TODAY'S LECTURE IS PART 1 OF 2; PART 2 BY SEIBERG ON SATURDAY.

CONNES, DOUGLAS AND A. SCHWARZ (NOV., 1997) SHOWED THAT "NONCOMMUTATIVE YANG-MILLS THEORY" (IN THE SENSE OF A. CONNES) IS RELEVANT TO A VERY SPECIFIC PROBLEM IN STRING THEORY

MANY SUBSEQUENT DEVELOPMENTS

DOUGLAS - HULL - derivation

e.g. SCHWARZ

BRACE, MORARIU, ZUMINO } MORE ON T-DUALITY

SCHOMERUS } OPEN STRING VERTEX OPERATORS

NEKRASOV AND SCHWARZ } NON COMM. INSTANTONS

KONTSEVICH
CATANEO & FELDER } DEFORMATION QUANTIZATION BY OPEN STRINGS

HOFMAN, H. VERLINDE, ZWART

(c.f. RECENT REVIEW BY DOUGLAS)

THE ORIGINAL APPLICATION ③

ON NON-COMM YANG-MILLS

WAS TO COMPACTIFICATION

ON T^2 (OR T^n WITH $n>2$)

IN THE LIMIT OF SMALL

AREA, WITH FIXED, NONZERO

$$\Phi = \int_{T^2} B \qquad B = \text{N.S. B-FIELD}$$

NEKRASOV & SCHWARZ GAVE A FASCINATING APPLICATION THAT DID NOT INVOLVE SMALL AREA AT ALL:

INSTANTONS AT $B \neq 0$

CONSIDER A Type IIB THREEBRANE OF WORLD-VOLUME

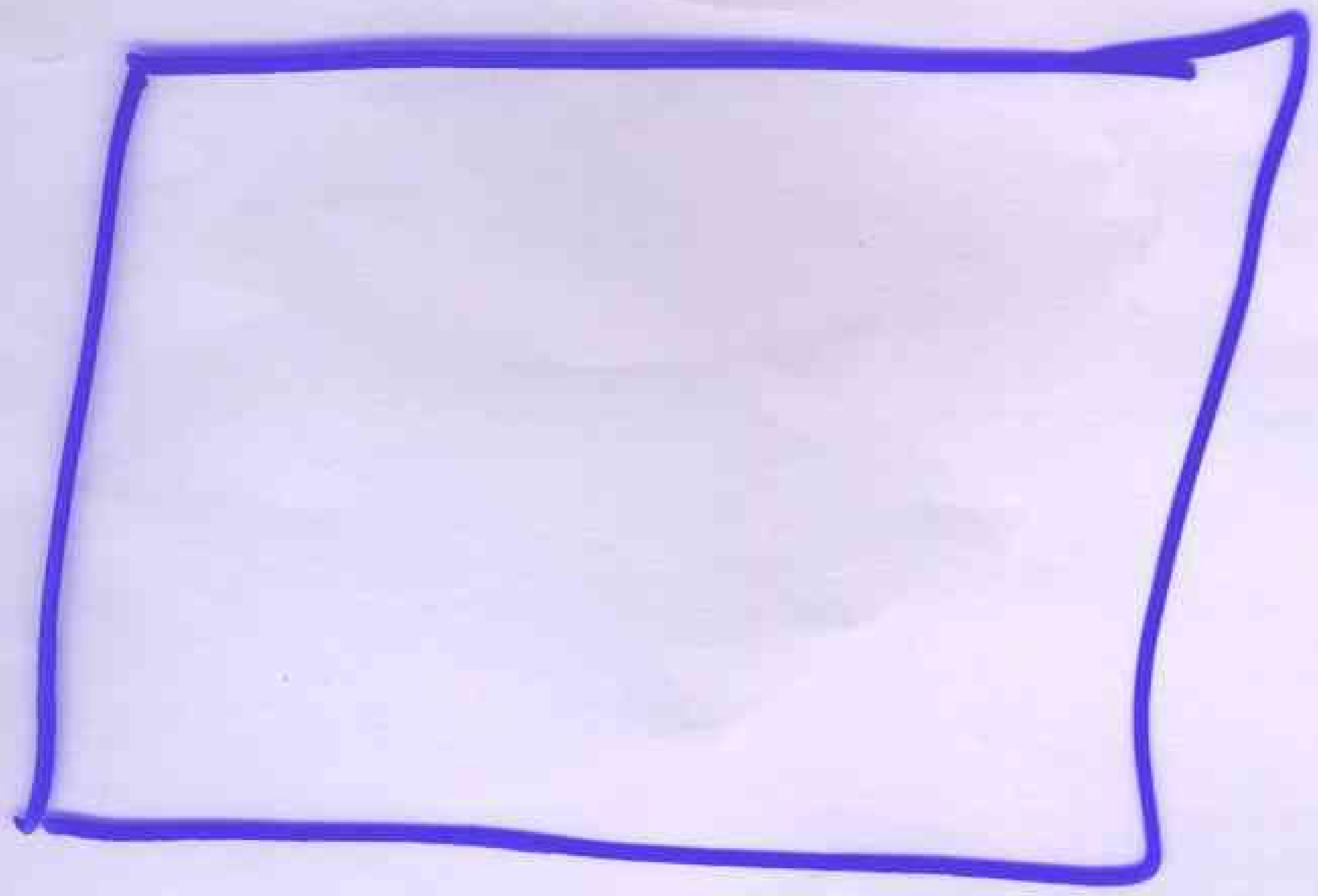
$$\mathbb{R}^4 \subset \mathbb{R}^{10}$$

