Renormalization Group Flows
from
Five-Dimensional Supergravity

N.P. Warner, July1999

- **Based on:**

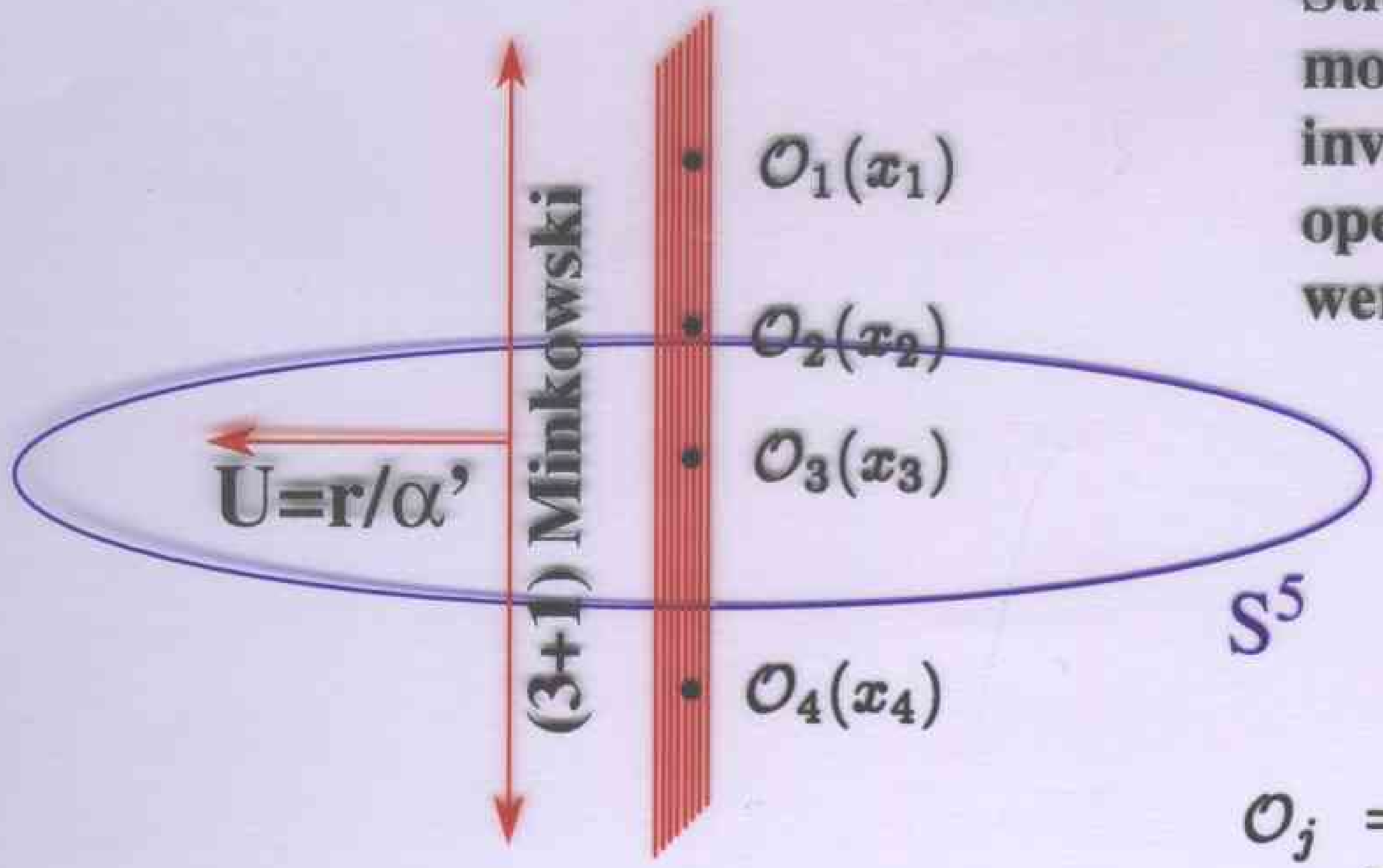
A. Khavaev, K. Pilch and N.P. Warner, hep-th /9812035
D. Freedman, S. Gubser, K. Pilch and N.P. Warner, hep-th /9904017, hep-th /9906194

- **Bulk/Boundary Correspondence**
- **Gauged $\mathcal{N}=8$ Supergravity in five dimensions**
- **$\mathcal{N}=1$ Supersymmetric phase of $\mathcal{N}=4$ Yang-Mills**
- **Renormalization group flow between supersymmetric phases**
- **Flows to *Hades* and the Coulomb branch**
- **Conclusions**

Bulk/Boundary Correspondence

D3 branes in IIB :

For references see the review hep-th /9905111



String/supergravity modes couple to gauge invariant Yang-Mills operators as if they were sources

