Calibrating Branes, or PhreMology

- Calibrations & the susy algebra
- Generalized Calibrations
- Kahler → co-associative – Cayley
- WZ realization: Domain walls & their intersections
- WZ 'wallpaper'

- J. Gutowski, G. Papadopoulos & P.K.T.
  "Supersymmetry & generalized calibrations"
  hep-th/9905156

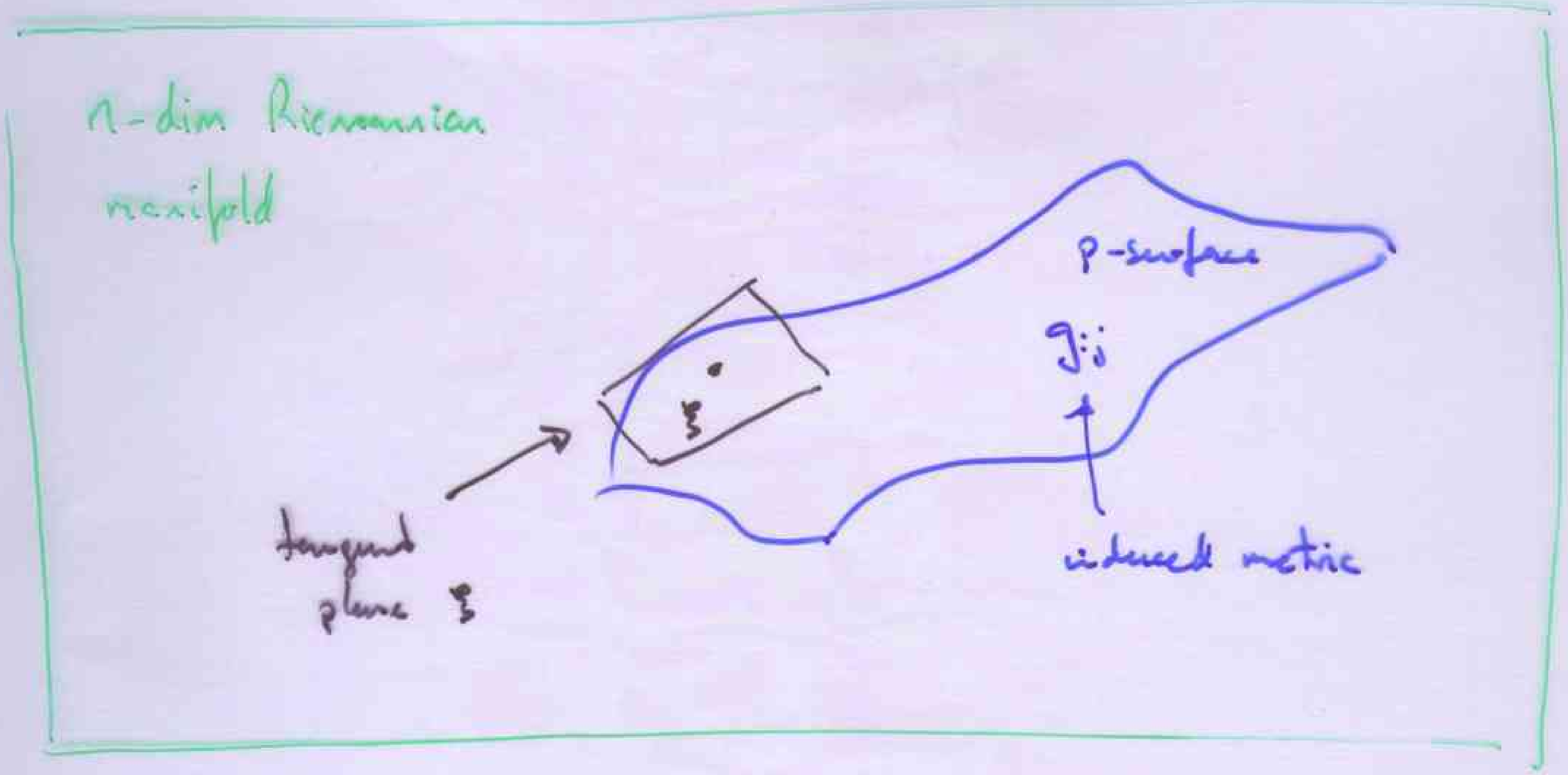
- G. Gibbons and P.K.T.
  "A Bogomol'nyi equation for intersecting domain walls"
  hep-th/9905196

Also

P. Saffin
"Tiling with almost-BPS junctions"
hep-th/9907066

P. Townsend
Strings '99

Calibrations & Minimal Surfaces (Harvey & Lawson)