N SUCH THREEBRANES

$\Rightarrow$ U(N) GAUGE SYMMETRY

ON $\mathbb{R}^4$

$\mathbb{R}^4$



$$\delta B_{ij} = \partial_i \Lambda_j - \partial_j \Lambda_i$$

$$\delta A_i = -\Lambda_i$$

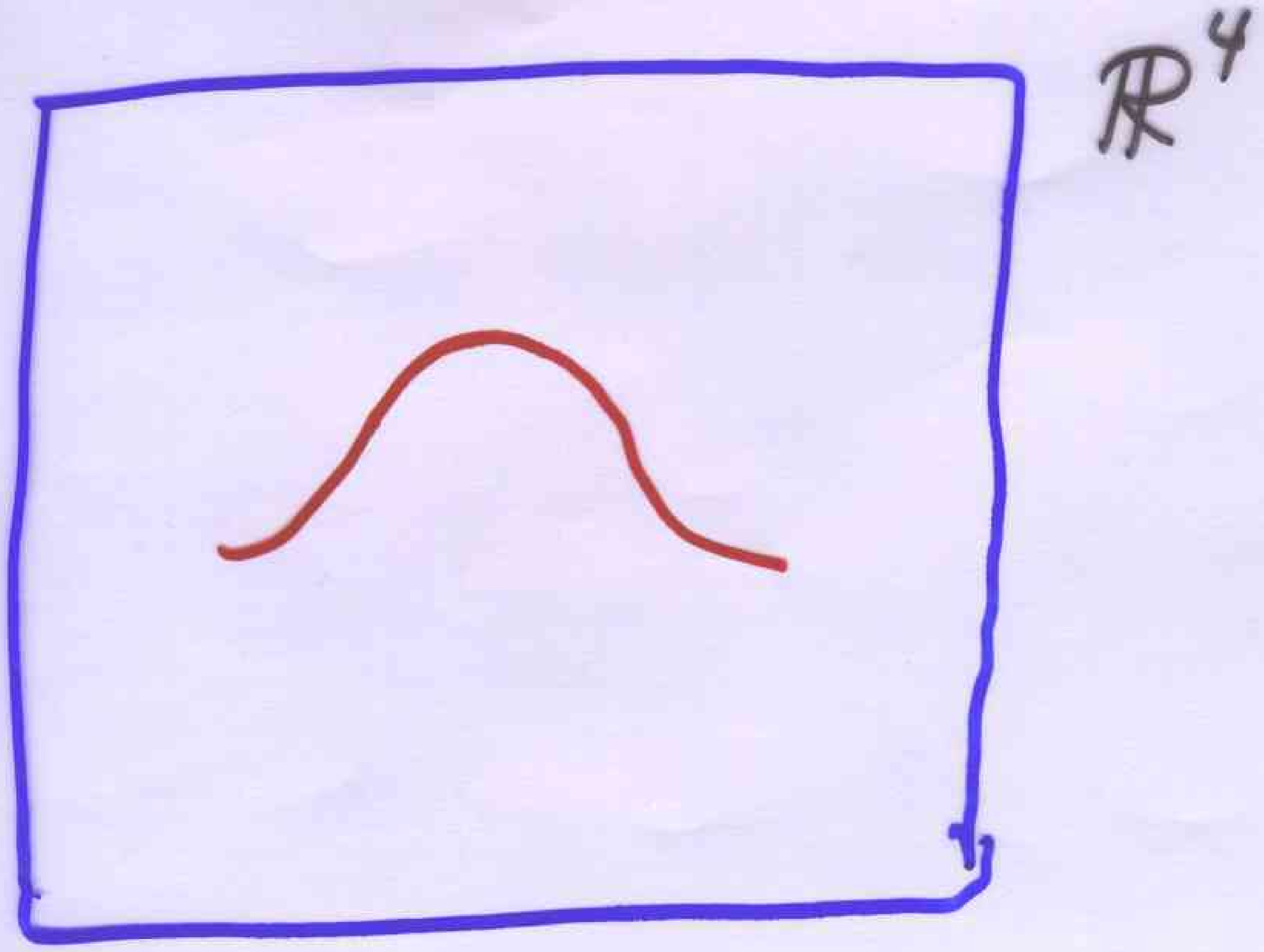
WE TAKE A CONSTANT B FIELD

$$B_{ij} = \text{CONSTANT}, \quad i, j = 1 \dots 4$$

(WHICH IN THE PRESENCE OF THE BRANE CANNOT BE GAUGED AWAY)

6

WE LOOK AT SUPERSYMMETRIC GAUGE CONFIGURATIONS ON THE BRANE, i.e. INSTANTONS



$F_{ij}^{+} = 0$

FIRST RECALL WHAT HAPPENS AT $B = 0$

FOR A BIG INSTANTON, (7)

THE CLASSICAL EQUATION

$$F_{ij}^{+} = 0$$

IS EXACT. AS THE INSTANTON SHRINKS, WE EXPECT α' CORRECTIONS.

HOWEVER, AT $B = 0$ (WITH A SUITABLE REGULARIZATION) THE INSTANTON EQUATION IS UNCORRECTED.

ONE SHOWS THIS USING THE OPEN STRING σ- MODEL:

$$\mathcal{L} = \int_{\Sigma} d^2 z \left[\frac{1}{4\pi\alpha'} \partial_z X^i \partial_{\bar{z}} X^i + \Psi \not{\partial} \psi \right] + \int_{\partial\Sigma} \left[A_i \frac{dX^i}{d\tau} + F_{ij} \psi^i \psi^j \right]$$

SPACETIME SUSY

$\Longleftrightarrow$

$SU(2)_R$ SYMMETRY

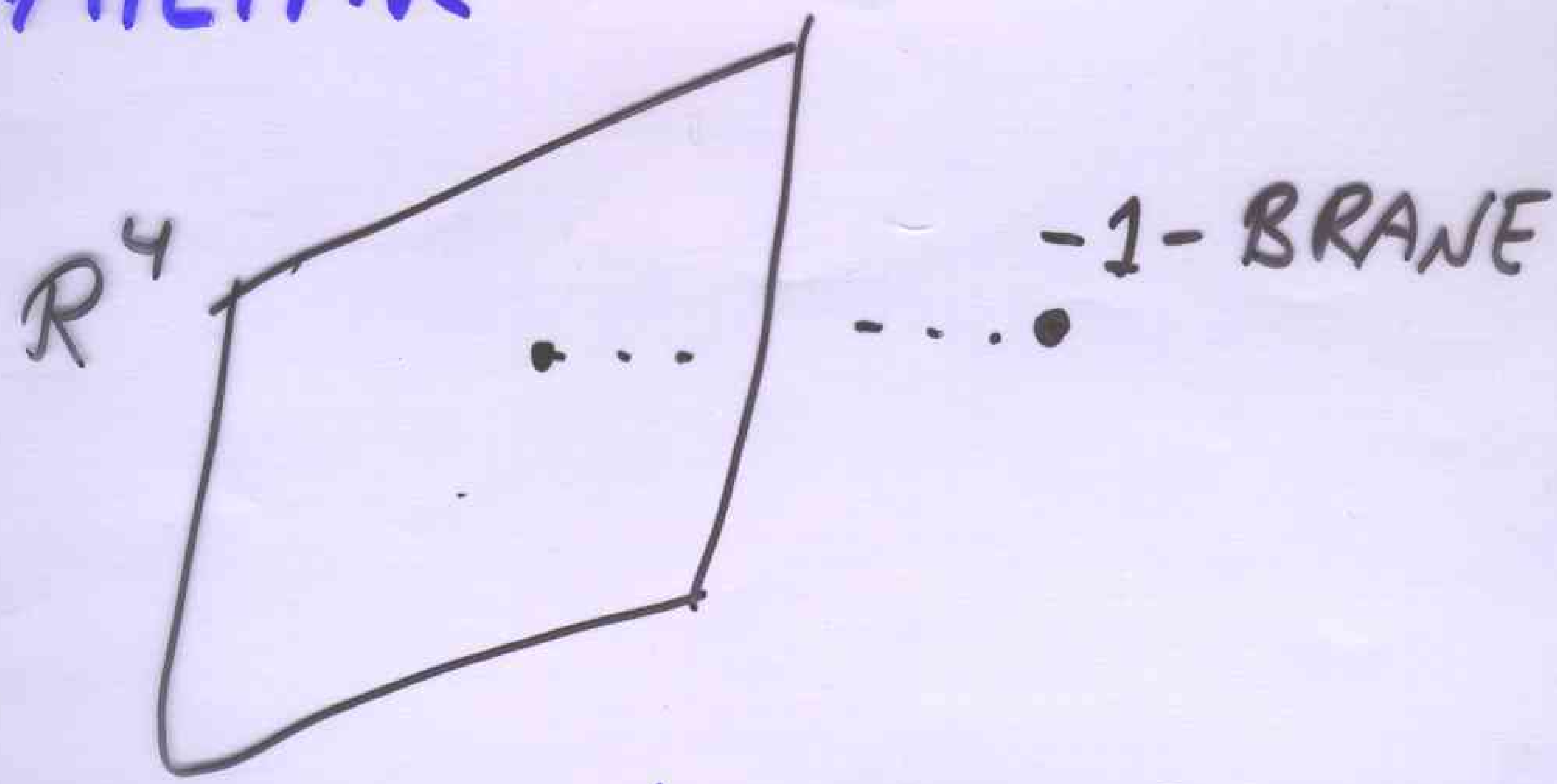
$\Longleftrightarrow$ $F^+ = 0$.

