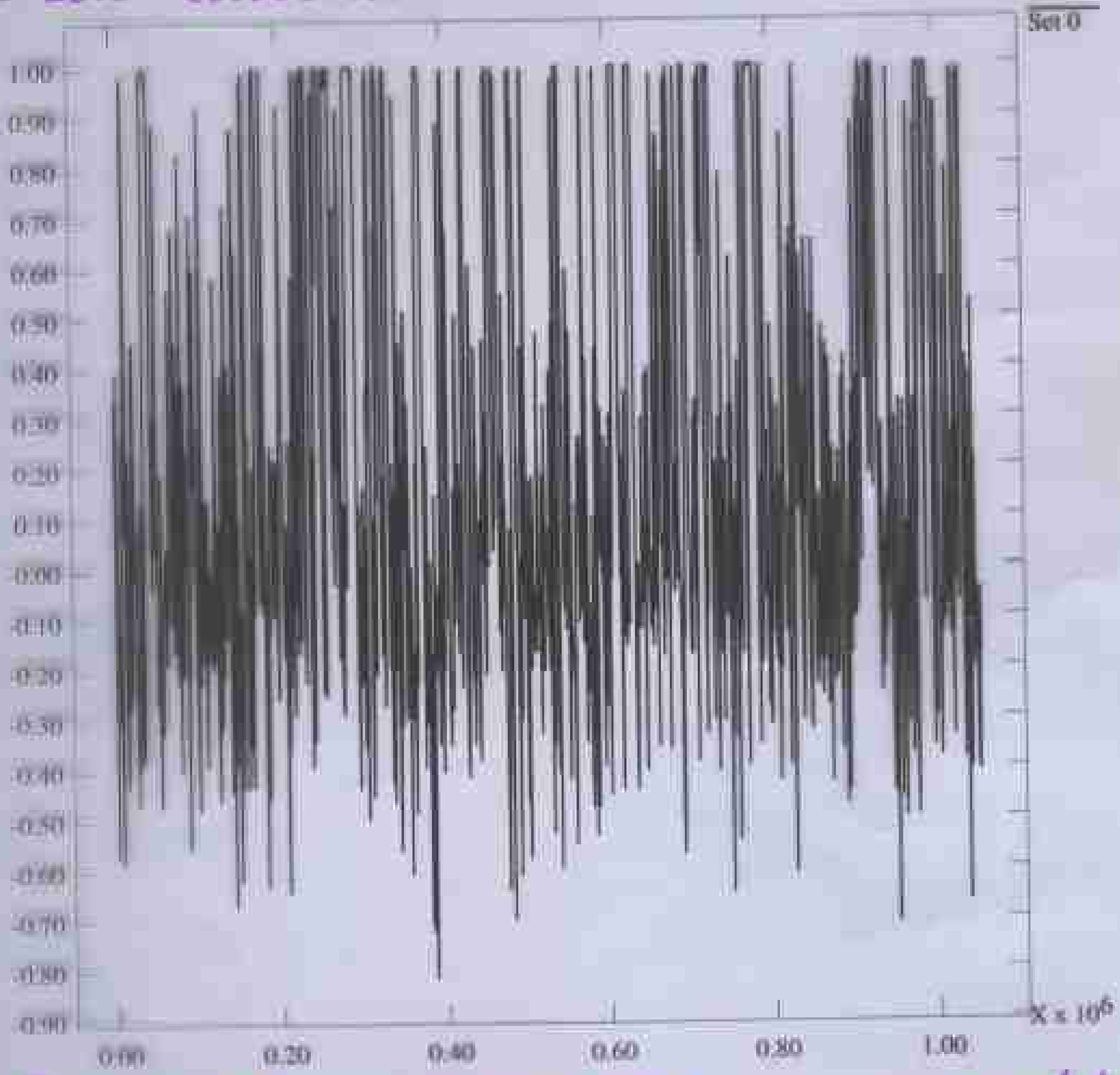
$SU(2)$ convergence:

Bosonic:	Susy:
$D\geqslant 5$	$D=4,6,10$

$N\geq 3$ convergence ??