Tangent plane characterized by p coform $\boxed{\xi = \frac{\partial}{\partial\sigma^1} \wedge \cdots \wedge \frac{\partial}{\partial\sigma^p}}$
where $\left\{\frac{\partial}{\partial\sigma^i}\,;\, i=1,\dots p\right\}$ span plane

A <u>calibration</u> is a p-form ϕ such that

(i) $\xi\cdot\phi \leq \mathrm{vol}_\xi$ ← vol. element on ξ

(ii) $d\phi = 0$

<u>Defn</u>: 'Contact set' $G = \{\xi \,;\, \xi\cdot\phi = \mathrm{vol}_\xi\}$

<u>Theorem</u>: (Harvey & Lawson) G = set of tangent planes of minimal p-surface

Derivation from super p-brane superalgebra

(Gutowski, Papadopoulos & PKT)

In simple cases, e.g. p-brane $\doteq \mathbb{E}^{(n,1)}$

$$\{Q,Q\} = \int_{p\text{-brane}} \Big[\underbrace{\mathrm{vol}(g)}_{\substack{\text{vol form}\\ \doteq \text{ induced}\\ \text{worldspace metric}}} \pm \Gamma_0 \Gamma_{I_1 \cdots I_p} \underbrace{dx^{I_1} \wedge \cdots \wedge dx^{I_p}}_{\substack{\text{closed p-form}\\ \text{'central' charge}}}\Big]$$

Let ϵ be constant spinor with $\underline{\epsilon^\dagger \epsilon = 1}$.

$$(\epsilon^\dagger Q)^2 = \int_{p\text{-brane}} \Big[\mathrm{vol}(g) \pm \phi\Big] \qquad \underline{\phi = \bar{\epsilon}\,\Gamma_{I_1 \cdots I_p}\,\epsilon\, dx^{I_1}\wedge\cdots\wedge dx^{I_p}}$$

Valid for any open set in worldspace so positivity of $(\epsilon^\dagger Q)^2$ implies

(i) $\xi \cdot \phi \leq \mathrm{vol}_\xi$

Topological nature of 'central' charge implies

(ii) $d\phi = 0$

$\Rightarrow$ ϕ is a calibration

Generalized calibrations

(Gutowski & Papadopoulos
G, P & PKT)

The 'simple' cases are those for which

$$\mathcal{H} = \mathrm{vol}(g)$$

p-brane energy density

More generally one has

(p+1)-form potential

$$\boxed{\mathcal{H} = \nu\, \mathrm{vol}(g) + i_k A}$$

'redshift' factor

contraction with timelike Killing vector field k.

In this case one finds, as before, that

(i) $\xi \cdot \varphi \leq \mathrm{vol}_\xi$

but now

(ii) $d\varphi = d\, i_k A \neq 0$

— defines a 'generalized' calibration

Theorem: The contact set of a generalized calibration is the set of p-planes tangent to an <u>$\mathcal{H}$-minimizing</u> p-surface

Supersymmetry & calibrations

(Becker², Morrison, Ooguri, Oz, Yin; Gibbons & Papadopoulos; Gauntlett, Lambert & West; Acharya, Figueroa-O'Farrill, Spence)

Calibrations deduced from susy algebra are

$$\boxed{\varphi = \mathrm{vol}(g)\, \epsilon^{+} \Gamma \epsilon}$$

for some constant spinor ϵ, where

$$\boxed{\Gamma = \frac{1}{p!\,|\mathrm{vol}(g)|}\, \varepsilon^{i_1 \cdots i_p}\, \partial_{i_1} X^{I_1} \cdots \partial_{i_p} X^{I_p}\, \Gamma_0 \Gamma_{I_1 \cdots I_p}}$$

$$\left.\begin{array}{l} \Gamma^2 \equiv 1 \\ \epsilon^{+}\epsilon = 1 \end{array}\right\} \Rightarrow \xi \cdot \varphi \leq \mathrm{vol}_\xi$$

<u>saturated</u> iff $\boxed{\Gamma_\xi \epsilon = \epsilon}$

evaluation of Γ on tangent p-plane ξ

$$\left.\begin{array}{l} \Gamma_\xi^2 = 1 \\ \mathrm{tr}\,\Gamma_\xi = 0 \end{array}\right\} \Rightarrow +1 \text{ eigenspace of } \Gamma_\xi \text{ is } \underline{\text{half-maximal}}$$