IN PARTICULAR, AT $B=0$, THE STRINGY INSTANTON MODULI SPACE IS THE SAME AS THE CLASSICAL ONE, AND THUS THERE IS A

<u>SMALL INSTANTON</u>

<u>SINGULARITY</u>

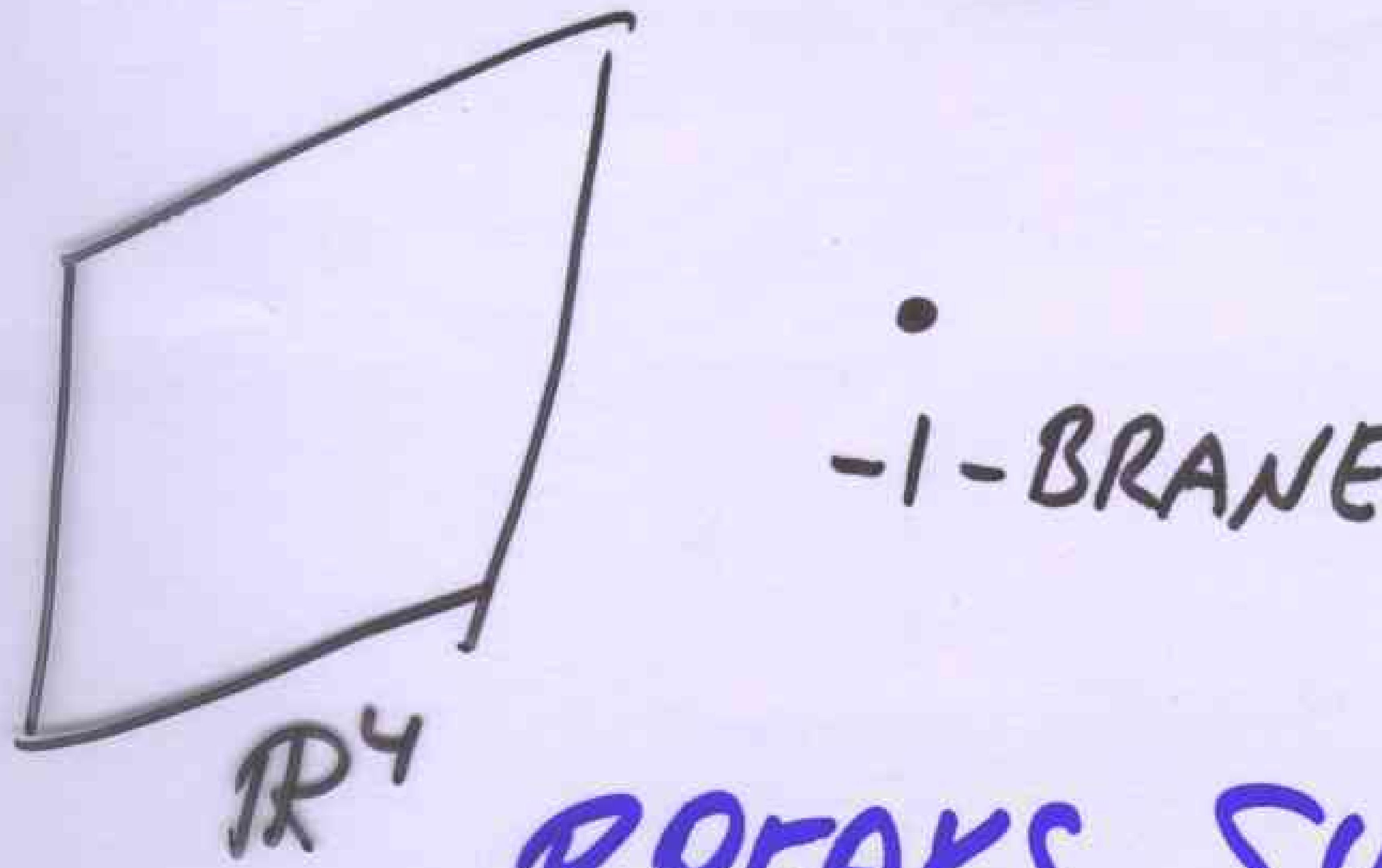
INDEED THE MEANING OF THE SMALL INSTANTON SINGULARITY IN STRING THEORY IS FAMILIAR:



AN INSTANTON SHRINKS TO A POINT AND THEN ESCAPES AS A -1- BRANE.

NOW TURN ON $B \neq 0$ 11

WE FIND, IF $B^+ \neq 0$,
THAT THERE CAN BE
NO SMALL INSTANTON
SINGULARITY SINCE
A 3-BRANE + (-1)-BRANE
ARE NOT BPS:



BREAKS SUSY IF
$B^+ \neq 0$.

SO THE INSTANTON MODULI

SPACE $\mathfrak{M}$ MUST HAVE

α' CORRECTIONS AT $B^+ \neq 0$

INDEED NEKRASOV AND

SCHWARZ PROPOSED THAT

AT $B \neq 0$ ONE GETS

THE MODULI SPACE OF

"NONCOMMUTATIVE

INSTANTONS"

WHAT ARE THEY?

NONCOMMUTATIVE YANG-MILLS

START WITH A POISSON BRACKET

$$\{f, g\} = \theta^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}.$$

(θ WILL BE DETERMINED IN TERMS OF B)

DEFORM THE f's TO OPERATORS WITH

$$\hat{f}\hat{g} - \hat{g}\hat{f} = i\widehat{\{f, g\}} + O(\theta^2)$$

OR, TO BE MORE PRECISE, DEFINE AN ASSOCIATIVE PRODUCT

$$f * g = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial u_i}\frac{\partial}{\partial v_j}} f(x+u)\, g(x+v)$$

$$= fg + i\{f,g\} + \ldots$$

GIVING AN ASSOCIATIVE ALGEBRA $\mathcal{A}$.

$$(f * g) * h = f * (g * h)$$

NOW IMITATE GAUGE THEORY. GAUGE PARAMETER $\hat{\lambda} \in \mathcal{A}$, GAUGE FIELD $\hat{A}_i$ TRANSFORMS

$$(\hat{\lambda} * \hat{A})_{ij} = \hat{\lambda}^i_k * A^k_j$$

$$\delta \hat{A}_i = \partial_i \hat{\lambda} + i \hat{\lambda} * \hat{A}_i - i \hat{A}_i * \hat{\lambda}$$

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i + \hat{A}_i * \hat{A}_j - \hat{A}_j * \hat{A}_i$$

TAKE $\mathcal{A} \to \mathcal{A} \otimes (N \times N \text{ MATRICES})$ TO GET A RANK N, i.e. $U(N)$ VERSION.

NONCOMMUTATIVE INSTANTONS:

$$\hat{F}^+ = 0.$$

NEKRASOV AND SCHWARZ SHOWED THAT THE MODULI SPACE $\hat{\mathfrak{M}}$ OF SUCH OBJECTS IS GIVEN BY A DEFORMATION OF THE ADHM EQUATIONS — STUDIED BY

NAKAJIMA

AHARONY BERKOOZ SEIBERG

JUMPING AHEAD OF OUR STORY A BIT, IT TURNS OUT THAT $\hat{\mathfrak{M}}$ IS INDEED AS CLAIMED, FOR ALL $B^+ \neq 0$, BUT NONCOMMUTATIVE YANG-MILLS IS MOST EFFECTIVE IN DESCRIBING THE PHYSICS FOR B VERY LARGE, i.e. $\alpha' B >> 1$

$B \to \infty$ IS PART OF THE CONNES-DOUGLAS-SCHWARZ STORY,

SINCE

$$\text{AREA}(\Sigma) \to 0$$

$$\int_{\Sigma} B = \text{CONSTANT}$$

IMPLIES $B \to \infty$, SO WE ARE CLOSE TO THE STARTING POINT.

I HAVE INTRODUCED THIS SUBJECT WITH INSTANTONS, BECAUSE THEY GIVE A VERY STRIKING APPLICATION, WHERE STRING THEORY IS HIGHLY NONCLASSICAL BUT CAN BE DESCRIBED IN GREAT DETAIL.

(in CONTRAST, SAY, TO CALABI-YAU's, WHERE A FULL DESCRIPTION IS WAY OUT OF REACH)

BUT NOW WE ARE GOING TO LEAVE THE INSTANTONS BEHIND; FOR THE REST OF THIS LECTURE, WE'LL TRY TO DERIVE NONCOMMUTATIVE YANG-MILLS SYSTEMATICALLY FROM OPEN STRINGS IN A MAGNETIC FIELD

(studied in mid-1980's
BY FRADKIN - TSEYTLIN
ABOUELSAOOD - CALLAN - NAPPI - [redacted] LOVELACE - YOST)

BOSONIC OPEN STRINGS:

$$\mathcal{L} = \frac{1}{4\pi\alpha'} \int d^2z \; g_{ij} \, \partial_z X^i \, \partial_{\bar{z}} X^j$$

$$g^2 \qquad + i \int B_{ij} \, dX^i \wedge dX^j$$

$$g^{ij} g_i g_j$$

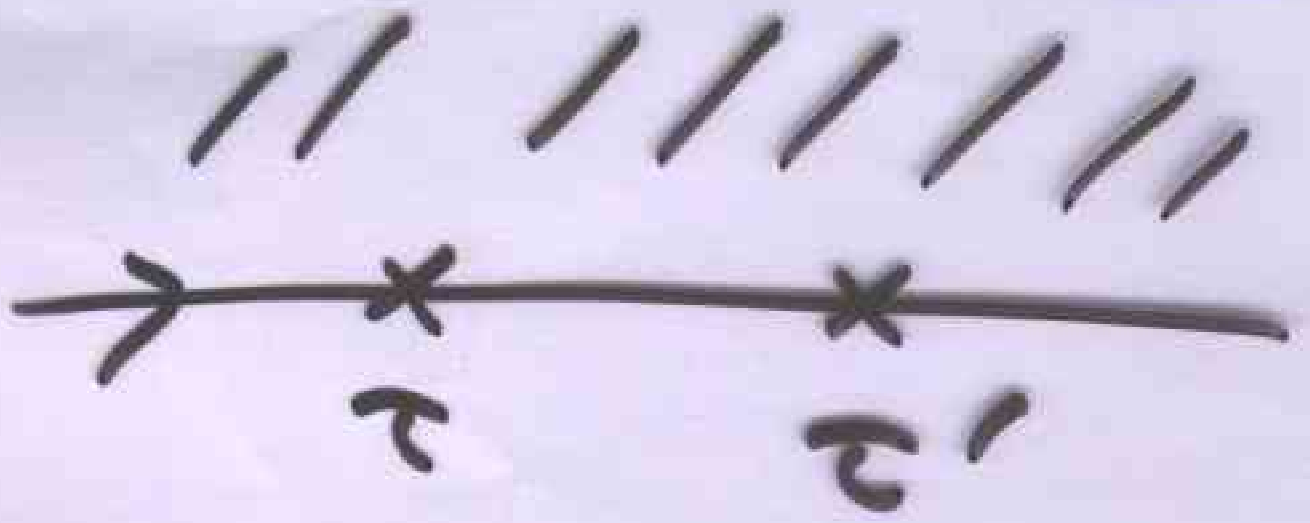
WE'LL BE CONSIDERING **OPEN** **STRING** VERTEX OPERATORS ONLY, SO WE NEED THE PROPAGATOR