A convergent Yang-Mills integral

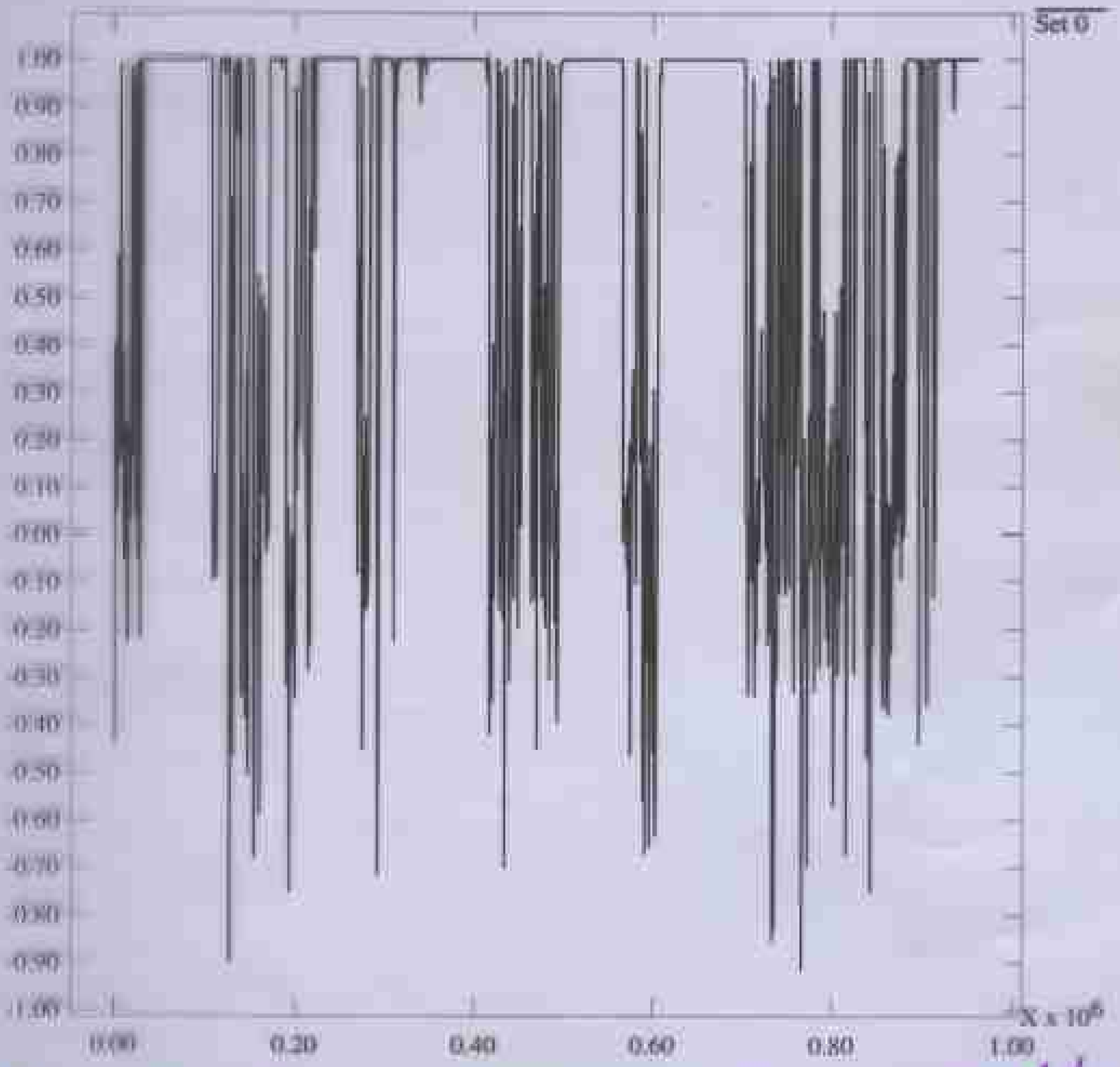
Y = auto-correlation



Example: Bosonic SU(2) $D = 5$ # updates

A (marginally) divergent Yang-Mills integral

y = auto-correlation

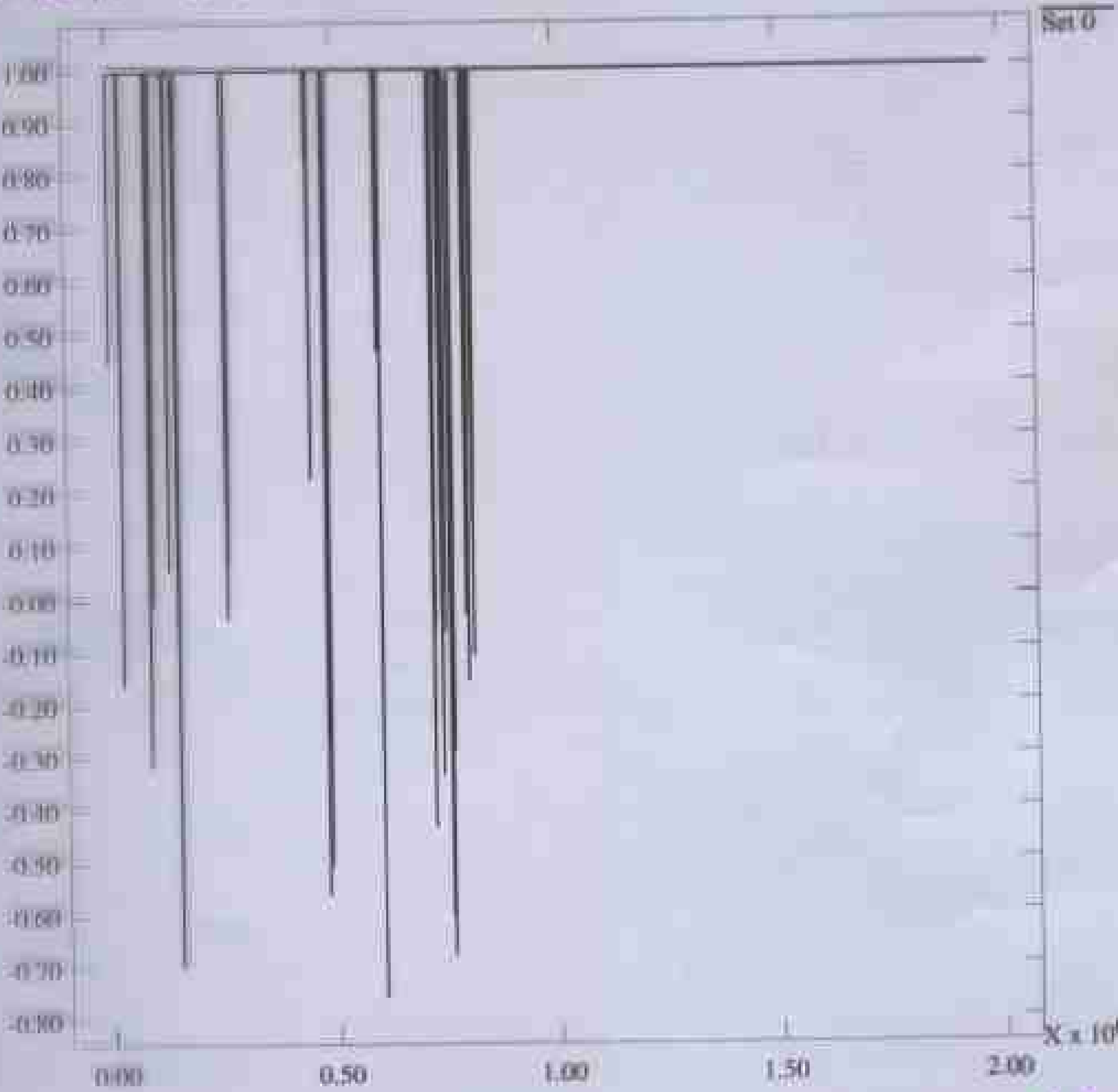


#updates

Example: Bosonic SU(2) $\underline{D=4}$

A (strongly) divergent Yang-Mills integral

Y = auto-correlation



Example: Bosonic $SU(2)$ $\underline{D=3}$ # updates

Convergence of SU(N) Yang-Mills integrals

Bosonic

$D=3 \quad N \geq 4$

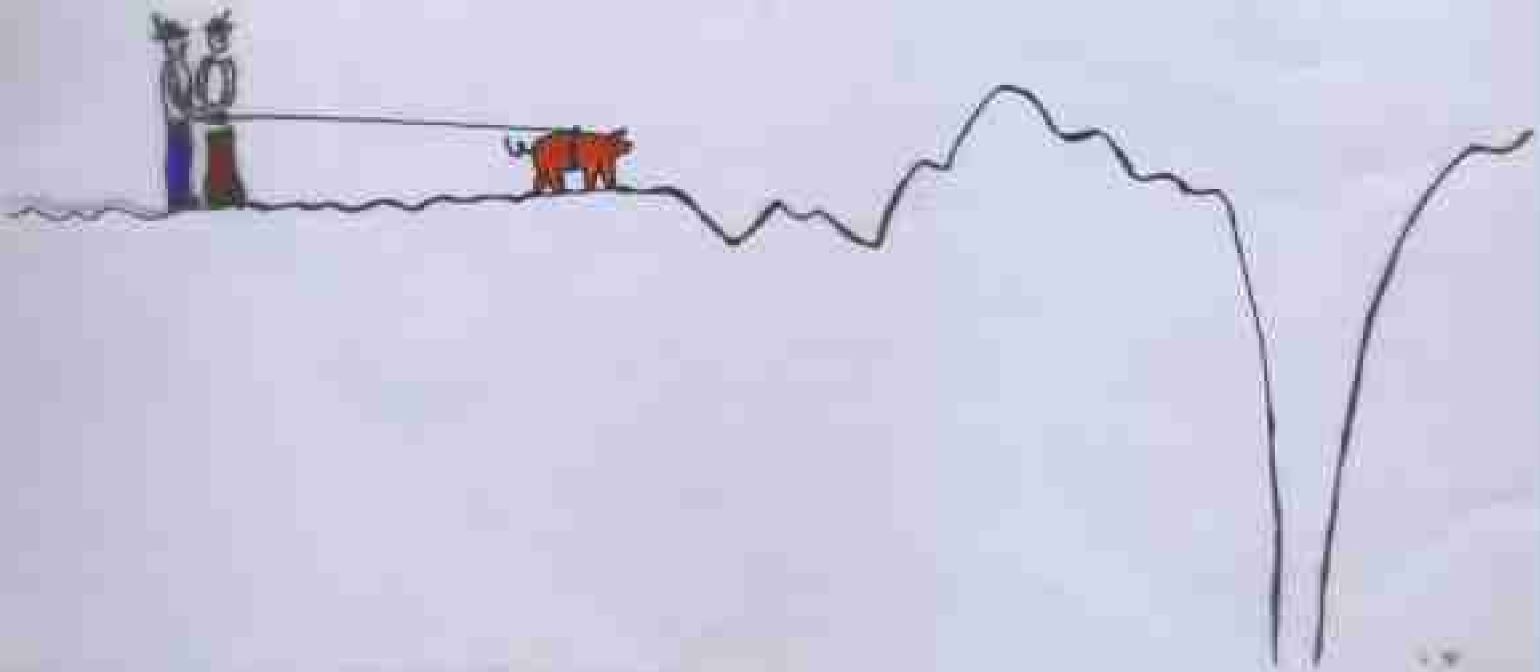
$D=4 \quad N \geq 3$

$D \geq 5 \quad N \geq 2$

Supersymmetric

$D=4,6,10 \quad N \geq 2$

All other cases are divergent. (in part.: Susy $D=3$)



One loop perturbative estimates

split $X_\mu = \begin{pmatrix} x_\mu^1 & & 0 \\ & \ddots & \\ 0 & & x_\mu^N \end{pmatrix} + Y_\mu$

$\mathcal{O}(Y^2)$ effective ensembles:

SUSY: (Aoki et.al. hep-th/9802085)

$$Z_{D,N} \sim \int \prod_{i,\mu}^{N,D} dx_\mu^i \left[\prod_\mu \delta\left(\sum_i x_\mu^i \right) \right] \times$$

$$\times \sum_{\substack{G:\,\text{maximal} \\ \text{tree}}} \prod_{\substack{(ij):\,\text{link} \\ \text{of } G}} \frac{1}{(x^i - x^j)^{3(D-2)}} + \ldots$$

Bosonic: (Hotta et.al. hep-th/9811220)

$$Z_{D,N}^{\mathcal{N}=0} \sim \int \prod_{i,\mu}^{N,D} dx_\mu^i \left[\prod_\mu \delta\left(\sum_i x_\mu^i \right) \right] \cdot \prod_{i<j} \frac{1}{(x^i - x^j)^{2(D-2)}}$$