i.e. <u>locally</u>, calibrated surface preserves ½ susy

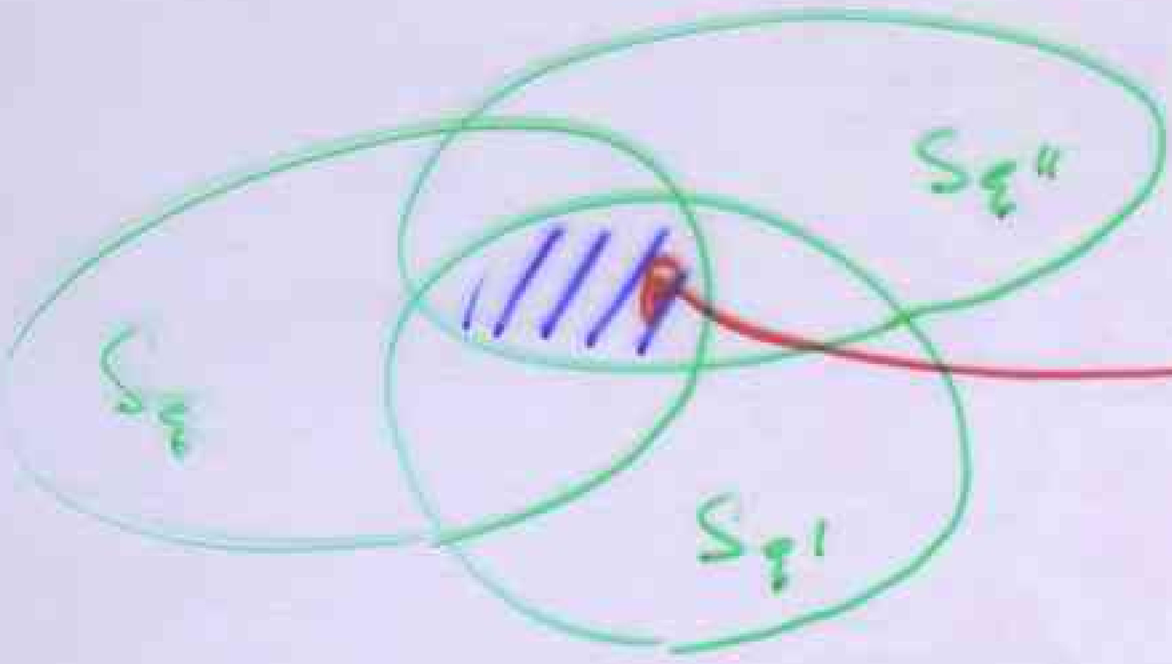
Classification

As ξ varies in contact set G

$$\Gamma_\xi \longrightarrow \Gamma_{\xi'} = R^{-1}\,\Gamma_\xi\, R$$

rotation in some subgroup of $SO(n)$

Let S_ξ be $+1$ eigenspace of Γ_ξ



solution space of $\Gamma_\xi \epsilon = \epsilon$ for all $\xi \in G$

Thus, in general we preserve $\nu \leq 1/2$ fraction of susy

$G = SU(m) \Big/ S\left[U(p)\times U(m-p)\right]$	(Kähler) $(n=2m)$	$\nu = 1/2^m$
$G = SU(m) \Big/ SO(m)$	(SLAG)	$\nu = 1/2^m$
$G = G_2 \Big/ SO(4)$	(Associative × Co-associative)	$\nu = 1/16$
$G = Spin\,7 \Big/ \left[SO(2)\times SU(2)\times S(U2)\right] \Big/ \mathbb{Z}_2$	(Cayley)	$\nu = \frac{1}{32}$

(Taken from Gibbons & Papadopoulos)

Examples (taken from Gauntlett, Lambert, West)

(i) Kähler $n=6$ $p=2$ $\nu = 1/8$

					$p=2$					
(probe)	M5:	1	2	3	4	5				
	M5:	1	2	3			6	7		
	M5:	1	2	3					8	9
		vacuum of $N=1$ $D=4$ theory.			$n=6$					

(ii) Co-associative $(\nu = 1/16)$

				$p=3$						
(probe)	M5:	1	2	3	4	5				
	M5:	1	2	3			6	7		
	M5:	1	2	3					8	9
	M5:	1	2		4		6		8	
				$n=7$						

interpret as domain wall in previous $N=1$ $D=4$ theory

e.g. in MQCD (Witten, Volovich, ...)