$$\langle X^i(\tau) \, X^j(\tau') \rangle$$

FOR τ, τ' ON **BOUNDARY**

IT WAS FOUND ~1985

TO BE



$$\langle X^i(\tau)\, X^j(\tau') \rangle$$

$$= -\alpha' G^{ij} \log(\tau-\tau')^2$$

$\tilde{g}^{ij}$

$$+ \frac{i}{2}\Theta^{ij}\, \epsilon(\tau-\tau')$$

WHERE

$$G^{ij} = \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_{SYM}$$

$$\Theta^{ij} = 2\pi\alpha' \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_{ANTI}$$

THE LOG TERM DETERMINES THE ANOMALOUS DIMENSIONS, FOR INSTANCE

$e^{ip\cdot X}$ HAS DIMENSION

$$\frac{\alpha'}{2} G^{ij} p_i p_j$$

SO G^{ij} IS THE EFFECTIVE METRIC SEEN BY THE OPEN STRINGS.

(OR $G_{ij} = (G^{ij})^{-1}$)

ON THE OTHER HAND, 23

AS SUGGESTED BY SCHOMERUS, θ^{ij} DETERMINES THE NONCOMMUTATIVITY:

WE HAVE

$$e^{ip\cdot X_{(\tau)}} e^{iq\cdot X_{(\tau')}} \Big|_{\substack{\tau > \tau' \\ \tau \to \tau'}}$$

$$\sim |\tau - \tau'|^{\alpha' G^{ij} p_i q_j} \cdot \left\{ \exp \frac{i}{2} \theta^{ij} p_i q_j \right\} e^{i(p+q)\cdot X_{(\tau)}}$$

IF WE COULD IGNORE THE FACTOR

$$|\tau - \tau'|^{\alpha' G^{ij} p_i q_j}$$

WE'D RECOGNIZE THE * PRODUCT

$$e^{ip\cdot X} * e^{iq\cdot X}$$

$$= \exp \frac{i}{2} \theta^{ij} p_i q_j \; e^{i(p+q)\cdot X}$$

THIS FACTOR IS, OF
COURSE, TIED UP WITH
THE USUAL WORLD-SHEET
STRUCTURE & ANOMALOUS DIM-ENSIONS

THE REST OF THIS TALK
HAS TWO PARTS:

I) WHAT WE CAN SAY
IN GENERAL

II) WHAT WE CAN SAY
WHEN WE GET RID OF
THE ANOMALOUS
DIMENSIONS.

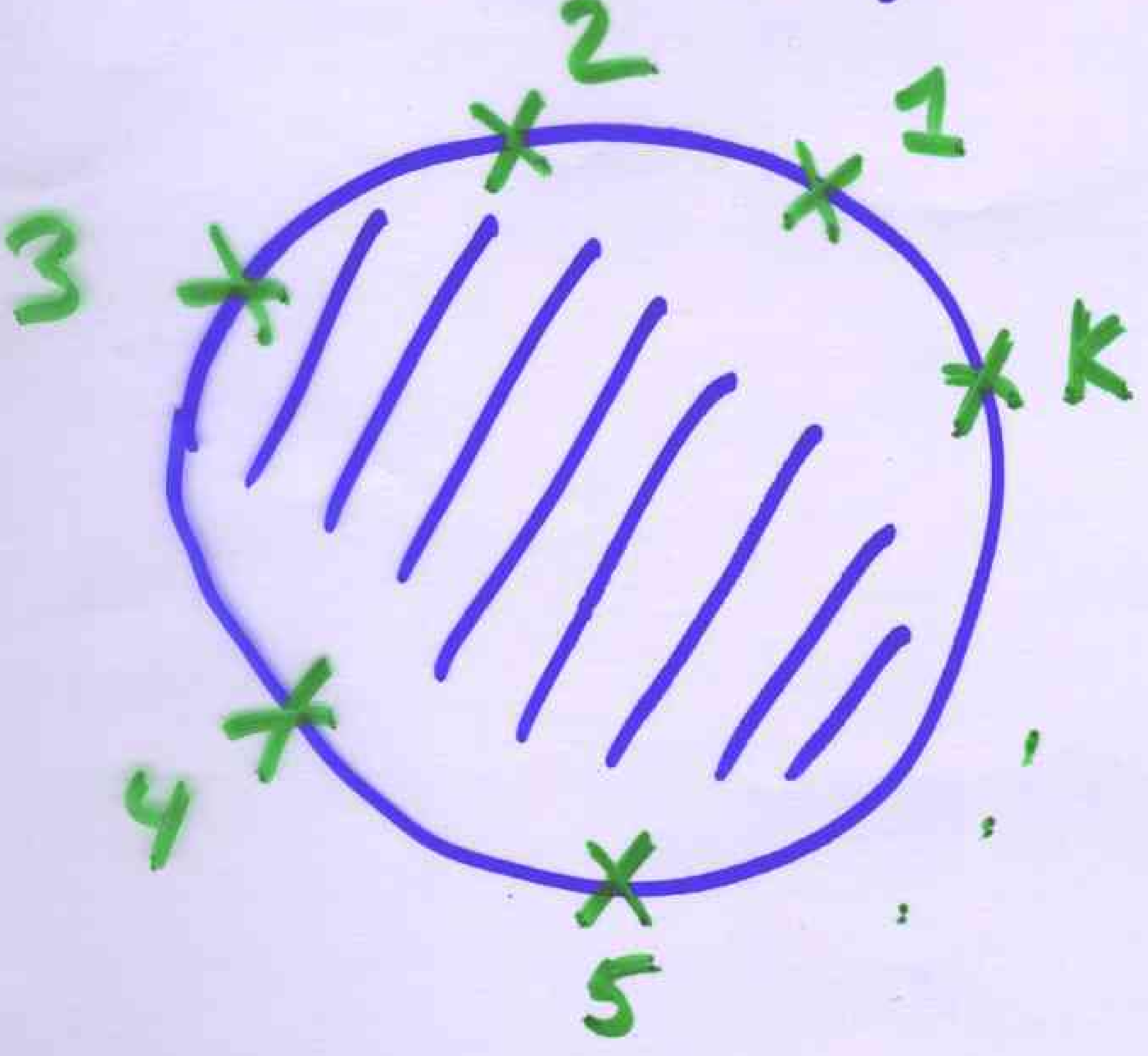
I) THE θ TERM

DOES NOT CONTRIBUTE TO CORRELATION FUNCTIONS OF

$$\frac{dx^i}{d\tau} \quad \text{OR} \quad \frac{d^m x^i}{d\tau^m}, \quad m>0$$

SUPPOSE FOR INSTANCE WE CONSIDER OPEN STRINGS WITH U(N) CHAN-PATON FACTORS

CONSIDER SCATTERING OF k GAUGE BOSONS OF MOMENTA p_i, POLARIZATIONS ε_i, CHAN-PATON WAVE FUNCTIONS λ_i, $i = 1 \ldots k$



THIS IS

$$A(\lambda_i, \varepsilon_i, p_i)_{G,\theta}$$

$$= \mathrm{Tr}\, \lambda_1 \lambda_2 \cdots \lambda_K$$

$$\int d\tau_i \left\langle \prod_{i=1}^{K} \epsilon_{(i)} \cdot \frac{dX}{d\tau} e^{i p_{(i)} \cdot X_{(\tau_i)}} \right\rangle_{G,\theta}$$

THE ONLY θ DEPENDENCE OF THE CORRELATION FUNCTIONS COMES FROM A FACTOR

$$\exp\left(-\frac{i}{2} \sum_{s>r} p^{(s)}_i p^{(r)}_j \theta^{ij}\right)$$

THIS FACTOR DEPENDS ONLY ON THE CYCLIC ORDERING OF THE VERTEX OPERATORS -WHICH IS KEPT FIXED - SO IT FACTORS OUT:

$$A(\lambda_i, \varepsilon_i; p_i)_{G, \theta}$$

$$= A(\lambda_i, \varepsilon_i; p_i)_{G, \theta=0}$$

$$\cdot \exp\left(-\frac{i}{2} \sum_{s>r} p_i^{(s)} p_j^{(r)} \theta^{ij}\right)$$

GIVEN THE TREE LEVEL S-MATRIX, ONE CAN, AT $\Theta = 0$, FIND A LOCAL EFFECTIVE ACTION

$$\underset{(\theta=0)}{S_G} = \frac{1}{g_{st}} \int d^n x \sqrt{G} \, \mathrm{Tr}\left(F_{ij} F^{ij} + \alpha' \text{ CORRECTIONS}\right)$$

THAT GENERATES IT.