- Powercounting reproduces convergence conditions
- Far from a proof, breaks down if any two x^i, x^j get close.

An exact result for all N (for Susy)

$$Z^{\mathcal{N}}_{D,N} = \frac{2^{\frac{N(N+1)}{2}}\,\pi^{\frac{N-1}{2}}}{2\sqrt{N}\prod_{i=1}^{N-1} i!} \times \begin{cases} - & D=3 \quad \mathcal{N}=2 \\ \frac{1}{N^2} & D=4 \quad \mathcal{N}=4 \\ \frac{1}{N^2} & D=6 \quad \mathcal{N}=8 \\ \sum_{m|N} \frac{1}{m^2} & D=10 \quad \mathcal{N}=16 \end{cases}$$

- D=10 conjectured by Green + Gutperle
- D = (3), 4, 6, 10 calculated by Moore et.al. hep-th/9803265
- D = 4, 6, 10 checked to ~N=5 by Monte Carlo

Bosonic integrals ?

Only the SU(2) result is known:

D≥5: $$Z^{\mathcal{N}=0}_{D,N=2} = 2^{-\frac{3}{4}D-1}\,\frac{\Gamma\left(\frac{D}{4}\right)\Gamma\left(\frac{D-2}{4}\right)\Gamma\left(\frac{D-4}{4}\right)}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(\frac{D-2}{2}\right)}$$

$$Z^{\mathcal{N}=0}_{D,N>2} = ?$$

CohFT Approach to susy integrals

(Moore, Nekrasov + Shatashvili)

- Beautiful, but incomplete:

Euclidean light cone: $\Phi = X_1 + iX_2 \quad \bar{\Phi} = X_1 - iX_2$

Φ is a complex matrix, and $\bar{\Phi} = \Phi^+$.

In the CohFT method, $\Phi, \bar{\Phi}$ are independent hermitean; corresponding to a "Wick rotation":

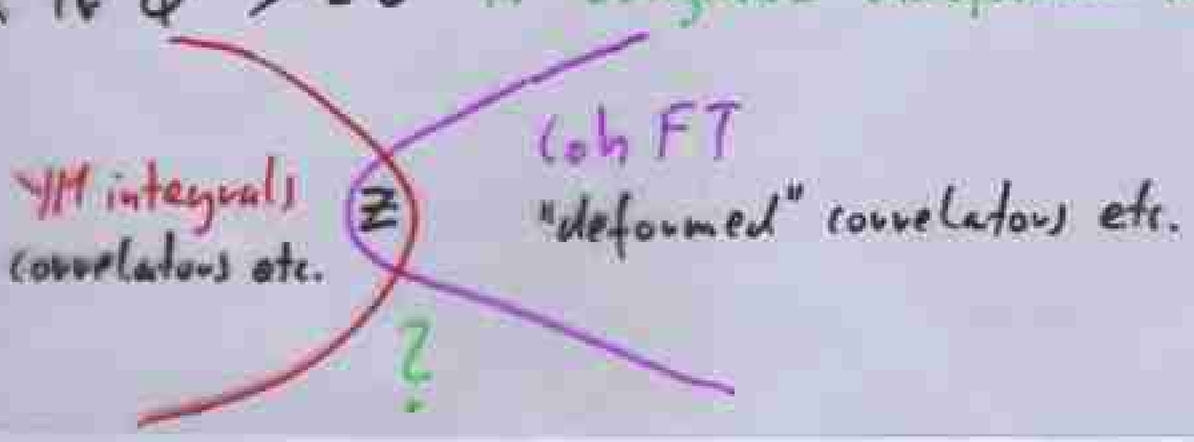
$$\Phi = X_1 - X_0 \quad \bar{\Phi} = X_1 + X_0$$