N=1 D=4 susy analysis

(Abraham, PKT
Gibbons, PKT)

$$\{Q, Q\} = H + \Gamma^{0i} P_i + \Gamma^{0ij} U_{ij} + \Gamma^{0ij} \gamma_5 V_{ij}$$

4 component spinor charge

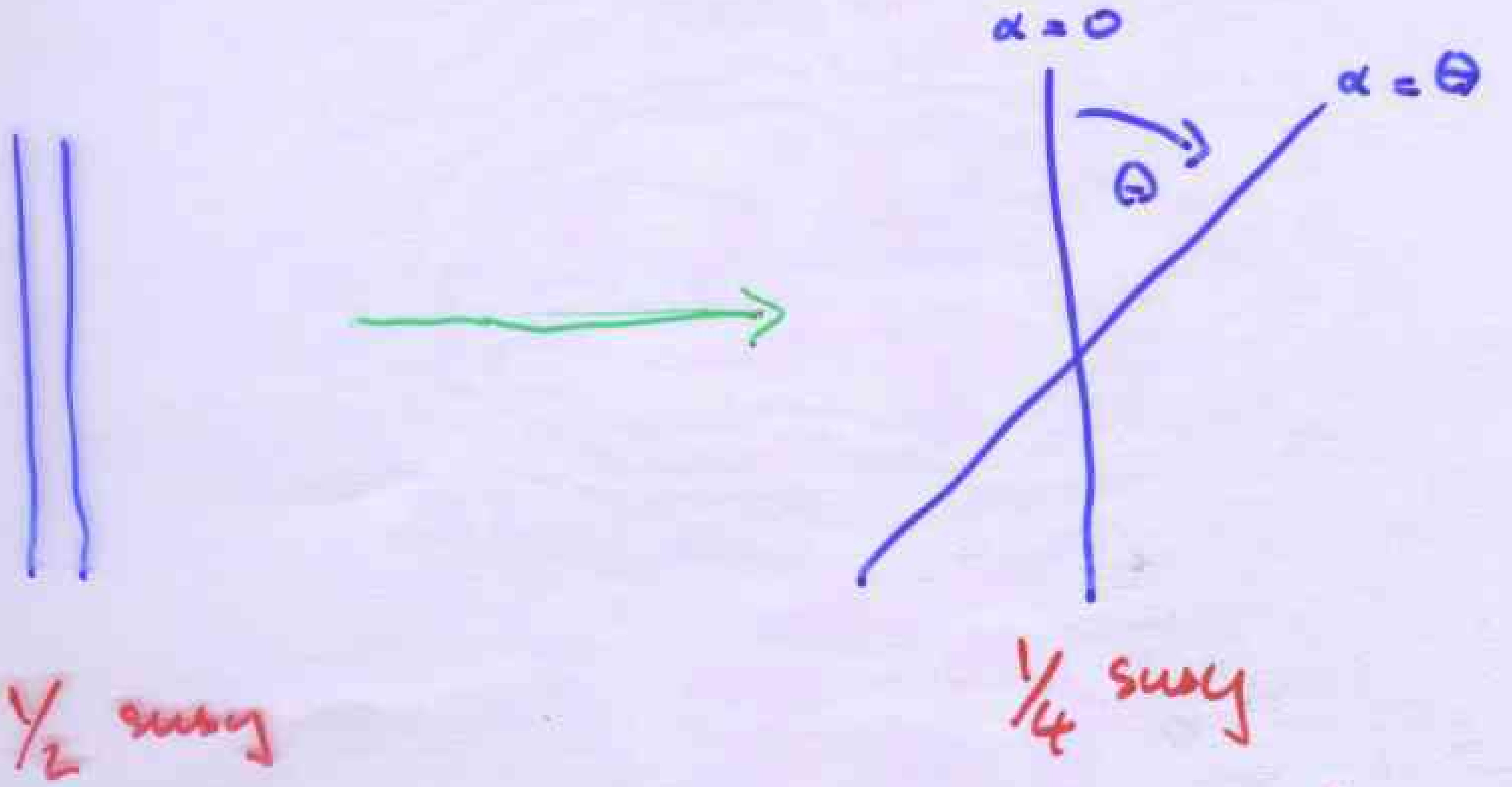
'electric' & 'magnetic' 2-form charges

Domain walls are associated with constraint

$$\Gamma^{0ij} e^{\alpha \gamma_5} \epsilon = \epsilon \quad \Rightarrow \quad \tfrac{1}{2} \text{ susy} \quad (\nu = 1/16)$$

some angle

Take 2 parallel domain walls. Get 1/4 susy by simultaneous rotation in space and charge space (α)



1/2 susy
$(\nu = 1/16)$

1/4 susy
$(\nu = 1/32)$

Relation to calibrations (Gauntlett)

Consider Cayley calibration via intersecting M5-branes

$p = 4$ (directions 2–5)

		1	2	3	4	5	6	7	8	9
(probe)	M5	1	2	3	4	5				
	M5	1	2	3			6	7		
	M5	1	2	3					8	9
	M5	1	2		4		6		8	
	M5	1		3	4		6			9

$n = 8$ (directions 2–9)

interpret as domain walls intersecting on 1-axis, preserving $\nu = 1/32$ susy $\Rightarrow$ $\frac{1}{4}$ of $\mathcal{N}=1$ $D=4$

Domain walls in $\mathcal{N}=1$ $D=4$ can be realized in WZ model with superpotential $W(\Phi)$ (Abraham, PKT; Cvetič, Quevedo, Rey; Dvali & Shifman)

$$\boxed{\mathcal{H} = \frac{1}{4} \underset{\sim}{\nabla}\phi \cdot \underset{\sim}{\nabla}\bar{\phi} + |W'(\phi)|^2}$$

What about intersecting domain walls?