A SINGLE TRACE SINCE WE HAD

$$\mathrm{Tr}\, \lambda_1 \lambda_2 \dots \lambda_N$$

WAVE FUNCTIONS MULTIPLIED IN CYCLIC ORDER.

FOR $\theta \neq 0$, WE MUST INCORPORATE THE PHASE FACTOR: REPLACE ORDINARY MULTIPLICATION OF WAVE FUNCTIONS BY THE $*$ PRODUCT

SO

$$\underbrace{\partial_i A_j + A_i A_j - (i,j)}_{F_{ij}} \Rightarrow \underbrace{\partial_i \hat{A}_j + \hat{A}_i * \hat{A}_j - (i,j)}_{\hat{F}_{ij}}$$

$$\hat{D}_k \hat{F}_{ij} = \partial_k \hat{F}_{ij} + \hat{A}_k * \hat{F}_{ij} - \hat{F}_{ij} * \hat{A}_k$$

SO

$$S_{G,\theta} = \frac{1}{g} \int d^n x \sqrt{G}\, \mathrm{Tr}\left(\hat{F}_{ij} \hat{F}^{ij} + \ldots \right)$$

SAME EXPRESSION AS AT $\theta = 0$, BUT WITH "HATS"

THUS, NONCOMMUTATIVE YANG-MILLS CAN BE USED TO GIVE A SIMPLE DESCRIPTION OF THE θ - OR B - DEPENDENCE OF THE EFFECTIVE ACTION, TO ALL ORDERS IN α'!

(AND IMPLIES EXACTNESS OF $\hat{F}^+ = 0$ AT $\theta \neq 0$ GIVEN EXACTNESS OF $F^+ = 0$ AT $\theta = 0$)

ON THE OTHER HAND, BY STANDARD METHODS (e.g. σ- MODELS WITH PAULI-VILLARS REGULARIZATION) ONE CAN DESCRIBE THE EFFECTIVE ACTION VIA LOCAL GAUGE-INVARIANT INTERACTIONS WITH USUAL GAUGE INVARIANCE, TO ALL ORDERS IN θ & α!

EXAMPLE: U(1) WITH ALMOST CONSTANT FIELDS

BORN-INFELD:

$$S = \frac{1}{g_{st}(\alpha')^2} \int d^n x \sqrt{\det(g + \alpha' F + \alpha' B)}$$

STANDARD GAUGE-INVARIANT INTERACTIONS - BUT DEPEND EXPLICITLY ON θ AND G (OR B AND g) NOT JUST VIA $*$ PRODUCT

HOW CAN IT BE? 36

THE SAME EFFECTIVE ACTION IS DESCRIBED BY NONCOMMUTATIVE YM AND ALSO BY STANDARD YM THEORY!!

THE TWO FRAMEWORKS MUST BE EQUIVALENT (BY A TRANSFORMATION THAT CHANGES THE ACTION)

THERE IS A COMPLETELY EXPLICIT CHANGE OF VARIABLES THAT DOES THIS:

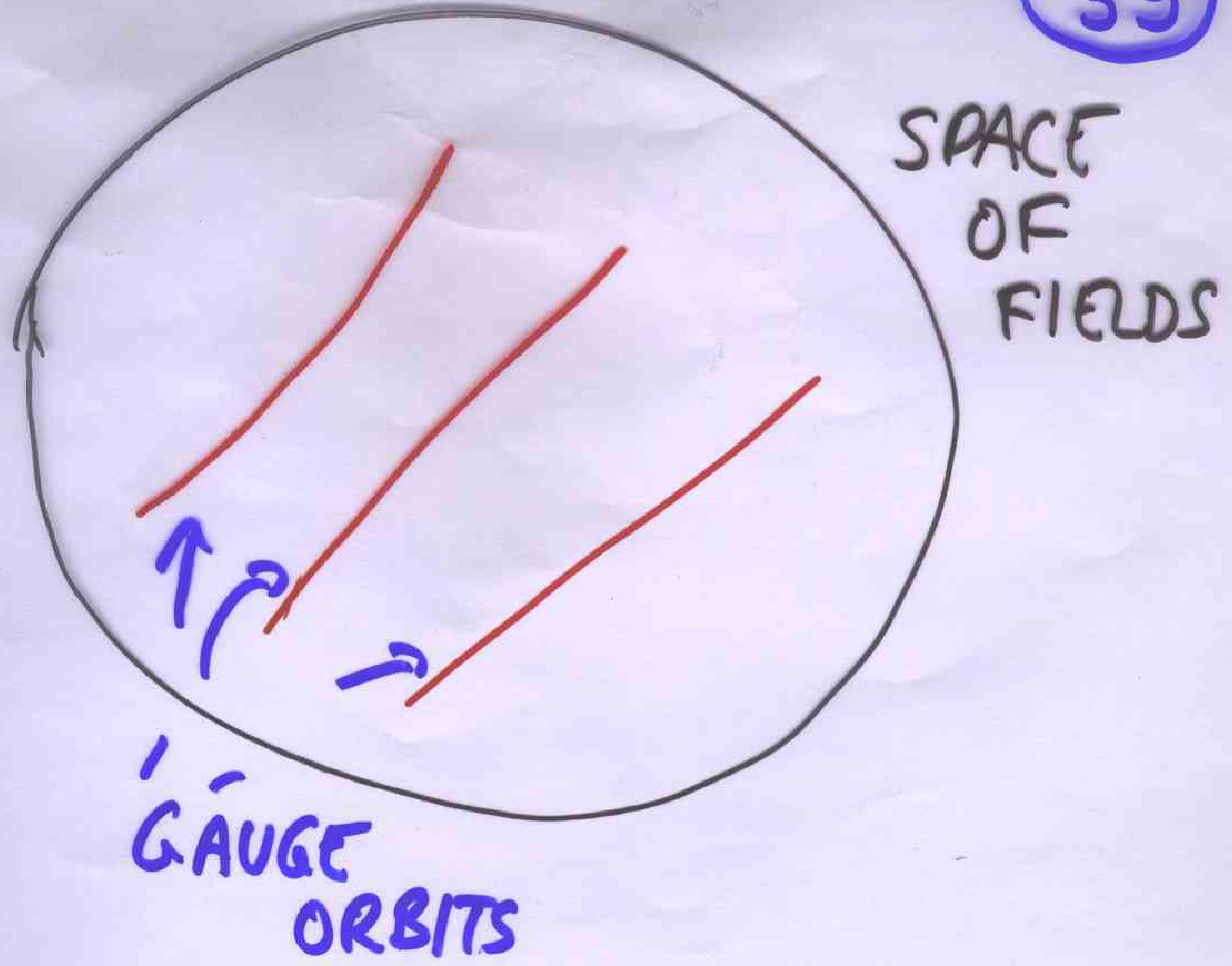
$$\hat{A}_i = A_i - \frac{1}{4}\theta^{kl}\{A_k, \partial_l A_i + F_{li}\} + O(\theta^2)$$

$$\hat{\lambda} = \lambda + \frac{1}{4}\theta^{ij}\{\partial_i \lambda, A_j\} + O(\theta^2)$$

(to all orders the transformation is generated by a differential eqn. of the same form)

NOTE THAT THE TRANSFORMATION OF λ DEPENDS ON A.

THE ORDINARY AND NONCOMMUTATIVE GAUGE GROUPS AREN'T EQUIVALENT, BUT THE TWO GAUGE EQUIVALENCE RELATIONS (ALL ONE NEEDS FOR PHYSICS) ARE THE SAME.



ORDINARY AND NONCOMM YM THEORY: DIFFERENT GENERATORS FOR SAME GAUGE ORBITS.

NOW OUR PATH TO
NONCOMMUTATIVE Y. M.
LED VIA THE S-MATRIX.