After elimination of all field(s) except Φ:

e.g. susy $D=4$: $$Z^{N=4}_{D=4,N} \sim \frac{1}{(N-1)!} \frac{1}{E^{N-1}} \oint \prod_{i=1}^{N-1} \frac{d\phi_i}{2\pi i} \prod_{\substack{i,j \\ i \neq j}} \frac{\phi_i - \phi_j}{\phi_i - \phi_j + E + i\epsilon}$$

- It would be nice to derive contour prescriptions from Wick rotation.
- $D=3$?? Euclidean Yang-Mills integral is ill-defined!
- Correlation functions:

$$\langle \mathrm{Tr}\, \Phi^K \rangle = 0$$ in original undeformed integral



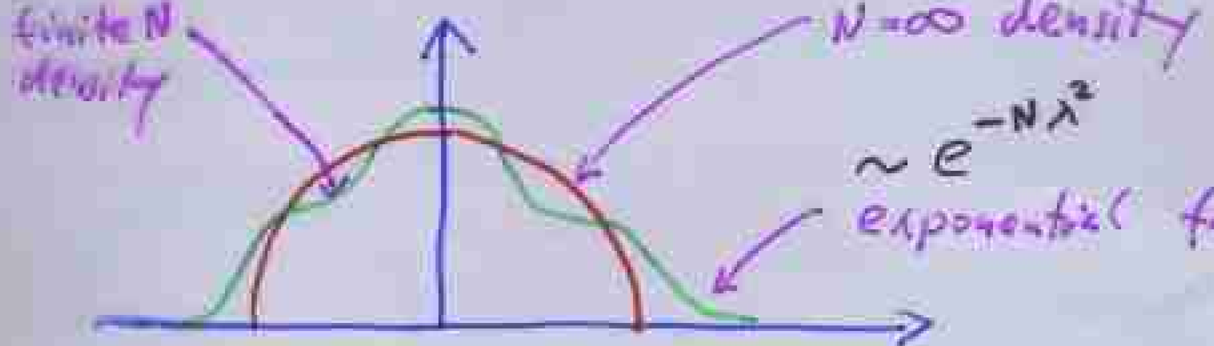
One-matrix correlators and the density of eigenvalues

- Simplest correlators: $\langle \frac{1}{N} \mathrm{Tr}\, X_D^{2K} \rangle$ = "moments"
- direct relation to eigenvalue density:

$$X_D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} \longrightarrow \rho(\lambda) \equiv \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle$$

$$\langle \frac{1}{N} \mathrm{Tr}\, X_D^{2K} \rangle = \int_{-\infty}^{\infty} d\lambda\, \rho(\lambda)\, \lambda^{2K}.$$

- "Wigner-type" matrix models:



- compute moments of YM integrals for $SU(2)$:
  Surprise! E.g. $D=4$ susy: all moments are ∞!
- Does density exist? Yes, but

$$\rho^{\mathcal{N}=4}_{D=4,N=2}(\lambda) \sim \frac{1}{\lambda^3} \quad \text{as } \lambda \to \infty$$

Exact results for $SU(2)$

Explicit $SU(2)$ eigenvalue densities and expectation values (Plefka, unpublished)

$$\rho_{D=4}^{SUSY}(\lambda) = \frac{3\cdot 2^{5/4}}{\sqrt{\pi}}\,\lambda^2\,U(\frac{5}{4},\frac{1}{2},8\lambda^4)$$
$$\rho_{D=6}^{SUSY}(\lambda) = \frac{105}{2^{3/4}\sqrt{\pi}}\,\lambda^2\left[U(\frac{9}{4},\frac{1}{2},8\lambda^4)-\frac{33}{16}U(\frac{13}{4},\frac{1}{2},8\lambda^4)\right]$$
$$\rho_{D=10}^{SUSY}(\lambda) = \frac{1287}{64\cdot 2^{3/4}\sqrt{\pi}}\,\lambda^2\left[546\,U(\frac{17}{4},\frac{1}{2},8\lambda^4)-147\,\frac{17\cdot 19}{8}U(\frac{21}{4},\frac{1}{2},8\lambda^4)\right.$$
$$\left.+45\,\frac{17\cdot 19\cdot 21\cdot 23}{256}U(\frac{25}{4},\frac{1}{2},8\lambda^4)-\frac{17\cdot 19\cdot 21\cdot 23\cdot 25\cdot 27}{2048}U(\frac{29}{4},\frac{1}{2},8\lambda^4)\right]$$

U is the Kummer-U function:

$$U\Big(\nu+\frac{1}{2},\frac{1}{2},p\,z\Big)=\frac{\sqrt{p}}{\Gamma(\nu+1)}\int_0^\infty dx\,\frac{x^\nu}{(x+z)^{\nu+1/2}}\,e^{-px}$$

- One has $\rho_{D=4}\sim\lambda^{-3}$, $\rho_{D=6}\sim\lambda^{-7}$, $\rho_{D=10}\sim\lambda^{-15}$

The susy one-matrix correlators for $D=6$

$$\langle \mathrm{Tr} X_D^2\rangle_{D=6}=\frac{1}{2}\sqrt{\frac{2}{\pi}}\qquad \langle \mathrm{Tr} X_D^4\rangle_{D=6}=\frac{25}{64}$$

as well as the susy $D=10$ ones

$$\langle \mathrm{Tr} X_D^2\rangle_{D=10}=\frac{8}{25}\sqrt{\frac{2}{\pi}}\qquad \langle \mathrm{Tr} X_D^4\rangle_{D=10}=\frac{9}{80}$$
$$\langle \mathrm{Tr} X_D^6\rangle_{D=10}=\frac{3}{32}\sqrt{\frac{2}{\pi}}\qquad \langle \mathrm{Tr} X_D^8\rangle_{D=10}=\frac{297}{4096}$$
$$\langle \mathrm{Tr} X_D^{10}\rangle_{D=10}=\frac{1089}{8192}\sqrt{\frac{2}{\pi}}\qquad \langle \mathrm{Tr} X_D^{12}\rangle_{D=10}=\frac{184041}{655360}$$