Bogomol'nyi eq for WZ domain wall junctions

(Gibbons, P.K.T.)
(Carroll, Hellerman & Trodden)

Reduce to 2D $\underline{\nabla} = (\partial_x, \partial_y)$

Let $\underline{z = x + iy}$. Then

$$\mathcal{H} = \left| \frac{\partial \phi}{\partial \bar{z}} \mp \bar{W}' \right|^2 \pm 2\,\mathrm{Re}\left(\frac{\partial W}{\partial z}\right) + \frac{1}{2} J(z, \bar{z})$$

$$J = \frac{\partial \phi}{\partial z}\frac{\partial \bar{\phi}}{\partial \bar{z}} - \frac{\partial \phi}{\partial \bar{z}}\frac{\partial \bar{\phi}}{\partial z}$$

Define

$$T = i \int dz\, d\bar{z}\, \frac{\partial W}{\partial z} \qquad Q = \frac{i}{4}\int d\bar{\phi} \wedge d\phi$$

complex domain wall charge

real domain wall junction charge

Then

$$H \geqslant |T| + |Q|$$

with equality when

$$\boxed{\frac{\partial \phi}{\partial z} = \bar{W}'}$$

have to solve this to find intersecting WZ domain walls preserving 1/4 susy

WZ junctions

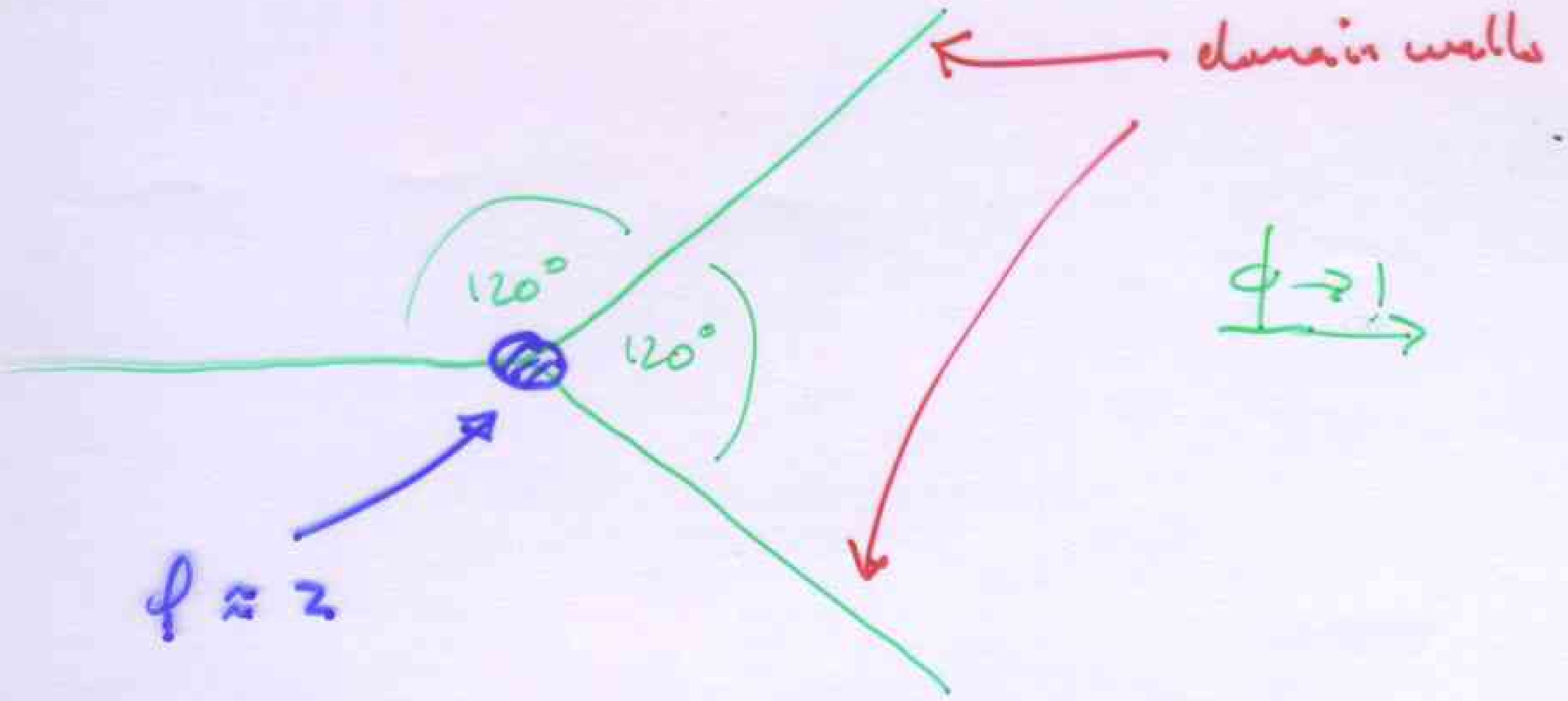
Take $W = \phi - \frac{1}{4}\phi^4 \Rightarrow W' = 1 - \phi^3$