$\mathcal{O}_j$ = Gauge Invariant Yang-Mills Operators

$$ds^2 = \alpha'\Big[\underbrace{\frac{U^2}{\sqrt{4\pi g_s N}}\big(\overbrace{-dt^2+dx_1^2+dx_2^2+dx_3^2}^{\text{D3-brane}}\big) + \sqrt{4\pi g_s N}\,\frac{dU^2}{U^2}}_{\text{Anti-de Sitter: AdS}_5} + \sqrt{4\pi g_s N}\,\underbrace{d\Omega_5^2}_{S^5}\Big]$$

$$\left\langle e^{-\int \varphi_j^{(0)}(x)\,\mathcal{O}_j(x)\,d^4x}\right\rangle_{brane} = Z_{string}[\varphi] \xrightarrow[\substack{\alpha'\to 0\\ g_sN\to\infty}]{} Z_{SG}[\varphi] \xrightarrow[g_s\to 0]{} e^{-S_{SG}[\varphi]}$$

where

- $\varphi(x,U) \to \varphi^{(0)}(x)$ on the "boundary" brane, which is located at $U=\infty$
- $S_{SG}[\varphi]$ is the classical IIB supergravity action evaluated on the classical solution with boundary conditions above

Gauged $\mathcal{N}=8$ Supergravity in Five Dimensions

- Consider the IIB Supergravity theory in terms of an effective five-dimensional field theory on AdS_5

Massless graviton supermultiplet on AdS_5
Gauged $\mathcal{N}=8$ Supergravity in Five Dimensions ↔ **$\mathcal{N}=4$ Supersymmetric Yang – Mills Energy – Momentum Tensor Supermultiplet**

42 Scalars of $E_{6(6)}/USp(8)$ ↔ **Masses and couplings**

SO(6) Reps

$$\mathbf{20'} \Leftrightarrow SL(6,\mathbb{R})/SO(6) \longleftrightarrow \mathcal{M}_{AB}\,\mathrm{Tr}(X^A\,X^B)$$

$$\mathbf{10} \oplus \bar{\mathbf{10}} \longleftrightarrow m_{ab}\,\mathrm{Tr}(\lambda^a\,\lambda^b) + c.c.$$

$$\mathbf{1} \oplus \mathbf{1} \Leftrightarrow SL(2,\mathbb{R})/SO(2) \longleftrightarrow \text{Couplings: } g_{YM},\ \theta$$

- Complete **non-linear structure** of five-dimensional supergravity theory determined by:
  - **Field Content**
  - **Supersymmetry**
  - **Gauge Symmetry** (SO(6))

- *Gauged $\mathcal{N}=8$ Supergravity in Five Dimensions* $\xrightarrow{\textbf{Embedded}}$ *IIB Supergravity*

$\Rightarrow$ • **Solution** ⟶ • **Solution**

Behaviour of $\mathcal{N}=4$ Yang-Mills to all orders under these mass perturbations will be determined by **Yang-Mills**

- **Field Content**
- **Supersymmetry**
- **Global Symmetry**

and by the (non-linear) equations of motion of

Gauged $\mathcal{N}=8$ Supergravity in Five Dimensions

Phases and the Supergravity Potential

- **The five-dimensional supergravity theory has a complicated non-linear potential for the scalars**

Critical point of potential ⟷ **AdS Vacuum of Supergravity** → **Conformal Phase of Yang-Mills**

Provided(?) AdS vacuum is stable

- Central charge of new fixed point follows from the cosmological constant

$$c_{IR} = \left(\frac{\Lambda_{UV}}{\Lambda_{IR}}\right)^{3/2} c_{UV}$$

Henningson and Skenderis, hep-th /9812032

- Supersymmetry $\Rightarrow$ Stability
  - General dogma: Supersymmetric vacuum of supergravity $\Rightarrow$ Good vacuum of the underlying string theory

$\Rightarrow$ **Supersymmetric critical points** ⟷ **IR phases of Yang-Mills at Finite N**

All phases with at least SU(2) symmetry:

Critical Point	Unbroken Gauge Symmetry	Cosmological Constant	Unbroken Supersymmetry	CentralCharge c_{IR}/c_{UV}
$(i)^{*}$	$SO(6)$	$-\frac{3}{4}g^2$	$\mathcal{N}=8$	1
$(ii)^{**}$	$SO(5)$	$-\frac{3^{5/3}}{8}g^2$	$\mathcal{N}=0$	$\frac{2\sqrt{2}}{3} \sim 0.9428$
$(iii)^{**}$	$SU(3)$	$-\frac{27}{32}g^2$	$\mathcal{N}=0$	$\frac{16\sqrt{2}}{27} \sim 0.8381$
$(iv)^{\dagger}$	$SU(2) \times U(1) \times U(1)$	$-\frac{3}{8}(\frac{25}{2})^{1/3}g^2$	$\mathcal{N}=0$	$\frac{4}{5} = 0.8$
$(v)^{*}$	$SU(2) \times U(1)$	$-\frac{2^{4/3}}{3}g^2$	$\mathcal{N}=2$	$\frac{27}{32} \sim 0.8438$