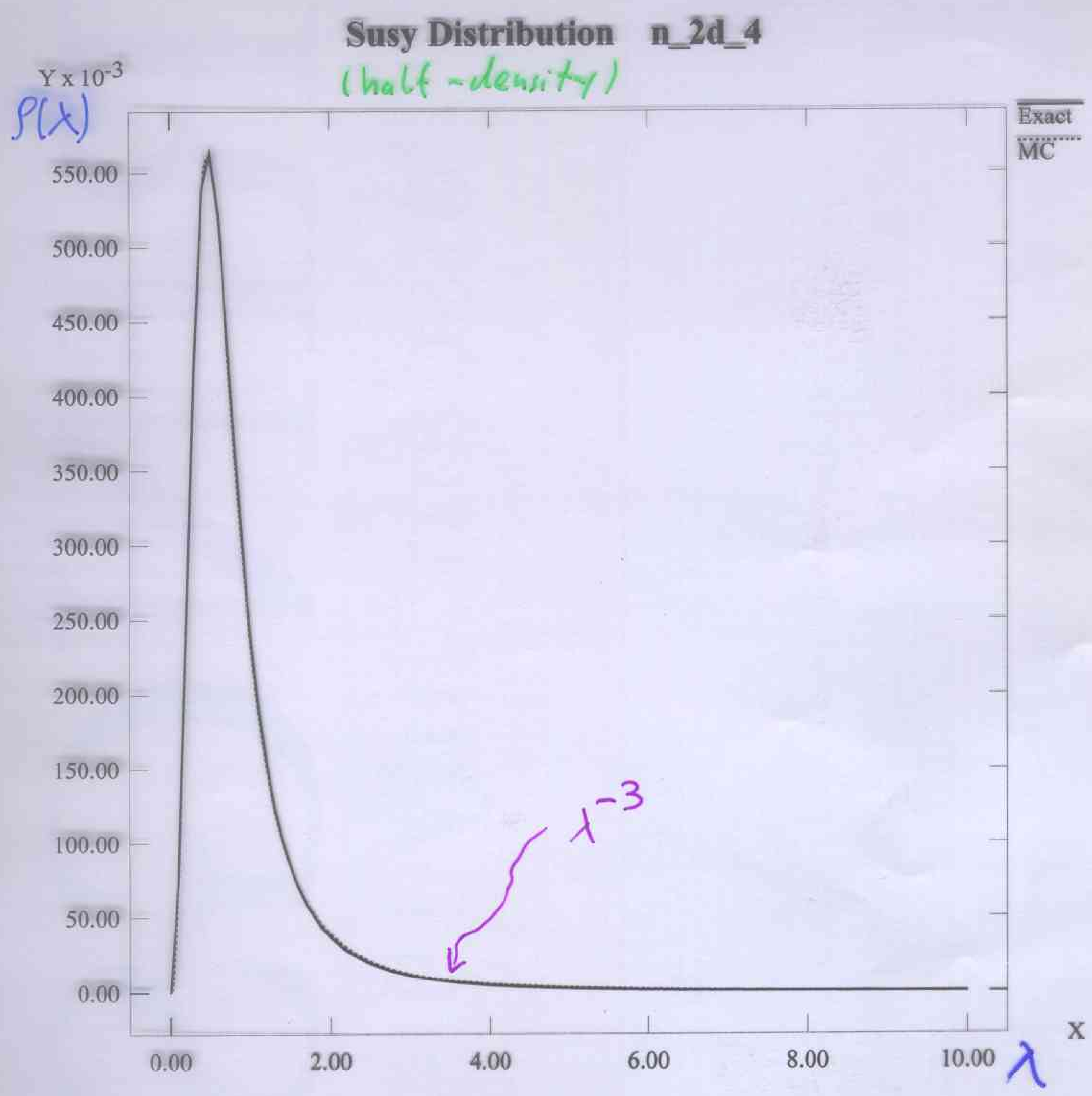
all other $SU(2)$ susy one-matrix correlators do not exist.

- Bosonic $SU(2)$ case:

$$\rho_{D,N=2}^{\mathcal{N}=0}(\lambda)\sim\frac{1}{\lambda^{D-3}}$$

Accuracy of Monte Carlo computation

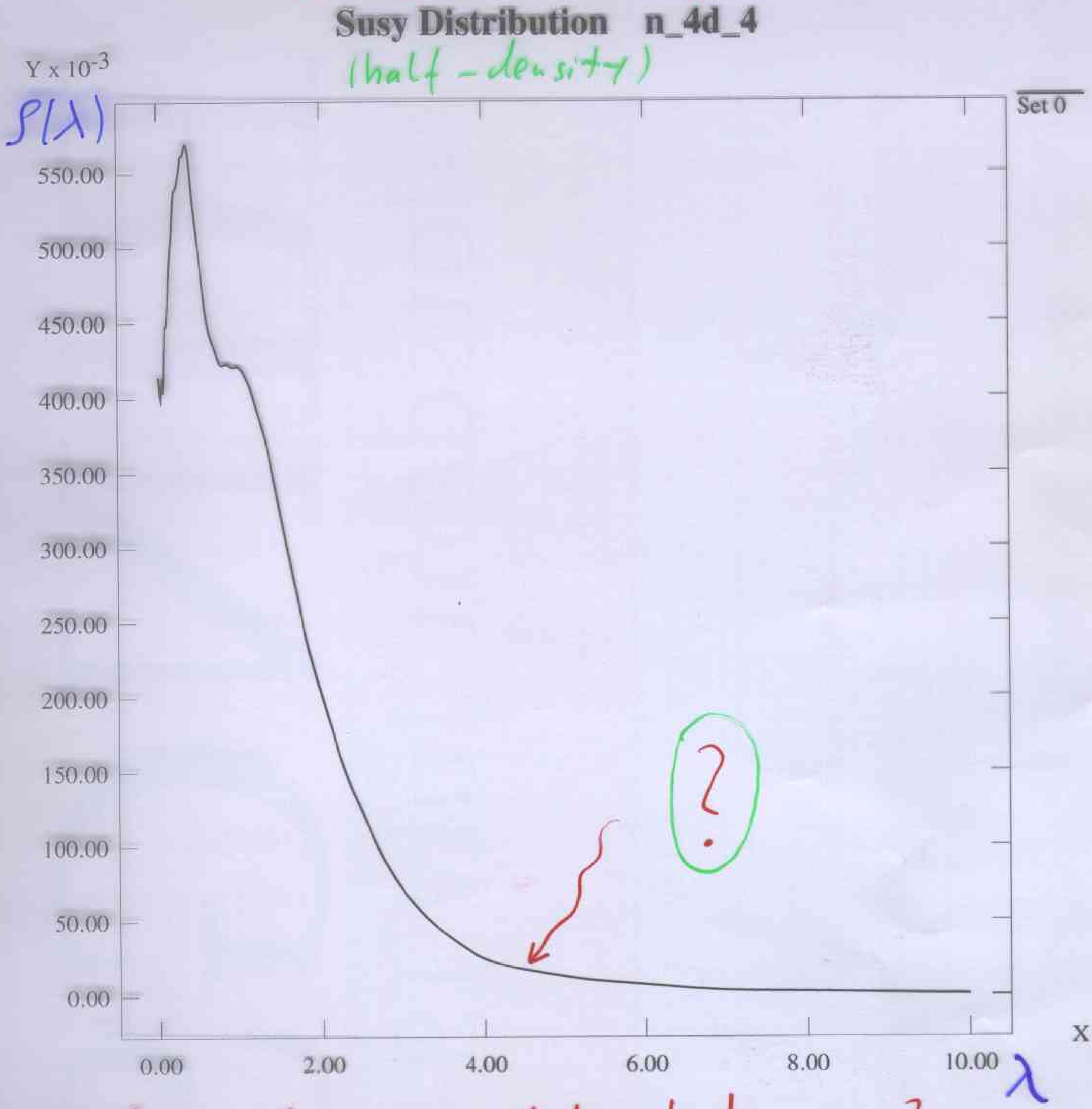
SU(2) D=4 Susy



"Kummer-U à la Monte Carlo"

N > 2 computations

SU(4) D=4 susy



What is the asymptotic behavior?