Then Bog. eq. is

$$\boxed{\frac{\partial \phi}{\partial z} = 1 - \bar{\phi}^3}$$

This has $\underline{\mathbb{Z}_3\text{-symmetry}}$: $(z, \phi) \to (\omega z, \omega \phi)$ $\omega^3 = 1$

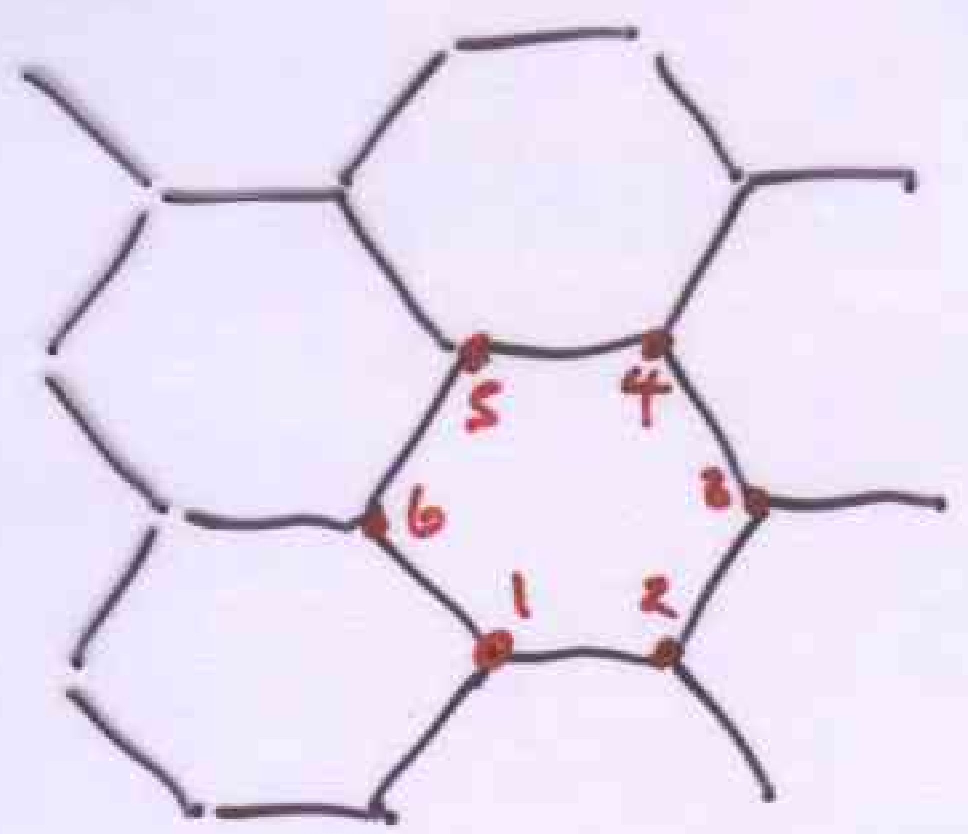
so seek solution of form



- Solution exists in thin-wall limit (Sternberg & Zeimer, Bronsard & Reitich)
- Numerical results confirm existence in general (Saffin)

WZ wallpaper (Saffin)

WZ junctions can be combined to get 'almost-BPS' networks.