* **stable**   ** **unstable**   † **stability unknown**

Non-linear predictions from Supergravity

- The potential of gauged $\mathcal{N}=8$ supergravity in five dimensions must determine the phase diagram of $\mathcal{N}=4$ Yang-Mills theory under soft-supersymmetry breaking (i.e. by masses)
- There must be an $\mathcal{N}=1$ supersymmetric phase of $\mathcal{N}=4$ Yang-Mills theory obtained by turning on some (very specific) masses and flowing to the IR:

$$\boxed{m\ \Phi_3^2 \text{ where } \Phi_i\ , i = 1,2,3 \text{ are the } \mathcal{N}=1 \text{ chiral superfields}}$$

- This phase must have an SU(2) global symmetry and a U(1) R-symmetry
- This phase must have: $c_{IR} = \frac{27}{32}\, c_{UV}$
  - General dogma suggests that this phase should be a feature at finite N

This $\mathcal{N}=1$ phase of $\mathcal{N}=4$ Yang-Mills has been identified as one of a class of fixed points discovered by Leigh and Strassler (hep-th /9503121) by purely field theoretic methods ..... ⇒ $\boxed{\Phi_i \text{ have } \gamma_i = -\frac{1}{4}\ , i = 1,2}$

S.S. Gubser, M.Strassler, private communications
Karch, Lust and Miemiec, hep-th/9901041
D. Freedman, S. Gubser, K. Pilch and N.P. Warner, hep-th /9904017

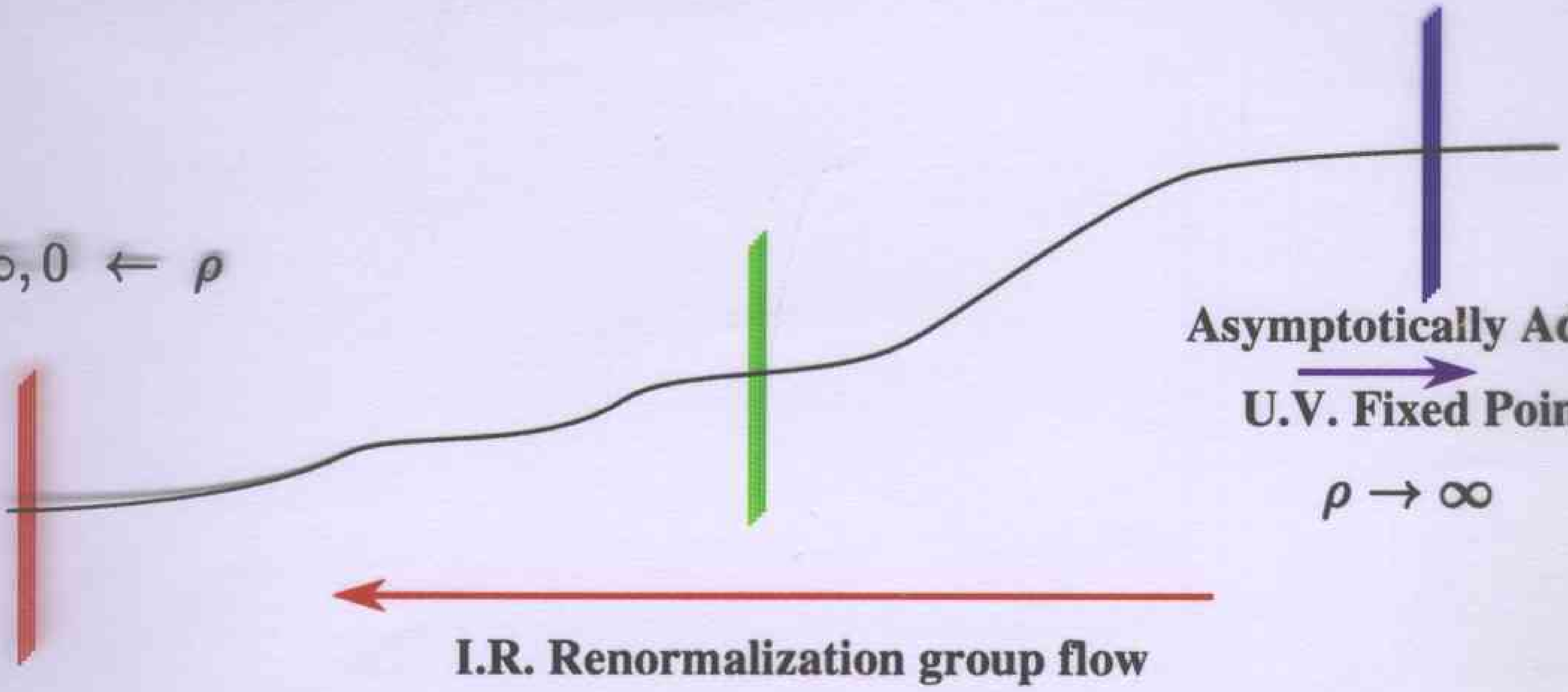
- Central charge matches perfectly
- Operators generating the Yang-Mills flow correspond properly with the supergravity fields defining the critical point
- Small oscillation analysis in supergravity correctly matches with the anomalous dimensions at the non-trivial fixed point
- Supergravity predicts more anomalous dimensions that those computed by field theory........