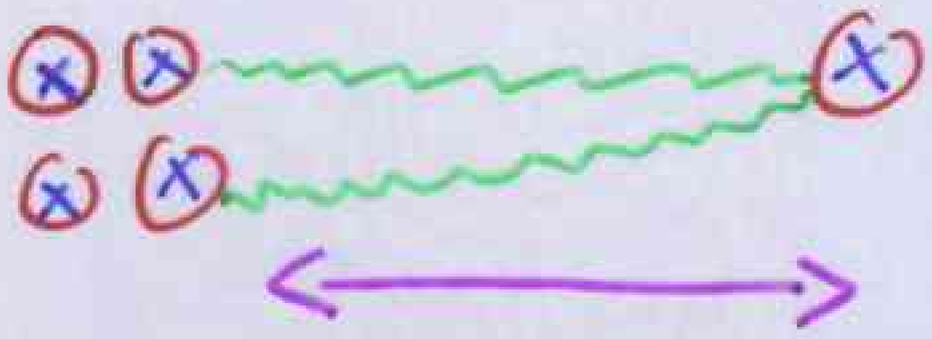
Intuition: One-loop estimates

We had:

$$Z^{susy}_{D,N} \sim \int \prod_i^{N-1} \prod_\mu^D dx^i_\mu \cdot \sum_{\text{trees}} \sum_{\substack{\text{links}\\(ij)}} \frac{1}{(x^i - x^j)^{3(D-2)}}$$

$$Z^{bosonic}_{D,N} \sim \int \prod_i^{N-1} \prod_\mu^D dx^i_\mu \cdot \prod_{i<j} \frac{1}{(x^i - x^j)^{2(D-2)}}$$

Most dangerous "infrared" configuration:



This suggests:

$$\left\langle \frac{1}{N} \operatorname{Tr} X_D^{2k} \right\rangle < \infty$$

if and only if

$$K < D - 3 \qquad \text{susy}$$

$$K < N(D-2) - \tfrac{3}{2} D + 2 \qquad \text{bosonic}$$

In bosonic case, in marked difference to susy, all moments exist as $N \to \infty$!

Is this true non-perturbatively?
$\Rightarrow$ Monte Carlo random walk

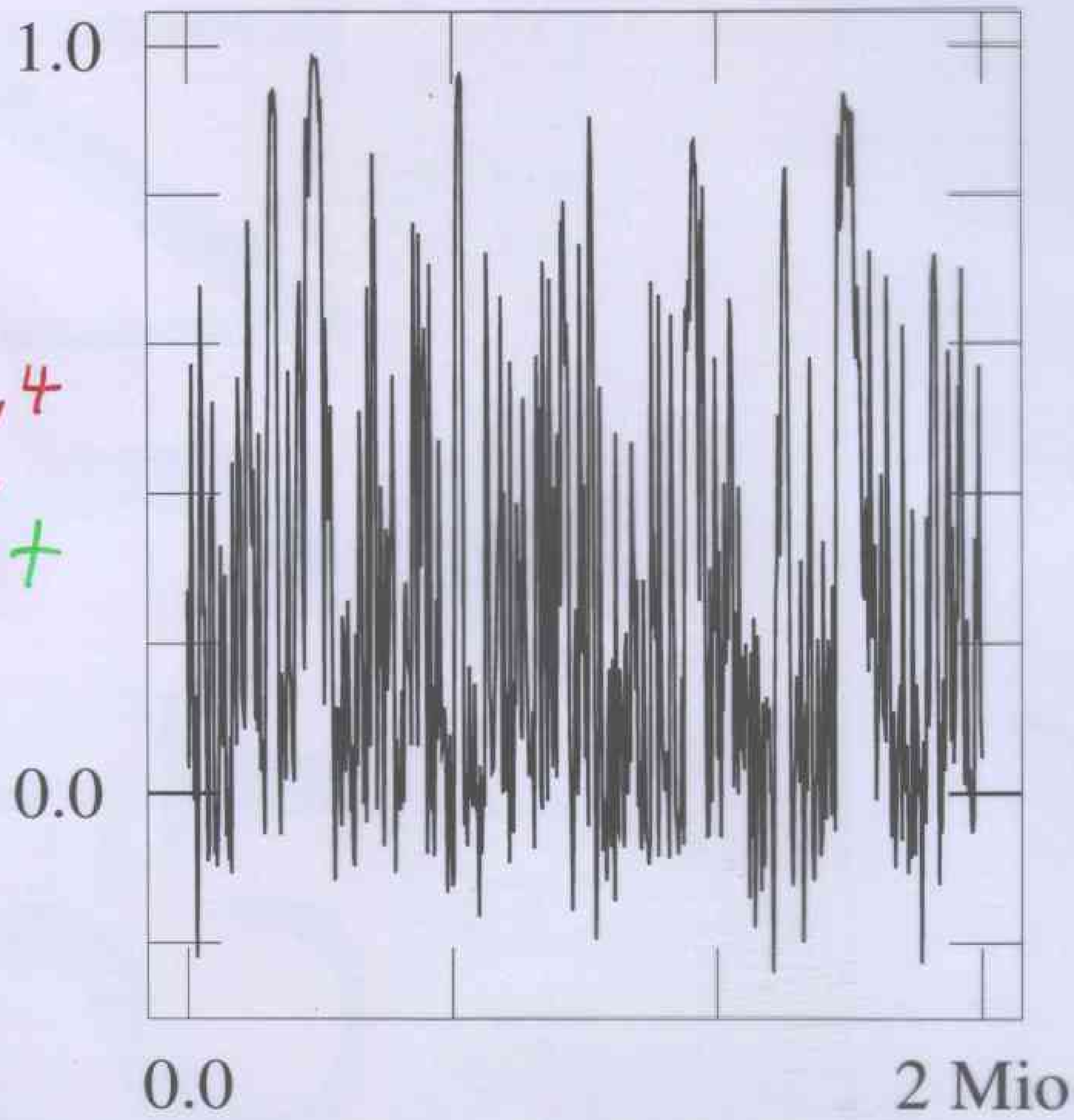
Auto correlation function:

Susy D=6 SU(4)

$\mathrm{Tr}\,X^4$

convergent

moment exists!