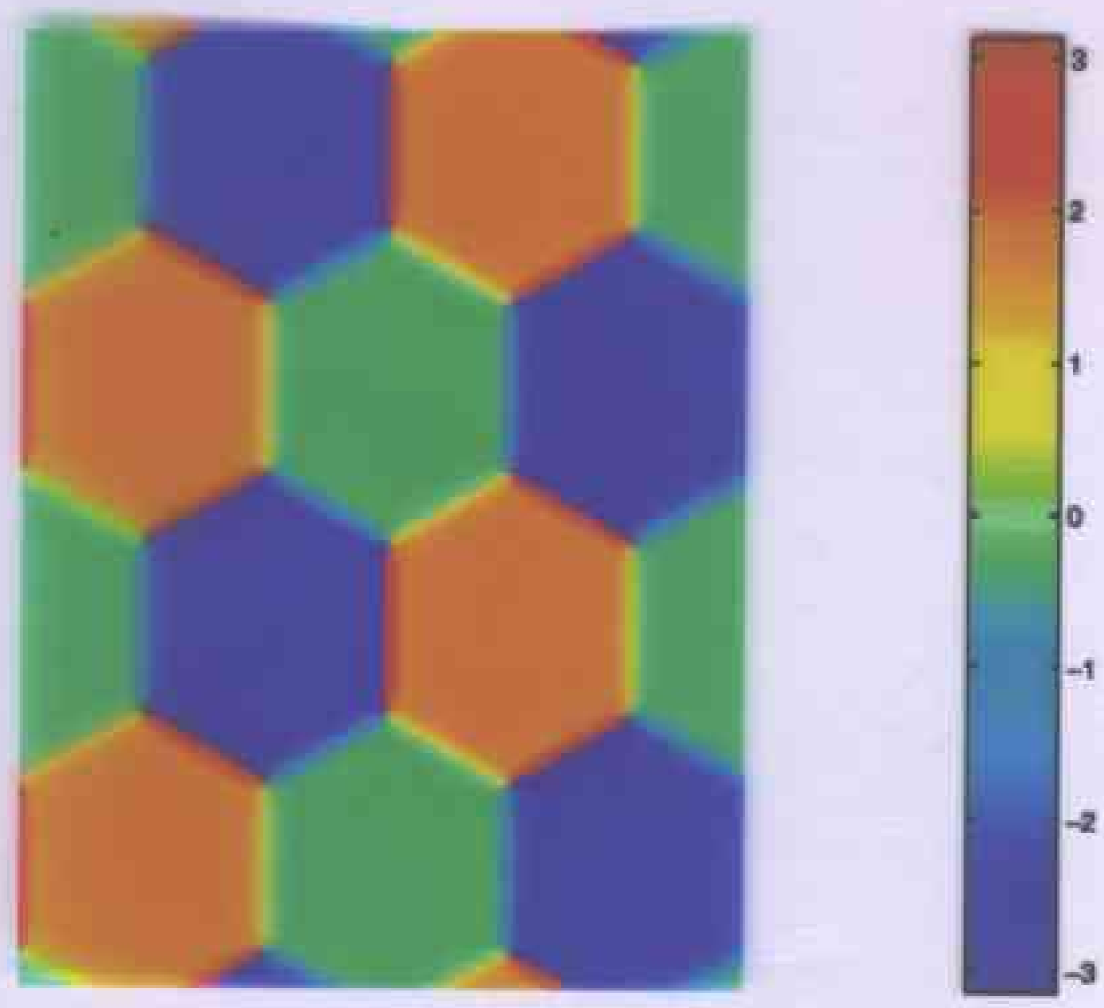
6 vertices are $\begin{cases} \mathbb{Z}_3 \text{ perms of junction} \\ \mathbb{Z}_3 \text{ perms of } \underline{\text{anti-junction}} \end{cases}$