Renormalization group flow from Supergravity

Girardello, Petrini, Porrati and Zaffaroni, hep-th /9810126
Distler and Zamora, hep-th/9810206

Focus of the 5-geometry Use a new radial variable, ρ, for which:

$$ds_5^2 = d\rho^2 + e^{2A(\rho)}\ \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right)\ , \quad \rho \sim \log(U), U \to \infty$$



$-\infty, 0 \Leftarrow \rho$

Asymptotically AdS$_5$

U.V. Fixed Point

$\rho \to \infty$

I.R. Renormalization group flow

- UV fixed point is $\mathcal{N}=4$ Yang-Mills theory
- Use the supergravity equations of motion to determine RG flow under the appropriate relevant perturbations
- Flow of masses (and couplings) are obtained from the (classical) behaviour of supergravity fields as a function of ρ
- RG flows under mass perturbations will be entirely determined by the equations of gauged $\mathcal{N}=8$ supergravity in five dimensions
- c-theorem for: $\boxed{c\ =\ (A'(\rho))^{-3}}$

Girardello *et al.* , hep-th /9810126
D. Freedman, S. Gubser, K. Pilch and N.P. Warner, hep-th /9904017

If this generalized holographic principle is correct then we must be able to find a flow (= supergravity kink) that interpolates between the $\mathcal{N}=4$ phase and the $\mathcal{N}=1$ phase and preserves $\mathcal{N}=1$ supersymmetry **all along the flow**........

Supersymmetric RG Flow

Strategy: Find supergravity kinks that obey the first order equations:

$$\delta\chi_{abc} = \sqrt{2}\left[\gamma^{\mu} P_{\mu\, abcd}(\varphi_j)\, \epsilon^d - \tfrac{1}{2} g\, A_{dabc}(\varphi_j)\, \epsilon^d\right] = 0$$

$$\delta\psi_{\mu a} = \mathcal{D}_{\mu}\epsilon_a - \tfrac{1}{6} g\, W_{ab}(\varphi_j)\, \gamma_{\mu}\epsilon^b = 0$$

for some choice of supersymmetry parameters ϵ^a

Generic result:

- First equation determines the evolution of the scalars, φ_j
- Second equation determines the 5-metric ($A(\rho)$) in terms of the supergravity scalars, φ_j

Typically find that there is an eigenvalue, $W(\varphi_j)$, of the tensor $W_{ab}(\varphi_j)$ such that the equations of motion become

$$\frac{d\varphi_j}{d\rho} = \frac{g}{2}\,\frac{\partial W}{\partial \varphi_j} \qquad\qquad A' = -\frac{g}{3}\, W$$

$c = (A'(\rho))^{-3} \sim W^{-3}$ ↔ Flow = steepest descent of c-function

Consider the two parameter, relevant perturbation:

$$\varphi_1\, \mathrm{Tr}(\lambda^4\, \lambda^4) + \varphi_2\, \mathrm{Tr}\big((X^5\, X^5) + (X^6\, X^6)\big)$$