IT WOULD BE MUCH
NICER TO EXTRACT THE
* PRODUCT, OR A MORE
GENERAL ASSOCIATIVE
PRODUCT INCLUDING EXCITED
STRINGS, DIRECTLY FROM
THE OPE:

$$e^{ik\cdot X}(\tau)\, e^{ip\cdot X}(\tau')$$

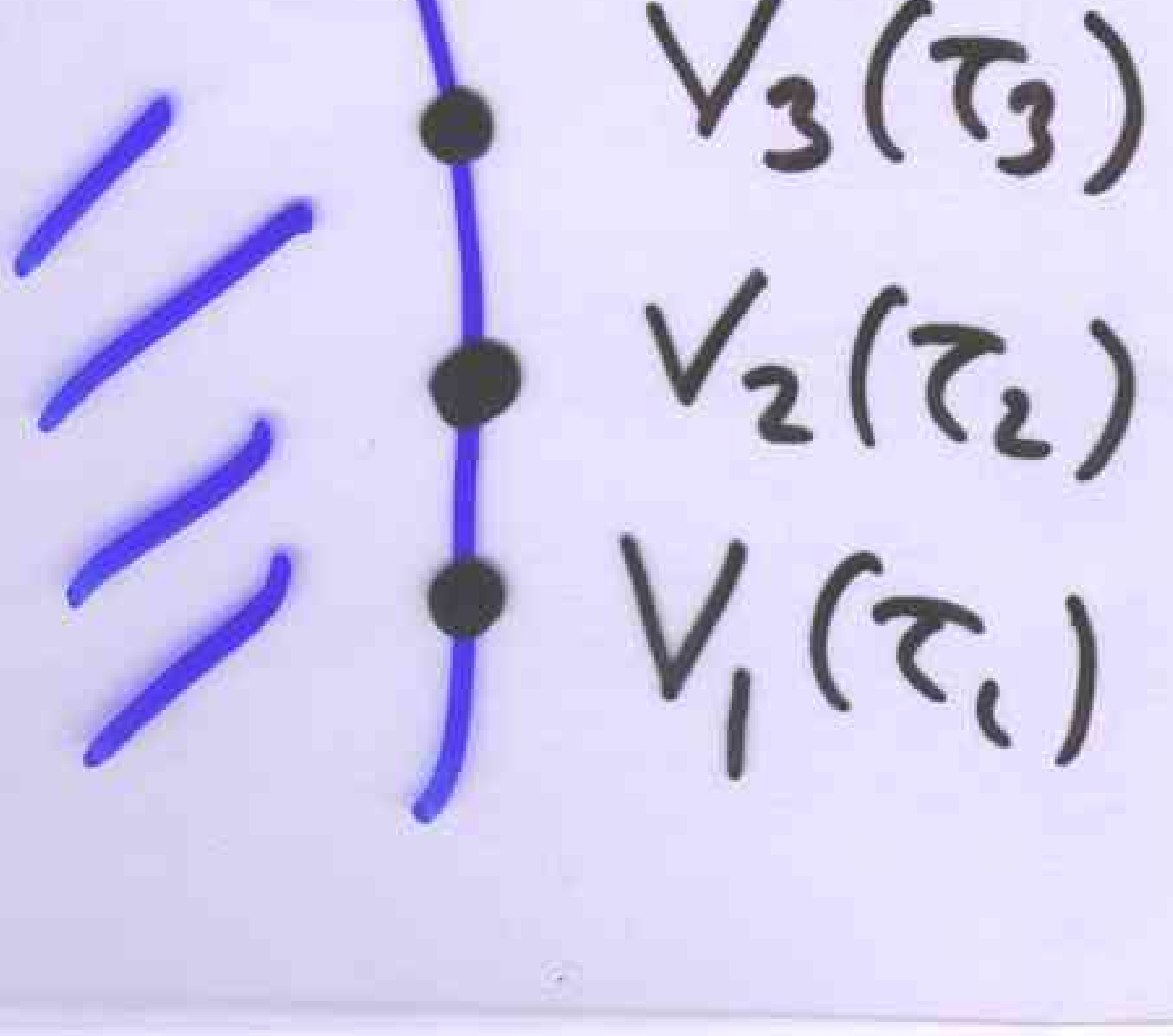
$$\to |\tau-\tau'|^{-\alpha' k\cdot p}$$

$$\exp -\frac{i\theta^{ij}}{2} k_i p_j\, \epsilon(\tau-\tau')$$

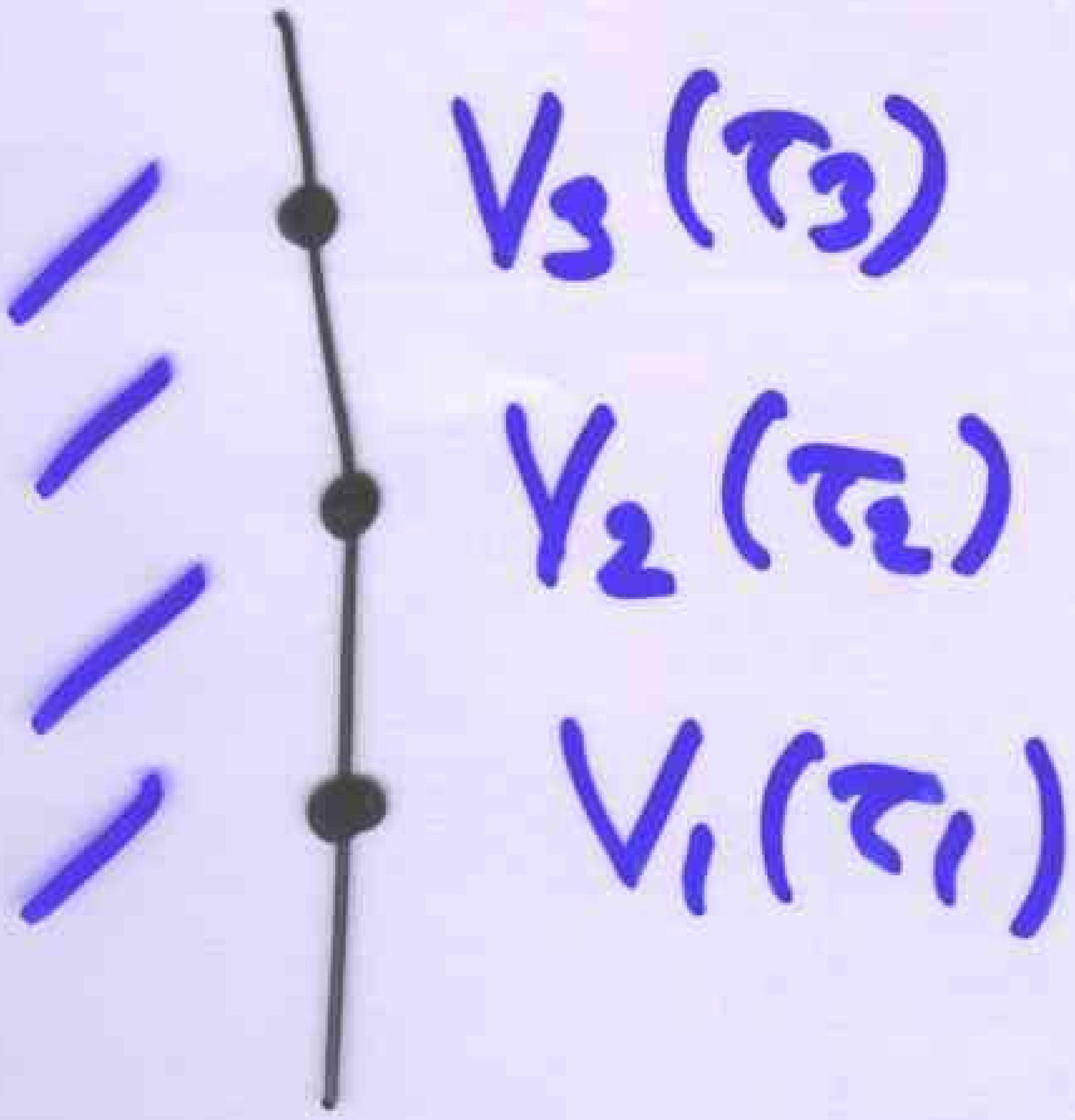
$$e^{i(k+p)\cdot X}$$

ASSOCIATIVITY:

MULTIPLY VERTEX OPERATORS IN ORDER

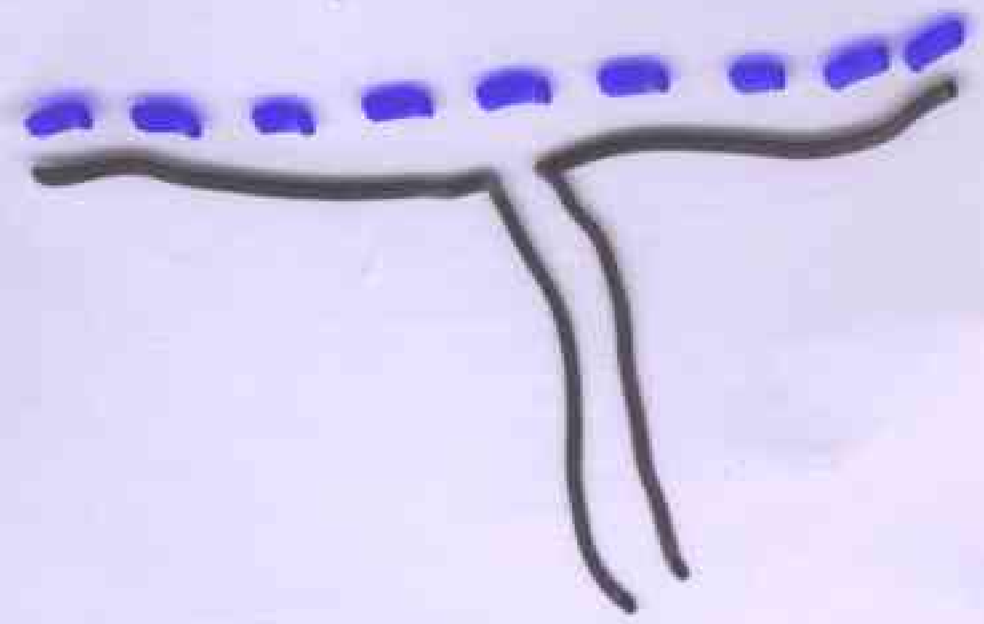


THIS DOESN'T WORK, BECAUSE MULTIPLYING VERTEX OPERATORS DEPENDS ON $\tau_3 - \tau_2$ AND $\tau_2 - \tau_1$, BECAUSE OF THE ANOMALOUS DIMENSIONS, AND THIS SPOILS ASSOCIATIVITY.



FOR THIS REASON, IN TRYING IN THE MID-80'S TO BASE OPEN STRING FIELD THEORY ON AN ASSOCIATIVE ALGEBRA AND CONNES-STYLE NONCOMMUTATIVE GEOMETRY, IT WAS NECESSARY TO USE, NOT DIRECTLY THE OPERATOR PRODUCT ALGEBRA, BUT A <u>MESSY</u>

ALGEBRA BASED ON GLUING STRINGS:



LIKEWISE,
THE ANOMALOUS DIMENSIONS
PREVENT ONE FROM
SEEING EITHER ORDINARY
(AT $\theta=0$) OR NONCOMM.
(AT $\theta\neq 0$) GAUGE INVARIANCE
DIRECTLY FROM OPE's

AT $\theta = 0$, A
STANDARD WAY TO MAKE
THE GAUGE INVARIANCE
MANIFEST INVOLVES
GOING TO LONG
WAVELENGTH – OR
TAKING $\alpha' \to 0$ AT
FIXED WAVELENGTH –

AND MAKING A

σ- MODEL EXPANSION IN

POWERS OF THE

PROPAGATOR

$$\langle X^i(\tau) X^j(\tau') \rangle = -\alpha' g^{ij} \log(\tau-\tau')^2.$$

AT $\theta \neq 0$, WE CAN DO

EXACTLY THE SAME THING:

$$\langle X^i(\tau) X^j(\tau') \rangle = -\alpha' G^{ij} \log(\tau-\tau')^2 + \theta^{ij} \varepsilon(\tau-\tau')$$

WANT $\alpha' \to 0$

WITH G^{ij}, θ^{ij} FIXED

WITH

$$G^{ij} = \left(\frac{1}{g + 2\pi\alpha' B} \right)^{ij}_{SYM}$$

$$\theta^{ij} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B} \right)^{ij}_{ANTI}$$

TO TAKE $\alpha' \to 0$ WITH G, θ FIXED WE NEED

B FIXED

$\alpha' \sim \epsilon^{1/2}$

$g_{ij} \sim \epsilon$ IF $B \neq 0$ IN i, j DIRECTIONS

$g_{ij} \sim 1$ NULL SPACE OF B.

IN THIS LIMIT, THE WORLD-SHEET ACTION REDUCES TO

$$\int_{\Sigma} B_{ij}\, dX^i \wedge dX^j$$

$$= \int_{\partial\Sigma} B_{ij}\, X^i \frac{dX^j}{d\tau}\, d\tau$$

AND THE SPACETIME ACTION REDUCES TO

$$S = \frac{1}{g_{st}} \int d^n x \sqrt{G}\, G^{ik} G^{jl} \operatorname{Tr} \hat{F}_{ij} \hat{F}_{kl}$$

WITH NO α' CORRECTIONS

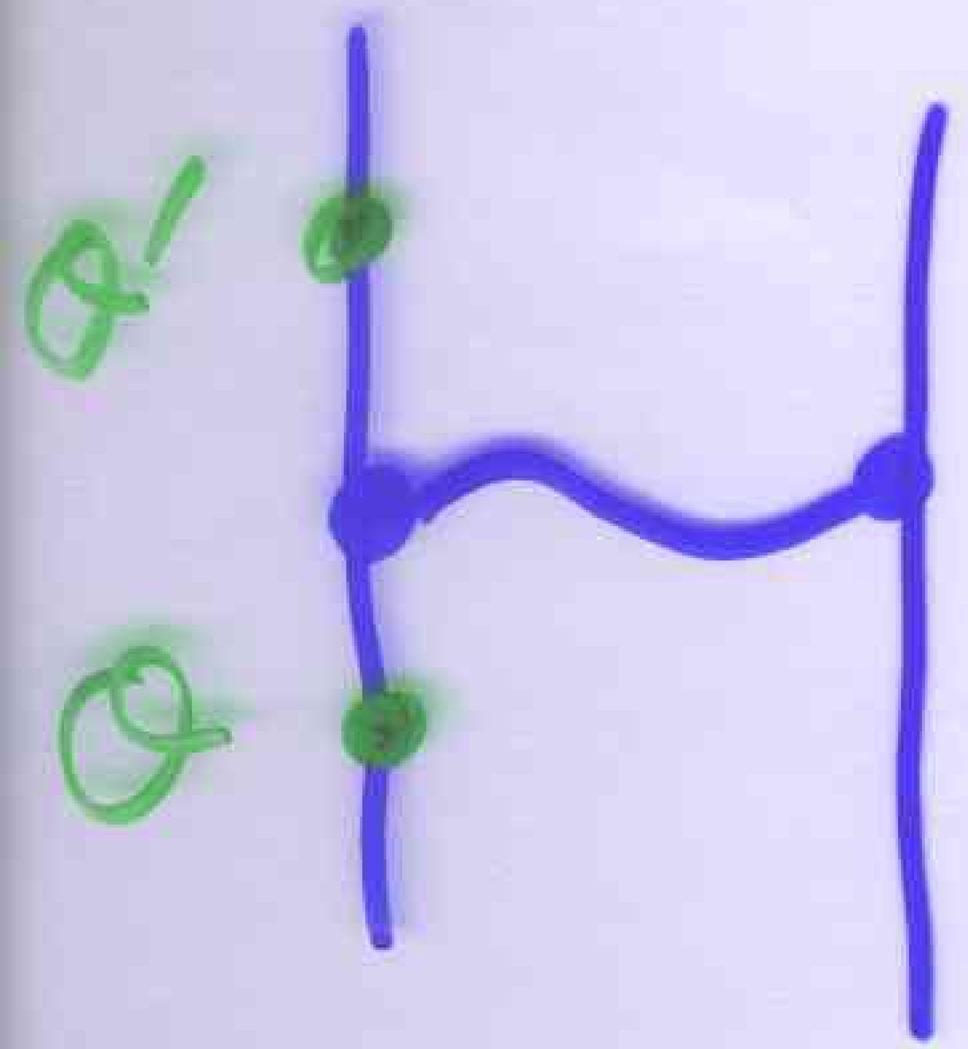
IF WE ARE ON A TORUS, THIS IS THE SMALL VOLUME LIMIT OF CONNES, DOUGLAS, AND SCHWARZ.