so not BPS

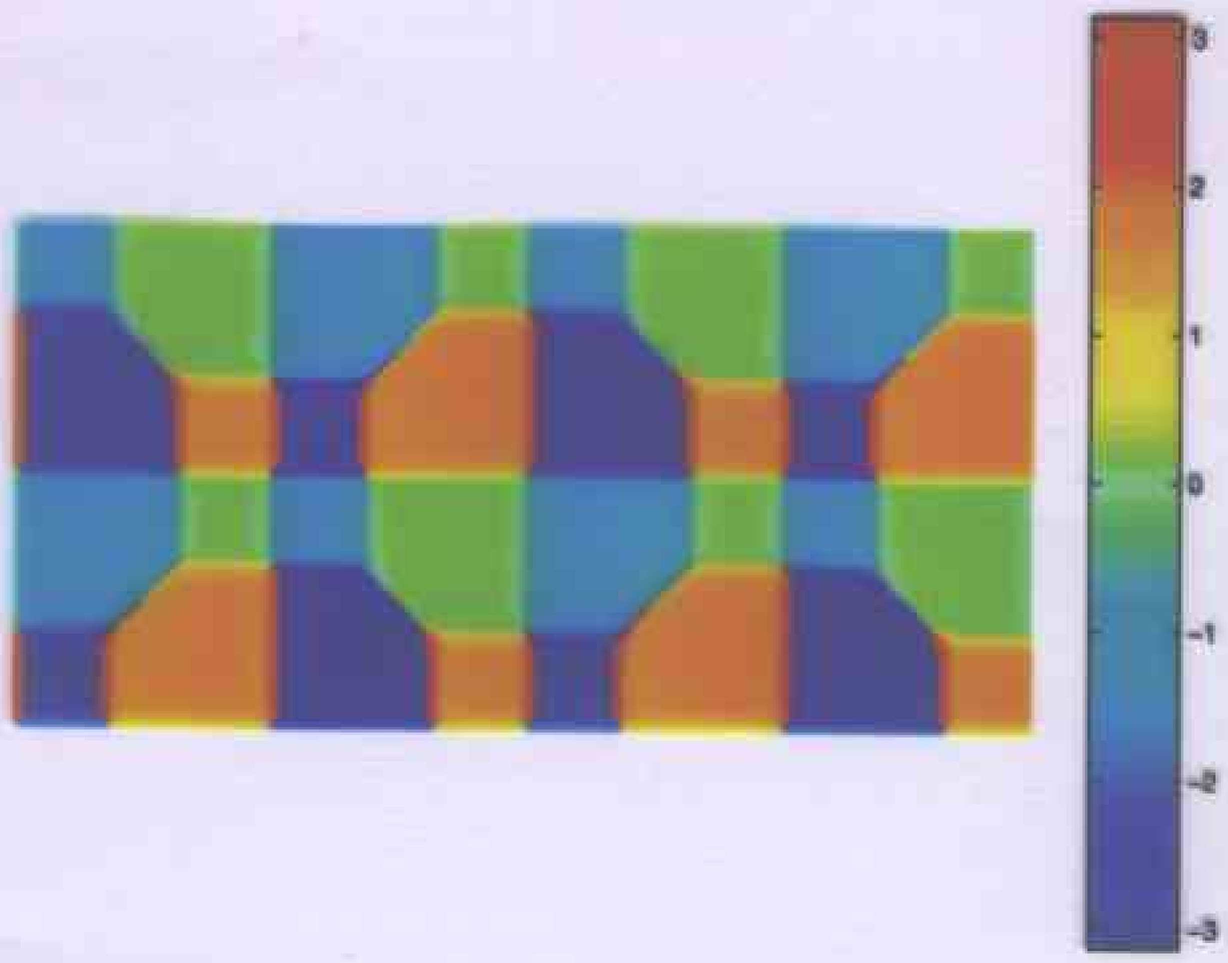
But network is meta-stable - decays by a cell-nucleation tunnelling (so classically stable)

For different choices of $W(\phi)$ different 'wallpaper' patterns are possible → tilings of plane

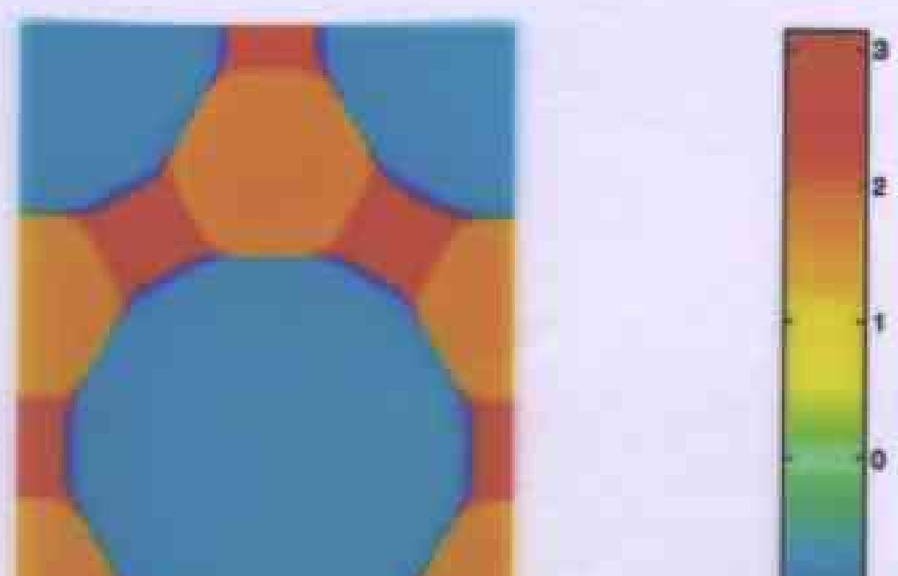
$W' = 1 - \phi^3$



$W' = 1 - \phi^4$



$W' = 1 - \phi^6$



$W' = 1 - \phi^6$