= making an $\mathcal{N}=1$ chiral superfield Φ_3 massive

On this sub-sector the supergravity potential is

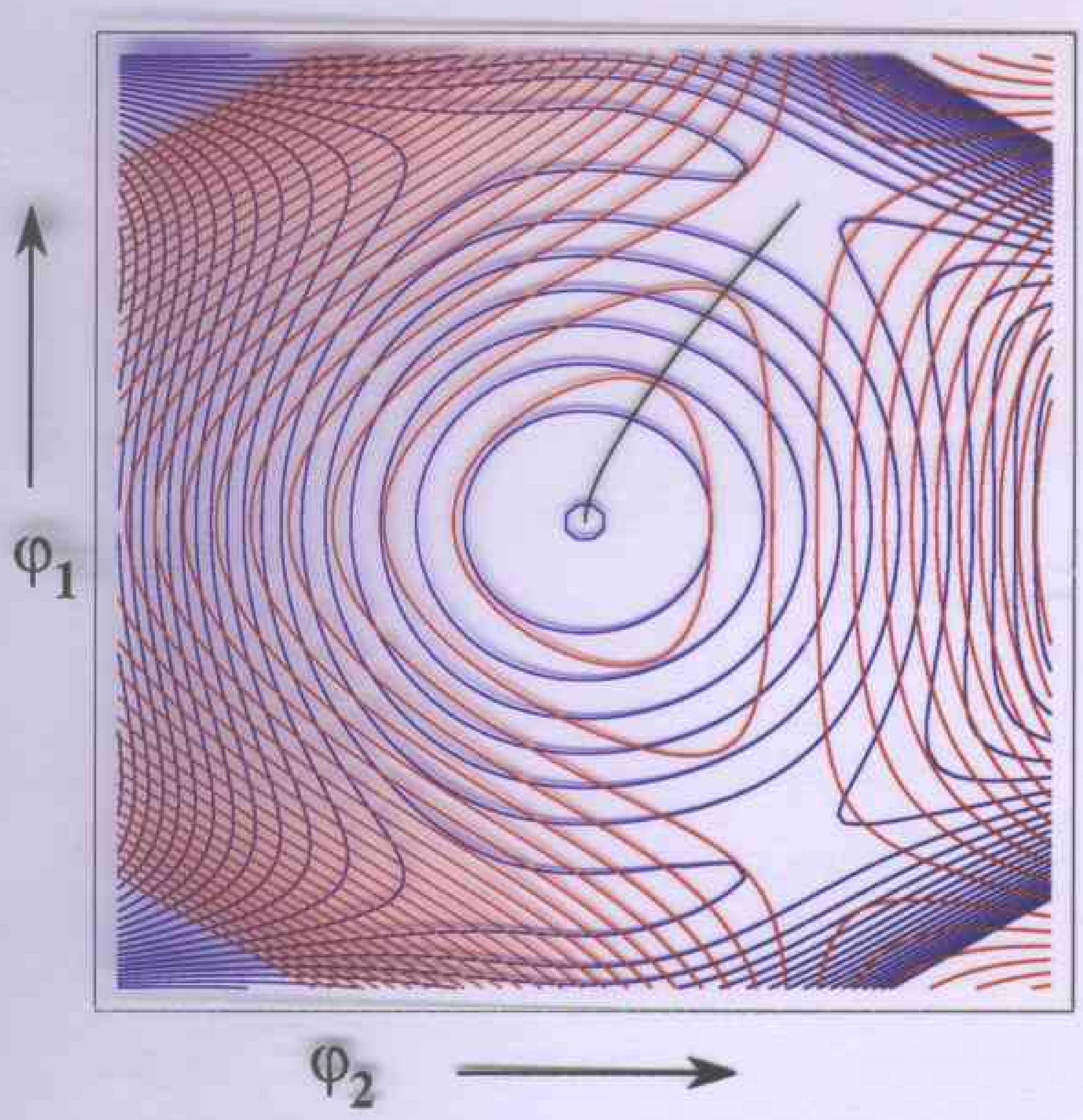
$$V = \frac{g^2}{8} \sum_{j=1}^{2} \left|\frac{\partial W}{\partial \varphi_j}\right|^2 - \frac{g^2}{3}\, |W|^2$$

where W is a superpotential:

$$W = \frac{1}{4\sigma^2}\left[\cosh(2\varphi_1)\,(\sigma^6 - 2) - (3\sigma^6 + 2)\right], \quad \text{and } \sigma \equiv e^{\frac{1}{\sqrt{6}}\varphi_2}$$