IN THIS LIMIT, THE NON-COMM YANG-MILLS GIVES A COMPLETE DESCRIPTION. FOR EXAMPLE, THE MULTIPLICATION OF OPEN STRING VERTEX OPERATORS IS JUST GIVEN BY A * PRODUCT:

$$e^{ik\cdot X}(\tau)\, e^{iq\cdot X}(\tau')$$

$$\rightarrow \left(e^{ik\cdot X} * e^{iq\cdot X}\right)(\tau')$$

THE OPEN STRINGS FORM A MODULE FOR THE $*$-ALGEBRA $\mathcal{A}$:



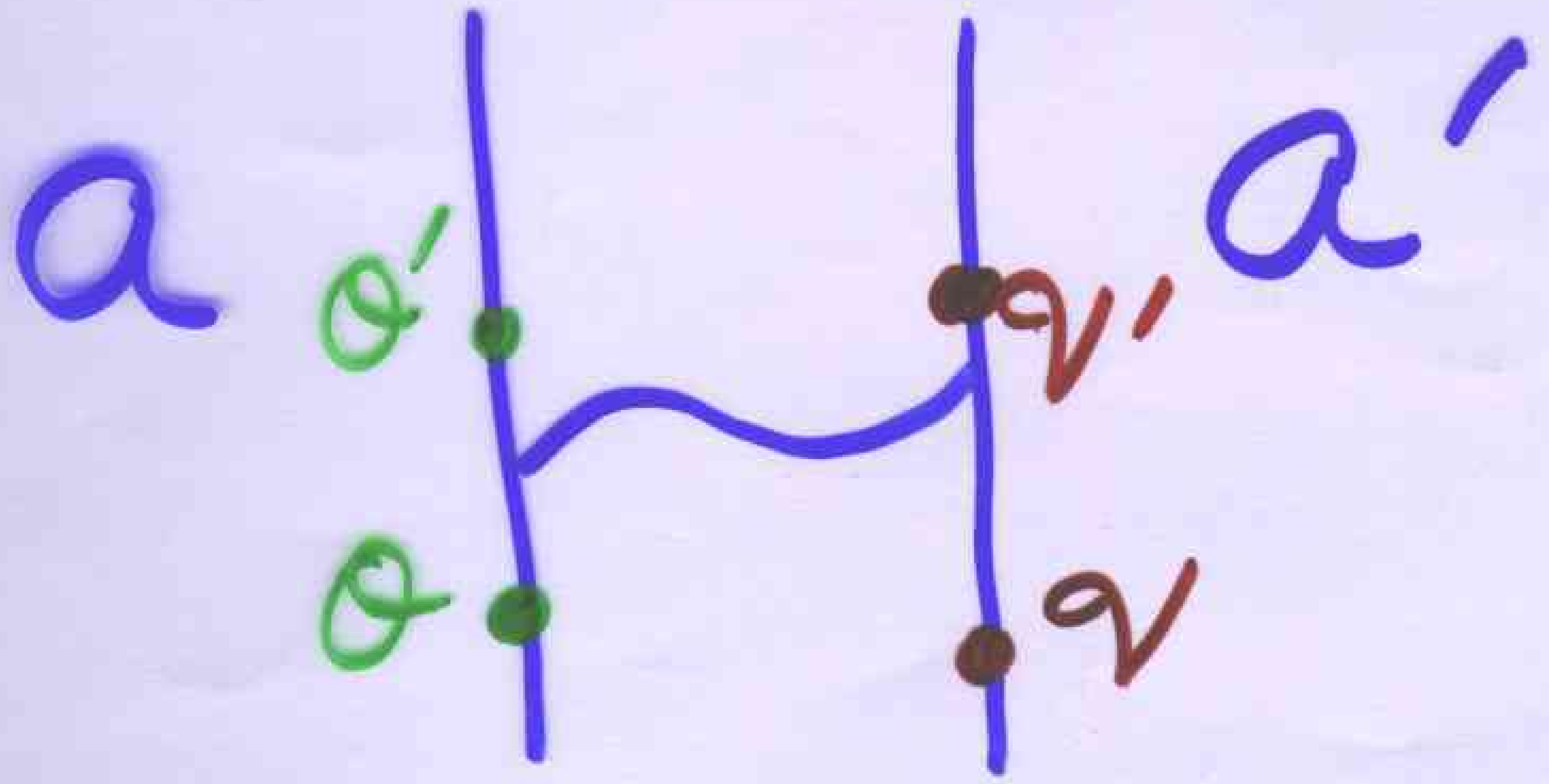
$$\mathcal{O}' = e^{ik\cdot X}$$

$$\mathcal{O} = e^{iq\cdot X}$$

IF ONE RESTRICTS THIS MODULE TO THE STRING GROUND STATES, ONE GETS THE USUAL "PROJECTIVE MODULES" STUDIED MATHEMATICALLY.

IN FACT, WE CAN READILY UNDERSTAND MANY ASPECTS OF THE MATHEMATICAL THEORY.

FOR EXAMPLE, IF $\mathcal{a}$ ACTS ON A MODULE M, IT COMMUTES WITH A SIMILAR ALGERA $\mathcal{a}'$:

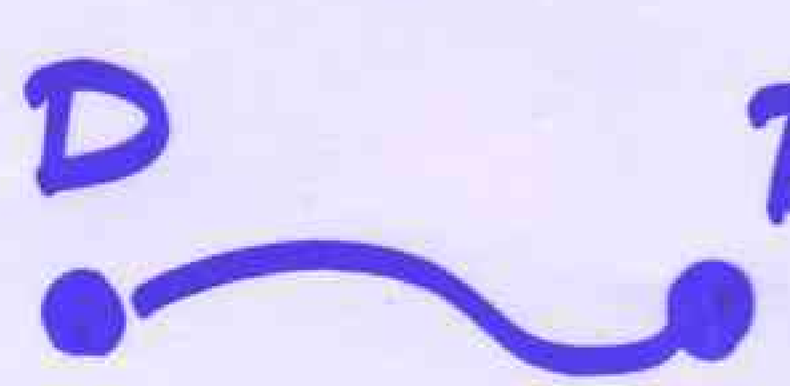


THAT IS OBVIOUS: VERTEX OPERATORS CAN BE INSERTED AT EITHER END OF THE STRING; CLEARLY,

θ, θ' COMMUTE WITH v', v

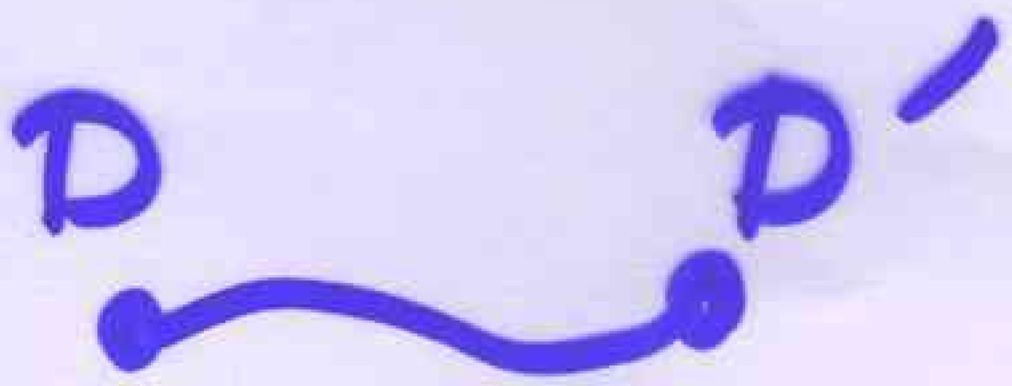
IN THIS SITUATION
"$\mathcal{A}$ AND $\mathcal{A}'$ ARE
MORITA-EQUIVALENT VIA $\mathcal{M}$."

IF D AND D' ARE
TWO D-BRANE STATES
THEN THE D-D' OPEN
STRINGS



GIVE A "MORITA EQUIVALENCE"
BETWEEN THE CORRESPONDING
ALGEBRAS $\mathcal{A}_D$, $\mathcal{A}_{D'}$.

IN PARTICULAR
EVERY D' GIVES A
MODULE FOR $\mathcal{a}_D$



SO WE CAN UNDERSTAND
THE MODULES FROM
STANDARD QUANTIZATION
OF OPEN STRINGS,
ALONG WITH THE
OTHER MATHEMATICAL
STATEMENTS

THAT HAVE BEEN USED

TO STUDY THE

$\hat{F}^2$ THEORY.

IN GENERAL, IN THE
$\alpha' \to 0$, θ, G FIXED
LIMIT, WHERE THE
$\hat{F}^2$ THEORY IS RELEVANT,
MANY CLAIMS ABOUT IT
THAT HAVE BEEN MADE
CAN BE SYSTEMATICALLY
DERIVED FROM OPEN STRINGS,
ALONG WITH NEW RESULTS
YOU'LL HEAR IN SEIBERG'S
TALK.