Autocorrelation function:

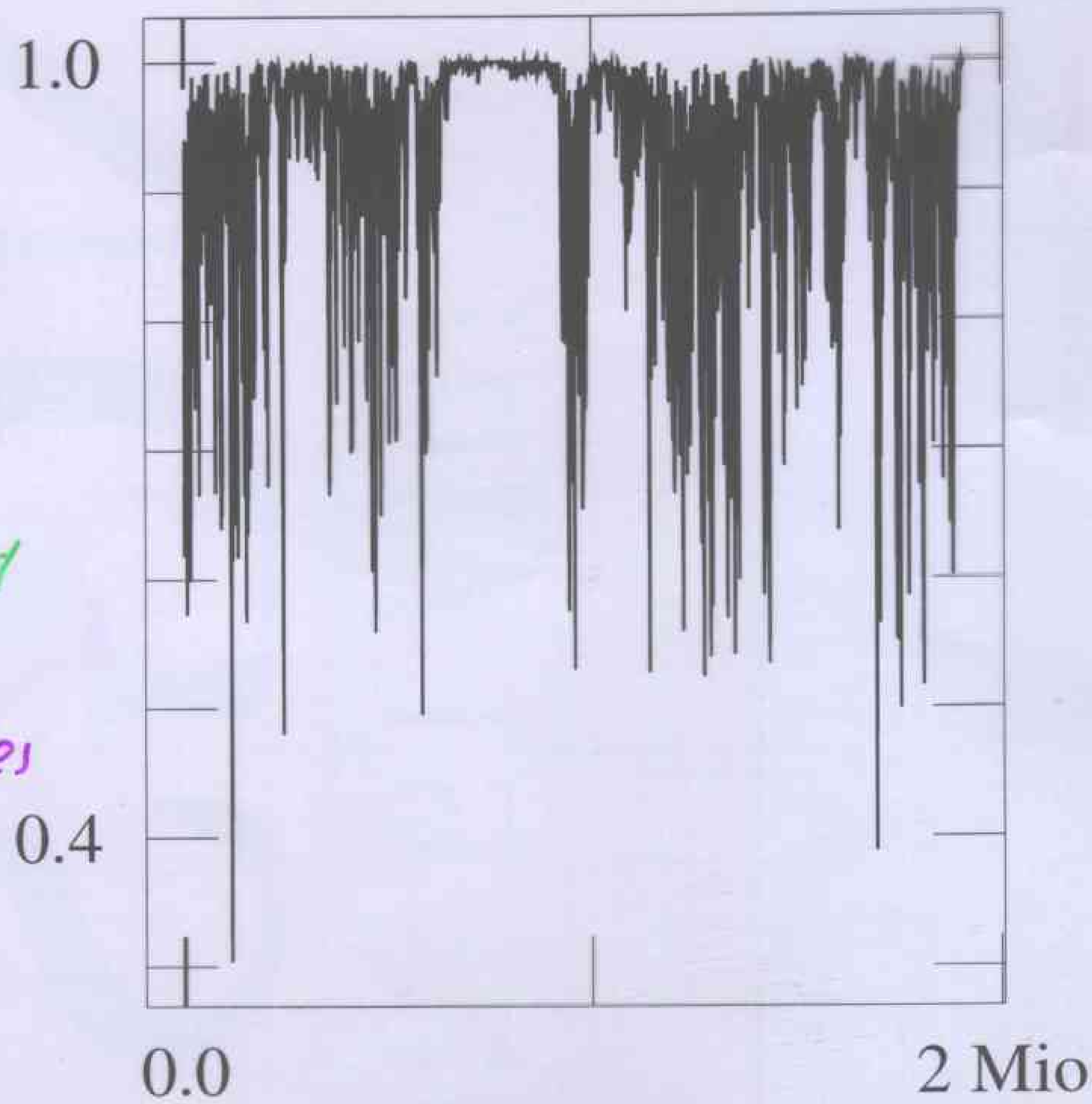
Susy D=6 SU(4)

$\mathrm{Tr}\, X^6$

marginally divergent

moment does

__not__ exist!



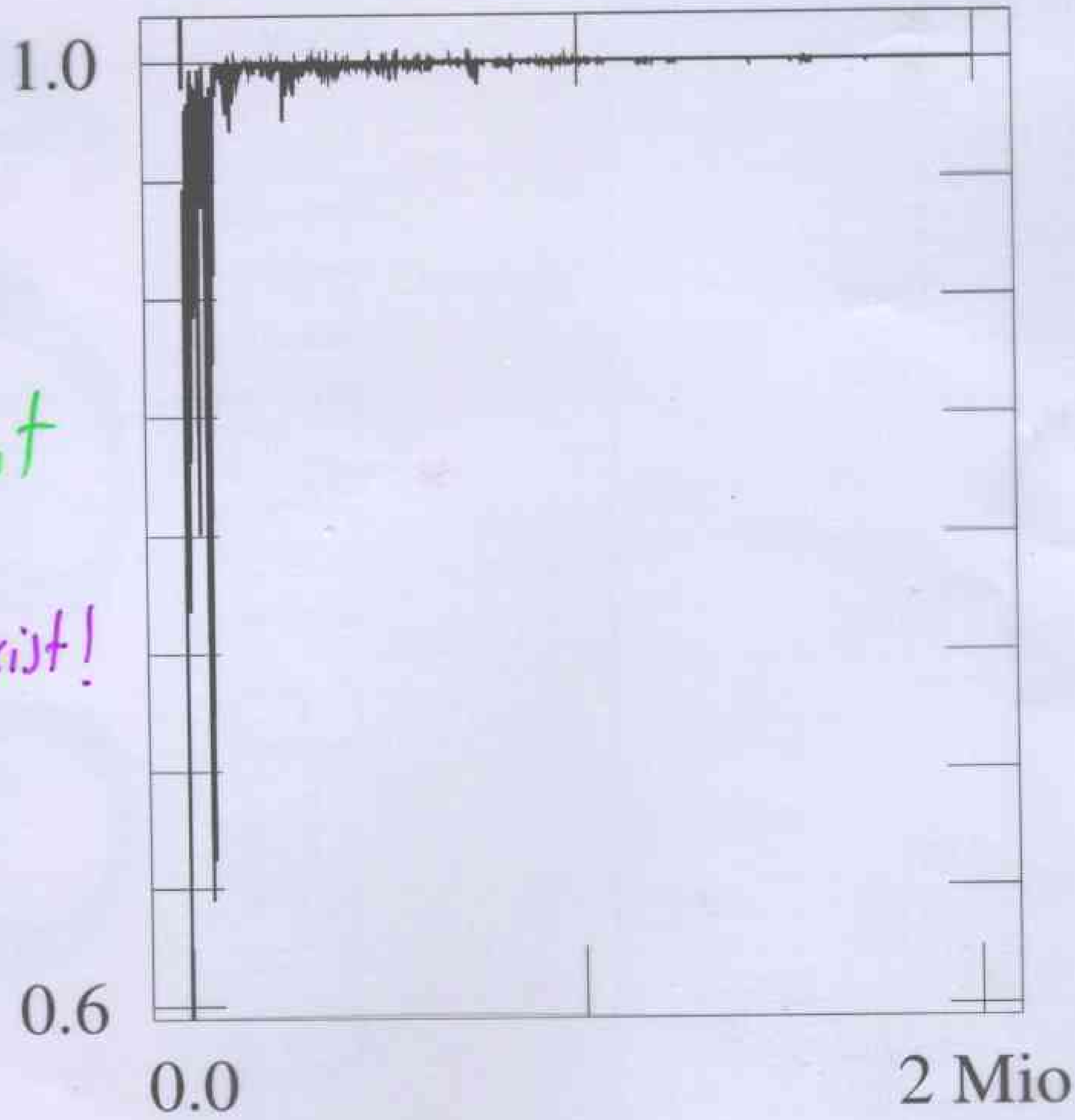
Autocorrelation function:

Susy $D=6$ $SU(4)$

$\mathrm{Tr}\, X^8$

divergent

Does not exist!



Asymptotic eigenvalue density

Susy:

$D=4 \qquad \rho(\lambda) \sim \lambda^{-3} \qquad$ all N

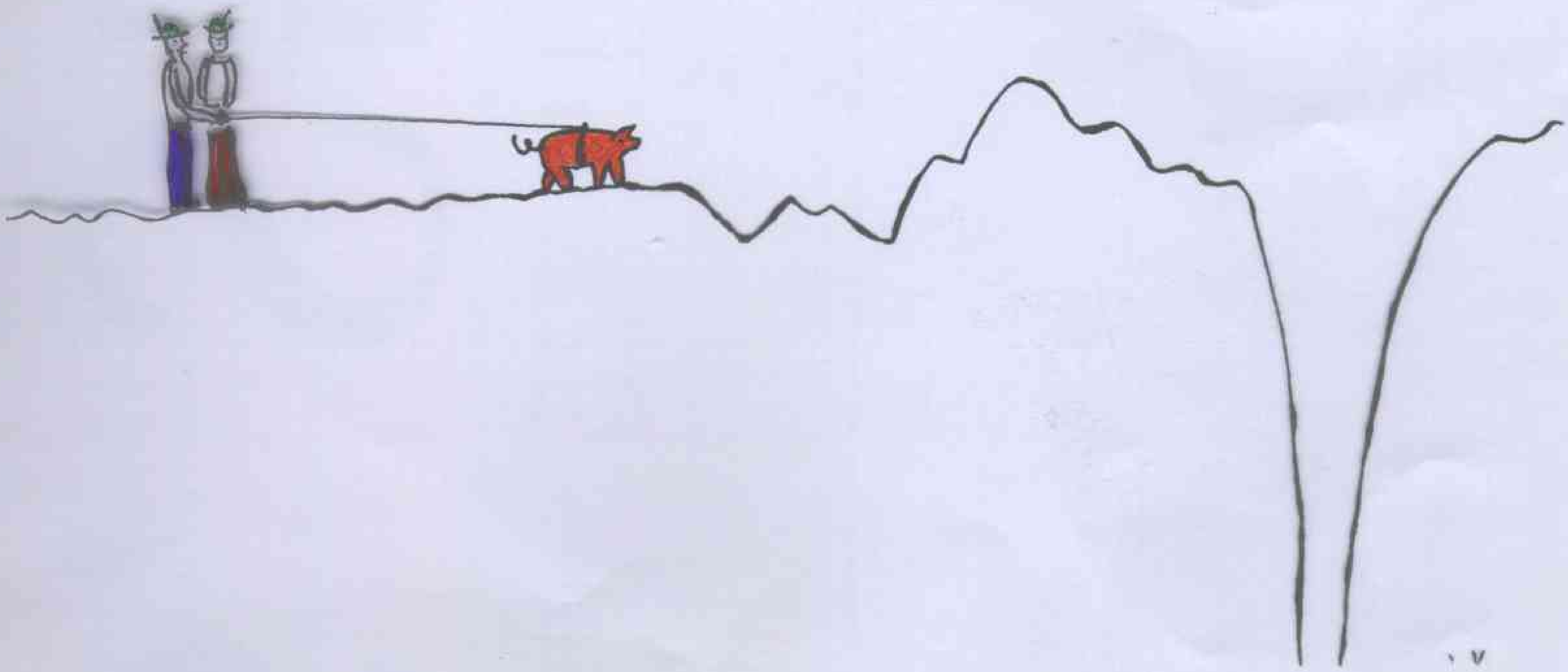
$D=6 \qquad \rho(\lambda) \sim \lambda^{-7} \qquad$ all N

$D=10 \qquad \rho(\lambda) \sim \lambda^{-15} \qquad$ all N

No clear defined edge !

Bosonic:

$D \geq 3 \qquad \rho(\lambda) \sim \lambda^{-2N(D-2)+3D-5}$



Conclusions

- Established non-perturbative convergence conditions for Yang-Mills integrals (w/without susy).
- Ditto for simplest correlators
  $\Rightarrow$ Non-perturbative eigenvalue densities.
- Demonstrated, that numerically precise results may be computed for "not too large" N.

Examples: partition functions,
correlation functions,
Wilson loops,
spectral distributions ...

Open Problems I: analytic

Yang-Mills integrals are

- a challenge to our "analytic toolbox",

and

- an ideal "laboratory".

- Rigorously prove convergence properties
- Calculate analytically partition functions and "observables":

$$\langle \mathrm{Tr}\, A_\mu^K \rangle, \langle \mathrm{Tr}\, F^4 \rangle, \langle \mathrm{Tr}\, P \exp i \oint A \rangle, \ldots$$

Open Problems II: numerical

- Learn to go to higher N, especially in the susy case, especially in the $D = 10$ case.
- Can we say more about the full effective action of diagonal elements ?

  → "Eguchi - Kawai" mechanism
  "self - compactification"

Exciting Prospect:

- Can we make further precise "predictions" for Yang-Mills integrals from "dual" formulations (string theory, Sugra)?
(especially at finite N!)

$\Longrightarrow$

As opposed to, say, Matrix QM, here we have a system that is amenable to non-perturbative analysis!