$\mathcal{N}=1$ Supersymmetric Flow to the IR Fixed Point

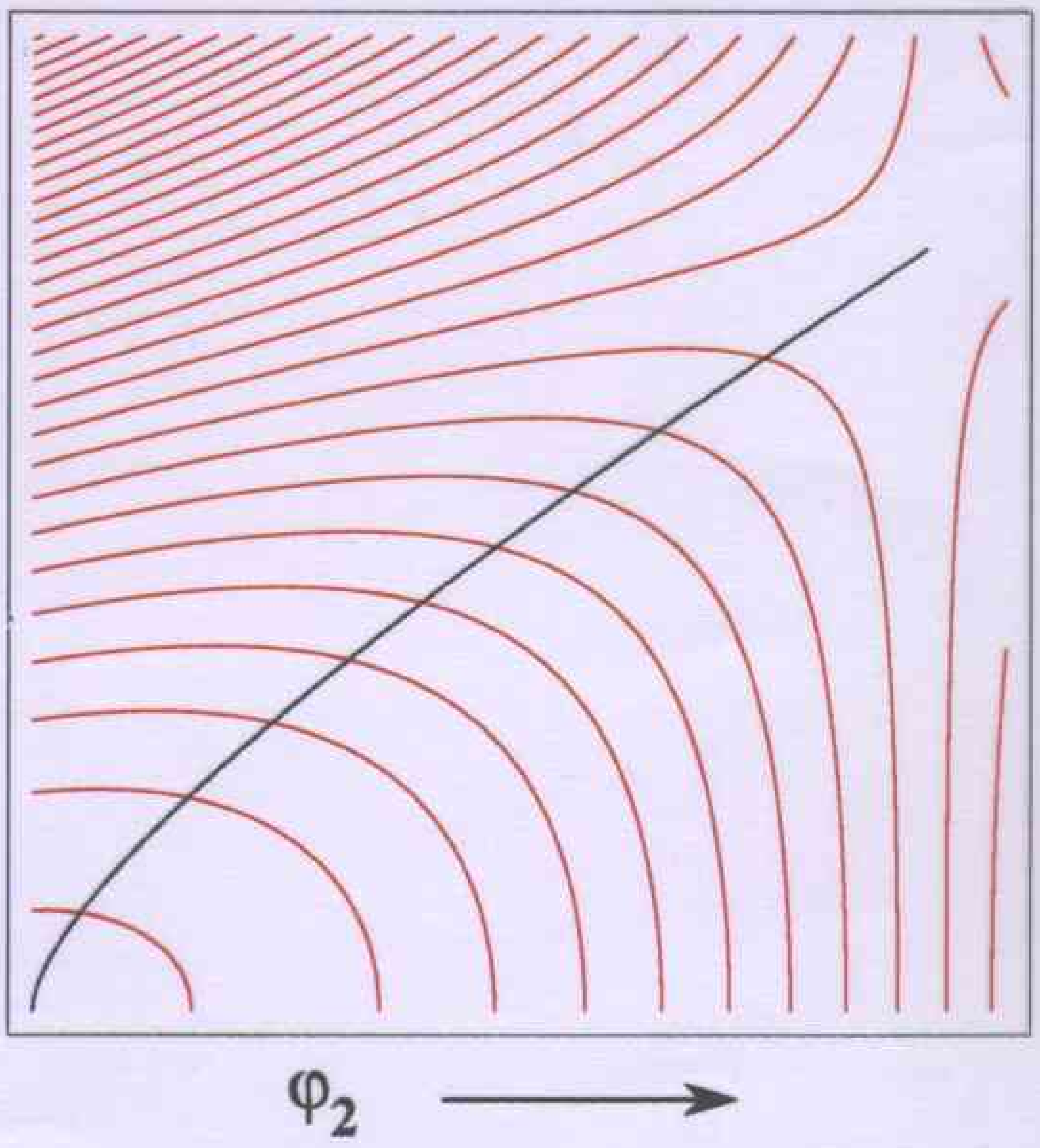
- Operator: $\varphi_1 \, \mathrm{Tr}(\lambda^4 \, \lambda^4) \; + \; \varphi_2 \, \mathrm{Tr}\big((X^5 \, X^5) + (X^6 \, X^6)\big)$
- Flow = steepest descent of superpotential, **W**



Contours of V and W

Showing path of steepest descent.

Origin, $\varphi_1 = \varphi_2 = 0$, in center.



Contours of W (detail)

Showing path of steepest descent

Origin, $\varphi_1 = \varphi_2 = 0$, bottom left.

Asymptotic behaviour of flow:

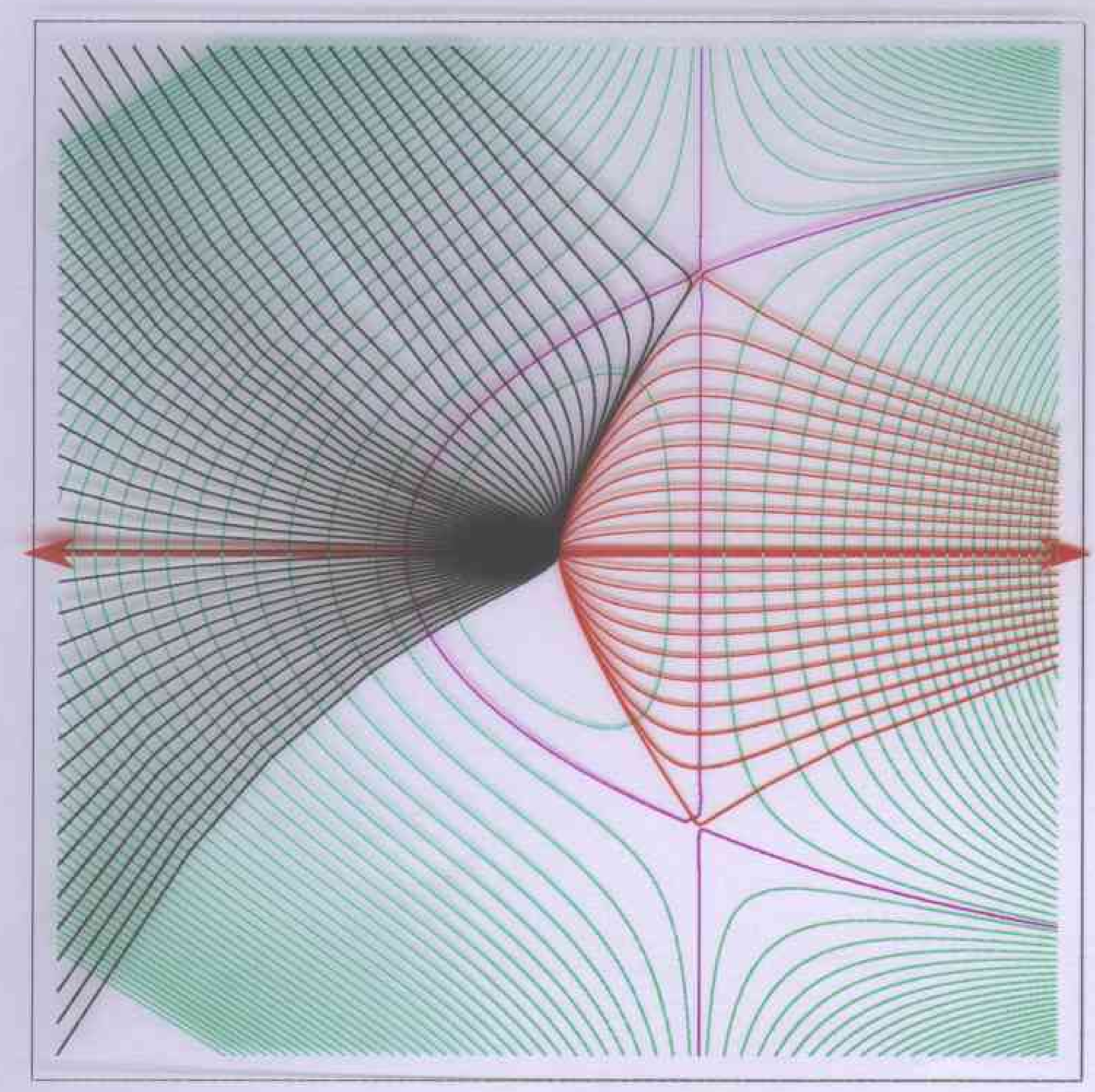
- **Near UV Fixed point** ($\varphi_1 = \varphi_2 = 0$):

$$\varphi_1 \;\sim\; a_0 \, e^{-1\rho} \qquad\qquad \varphi_2 \;\sim\; \sqrt{\frac{8}{3}} \, a_0^2 \, \rho e^{-2\rho} + a_1 e^{-2\rho}$$

- **Exponents = dimensions of operators**
- **Note the parabolic behaviour** $\Leftrightarrow$ $\mathcal{N}=1$ *Supersymmetry*

- **Near IR Fixed point** $\quad \Delta_{flow\ operator} \; = \; 3 + \sqrt{7}$

Flows to *Hades*



All steepest descents are $\mathcal{N}=1$ supersymmetric flows.

What do they represent?

Focus on the horizontal flows: $\varphi_1 = 0$

$$\varphi_2 \leftrightarrow \mathrm{Tr}\Big(\sum_{A=1}^{4}(X^A\,X^A) - 2(X^5\,X^5) - 2(X^6\,X^6)\Big)$$

$$\frac{d\varphi_2}{d\rho} = -\frac{g}{\sqrt{6}}\left(e^{\frac{4}{\sqrt{6}}\varphi_2} - e^{-\frac{2}{\sqrt{6}}\varphi_2}\right)$$

- Two possible (inequivalent) orientations
- These flows preserve SO(4) x SO(2)
- These flows preserve $\mathcal{N}=4$ *Supersymmetry*

$\Rightarrow$

- **These two flows cannot involve turning on a scalar mass**
- Flow to infinite **V** and **W** .... is this physical?

Purgatory.....

A place or state of punishment wherein the souls of those who die in God's grace may make satisfaction for past sins and so become fit for heaven

Bulk/Boundary relation:

$$\mathbf{20'} \ of \ SO(6) \ \leftrightarrow \ \mathcal{M}_{AB} \ \leftrightarrow \ \mathrm{Tr}\Big((X^A X^B)\Big) \ - \ \tfrac{1}{6}\,\delta_{AB}\,\mathrm{Tr}\Big((X^C X^C)\Big)$$

In supergravity: $S \equiv \exp(\mathcal{M}) \in SL(6,\mathbb{R})/SO(6)$

Define a superpotential, W, by: $W \equiv -\frac{1}{4}\,\mathrm{Tr}(SS^T)$

$$\frac{d\varphi_j}{d\rho} = \frac{g}{2}\,\frac{\partial W}{\partial \varphi_j} \qquad\qquad A' = -\frac{g}{3}\,W$$

preserves $\mathcal{N}=4$ Supersymmetry

where φ_j is an orthonormal basis for the scalars

Convenient gauge choice:

$$SS^T = diag\big(e^{2\beta_1}, e^{2\beta_2}, e^{2\beta_3}, e^{2\beta_4}, e^{2\beta_5}, e^{2\beta_6}\big) \qquad \sum_j \beta_j \equiv 0$$

General flows are complicated, but far from the UV fixed point the flows stabilize on one of five directions:

- $\vec{\beta} = \pm\,(1,1,1,1,1,-5)\mu$ SO(5) Invariant
- $\vec{\beta} = \pm\,(1,1,1,1,-2,-2)\mu$ SO(4) x SO(2) Invariant
- $\vec{\beta} = (1,1,1,-1,-1,-1)\mu$ SO(3) x SO(3) Invariant

Corresponding 5-metrics: Regular for the SO(5) and SO(4) x SO(2) flows with $\mu > 0$, and the + choice. All other flows have naked time-like singularities that can be reached in finite distance..... This however does not necessarily make them unphysical from the Yang-Mills perspective.... **(see talk by Steve Gubser)**

The Coulomb Branch

- Cannot be trurning on a mass - must be moving in the space of moduli of the $\mathcal{N}=4$ theory

Consider more general (non-supersymmetric) flows:

Equation of motion:

$$\frac{d^2\varphi_j}{d\rho^2} + 4A'(\rho)\,\frac{d\varphi_j}{d\rho} = \frac{\partial V}{\partial\varphi_j}$$

Two solutions, which for the scalars of the **20'** behave asymptotically as

$$\varphi_j \sim \alpha_j \underbrace{\rho\, e^{-2\rho}}_{\text{Non-normalizable AdS Mode}} + \beta_j \underbrace{e^{-2\rho}}_{\text{Normalizable AdS Mode}} \qquad \text{for } \rho \to \infty$$

Supersymmetric flows: First order equations select an initial velocity:

- $\mathcal{N}=1$ Flow to IR fixed point: $\varphi_2 \sim \rho\, e^{-2\rho}$
- $\mathcal{N}=4$ Flow to *Hades*: $\varphi_j \sim e^{-2\rho}$

Balasubramanian, Kraus and Lawrence, hep-th /9805171

Non-Normalizable Modes ⟷ **Couple to operators on boundary**

Normalizable Modes ⟷ **Represent states of the Yang-Mills theory**

Flows along non-normalizable modes: Massive RG flows

Flows along normalizable modes: Flows through States of the system

The $\mathcal{N}=4$ supersymmetric flows must be flows along the Coulomb branch.......**(see talk by Steve Gubser concerning brane distributions)**

Comments and Homework Problems ....

- There is a large family of $\mathcal{N}=1$ supersymmetric flows that involve fermion masses, and are not purely on the Coulomb branch
- There is a smaller, but similar family of $\mathcal{N}=2$ supersymmetric flows to *Hades*
- Are all supersymmetric flows determined by a superpotential, W?
- Supersymmetric flows are steepest descents of the c-function. Is this true for non-supersymmetric RG-flows?
- Critical points and Yang-Mills phases:
  - Are there stable, non-supersymmetric phases
  - What is the Yang-Mills meaning of unstable critical points?

    $$\Delta = 2 + i\,\mu$$

  - Can one compute the other anomalous dimensions in the $\mathcal{N}=1$ supersymmetric phase, e.g.

    $$\Delta_{flow\ operator} = 3 + \sqrt{7}$$

  - Arithemetic of $\mathbf{E}_{6(6)}$ and Yang-Mills phases

Conclusions

- AdS/CFT correspondence seems to work beyond the linear level, and beyond the conformal regime
    - Pertubative spectrum, symmetries and supersymmetries determine non-linear extensions AdS/CFT correspondence

- Gauged $\mathcal{N}=8$ supergravity is a powerful tool for studying $\mathcal{N}=4$ Yang-Mills
    - $\mathcal{N}=8$ supergravity potential gives the phase diagram of softly-broken $\mathcal{N}=4$ Yang-Mills
    - Supergravity kinks can be used to describe RG flows

- Gauged $\mathcal{N}=8$ supergravity knows about the $\mathcal{N}=1$ IR fixed point of Leigh and Strassler: Highly non-trivial test of the non-conformal, non-linear exetnsion of AdS/CFT.

- The Coulomb branch (or at least a small part of it) emerges directly out of gauged $\mathcal{N}=8$ supergravity
    - Coulomb branch physics is a necessary part of the original Maldacena conjecture
    - Useful for further study .....