Complex Gravity &
Noncommutative Geometry

Strings 2000

Ali Chamseddine
cams
AUB

- Noncommutative Geometry
- Gravity & Complex Gravity : Gauging $U(1,D-1)$
- Generalization to NC spaces with $*$ product
- open problems

- presence of constant background B-field for open strings or D-branes $\Rightarrow$ Noncommuting space-time coordinates

- More generally for branes on S^3 with non-constant B-field (but constant $H = dB$) yields a field theory on non-commutative fuzzy 2-sphere.

This is expected to be the result for a general background B-field:

The space is expected to be non-commut. & curved.

Q: How to describe the dynamics of gravitational field for such spaces?

- Use the NC geometry of Alain Connes as specified by spectral triples:

$(\mathcal{A}, \mathcal{H}, \mathcal{D})$

$\mathcal{A}$: associative algebra with $*$ product and identity

$\mathcal{H}$: Hilbert space

$\mathcal{D}$: self-adjoint operator on $\mathcal{H}$ such that $[\mathcal{D}, a]$, $a \in \mathcal{A}$ defines a bounded operator on $\mathcal{H}$.

It is possible, in this setting, to develop analogue of NC Riemannian geometry:

AHC + G. Felder + J. Fröhlich

Comm. Math. Phys. 155 205 ('93)

Ex: $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(S, M)$

$\mathcal{D} = e_a^\mu \gamma^a (\partial_\mu + \omega_\mu)$

A good example of the realization of NC spaces is the data encoded in superconformal field theory

A : the algebra of vertex oper. in a conf. field theory

H : Hilbert space of states

D : Ramond generator

The operator D encodes : metric, differential calculus, integration, dynamics

For simple NC spaces such as the NC space defined by Standard model all information about the action, bosonic & fermionic, is in the Dirac operator (i.e spectrum of D). The spectral action principle does work

A. C + Alain Connes
Comm. Math. Physics. '97

The difficulty in this approach is that to make progress, one must know the Dirac operator. Enough information about D must be available to define geometrical quantities.

In the problem at hand, it is not easy to guess what D should we start with.

Strategy: Gather information about NC spaces with background B-fields

- open strings or D-branes in presence of B-field can be realized by deforming algebra of functions on classical world volume.

  The O.P.E of vertex operator is identical to $*$ product of functions of NonComm. spaces.

For U(N) Y-M theory :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu * A_\nu - A_\nu * A_\mu$$

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\eta^\nu}} f(x+\xi)\, g(x+\eta)\Big|_{\xi=\eta=0}$$

The gauge transf. are :

$$A_\mu^g = g * A_\mu * g_*^{-1} - \partial_\mu g * g_*^{-1}$$

$$g * g_*^{-1} = g_*^{-1} * g = 1$$

Because of the θ - contributions in $*$ product only U(N) gauge theories are allowed : condition $A_\mu^+ = -A_\mu$ could be preserved.

$$A_\mu * A_\nu - A_\nu * A_\mu \equiv [A_\mu \overset{*}{,} A_\nu] =$$

$$\left(A_\mu^a A_\nu^b - \frac{1}{2}\theta^{\rho\sigma}\theta^{\kappa\lambda}\partial_\rho\partial_\kappa A_\mu^a \partial_\sigma\partial_\lambda A_\nu^b + O(\theta^4)\right)[T^a, T^b]$$

$$+ i\left(\theta^{\rho\sigma}\partial_\rho A_\mu^a \partial_\sigma A_\nu^b + O(\theta^3)\right)\{T^a, T^b\}$$

$\Rightarrow$ only U(N) theories

For open-strings in back. B-field the effective metric is ;

$$g^{ij} = (G_{ij} + 2\pi\alpha' B_{ij})^{-1}_{S}$$

$$\theta^{ij} = (G_{ij} + 2\pi\alpha' B_{ij})^{-1}_{A}$$

$$g_{ij} = G_{ij} - (2\pi\alpha')^2 (BG^{-1}B)_{ij}$$

one can imagine a general setting where the closed string theory metric arises as an effective metric coming from open strings, or where the D-branes become dynamical. Under such circumstances one can get an effective metric of the form

$$g_{\mu\nu} = e_\mu^{\ a} * e_{\nu a}$$

Because of θ-contributions, the metric must become complex!

Q: Can the metric become complex?

To answer this question let us briefly review the simplest construction of gravity based on gauging Lorentz group $SO(1, D-1)$:

$$D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}$$

$$[D_\mu, D_\nu] = \frac{1}{4}R_{\mu\nu}{}^{ab}\Gamma_{ab}$$

$$R_{\mu\nu}{}^{ab} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} + \omega_\mu{}^{ac}\omega_{\nu c}{}^{b} - \omega_\nu{}^{ac}\omega_{\mu c}{}^{b}$$

Introduce $e_\mu{}^a$ & inverse $e_a{}^\mu$:

$$e_\mu{}^a e_a{}^\nu = \delta_\mu{}^\nu$$

under gauge transf.:

$$\tilde\omega_\mu{}^{ab} = M^a{}_c M^b{}_d\,\omega_\mu{}^{cd} - M^a{}_c\,\partial_\mu M^b{}_d$$

$$MM^T = 1$$

$$\tilde e_\mu{}^a = M^a{}_c\, e_\mu{}^c$$

The action

$$I = \int d^Dx\, e\, e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}$$

is gauge invariant.

One can integrate $\omega_\mu{}^{ab}$ (gaussian integ) so that I becomes $I(e)$.

Because of gauge invariance:

$$M^a{}_b = \delta^a_b + \Lambda^a{}_b \,, \quad \Lambda^{ab} = -\Lambda^{ba}$$

Λ^{ab} could be used to eliminate antisymm. part of $e_\mu{}^a$. Only symmetric part in $g_{\mu\nu} = e_\mu{}^a e_{\nu a}$ remains

$$\therefore \quad I = I(g) = \int \sqrt{g}\, R(g)\, d^Dx$$

note: It is a coincidence that torsion condition:

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b = 0$$

can be solved uniquely for $\omega_\mu{}^a{}_c$ in terms of $e_\mu{}^a$:

$\omega_\mu{}^{ab}$: $D \cdot \frac{D(D-1)}{2}$ variables

$T_{\mu\nu}{}^a$: $\frac{D(D-1)}{2} \cdot D$ conditions.

- provided $T_{\mu\nu}{}^a = 0$, the action I is invariant under the transf:

$$\delta e_\mu{}^a = D_\mu \xi^a$$
$$\simeq \partial_\mu \xi^\nu e_\nu{}^a + \xi^\nu \partial_\nu e_\mu{}^a$$
$$\underline{\xi^a = \xi^\nu e_\nu{}^a}$$

Diffeomorphisms.

I will make 2 assumptions

- $g_{\mu\nu}$ is a complex field
- The gauge group is $U(1, D-1)$

Set of transf. leaving quadratic form $Z^{a+} \eta^a_b Z^b$ invariant

$$\eta^a_b = \mathrm{diag}(-1, \underbrace{1, \ldots 1}_{D-1})$$

For simplicity, will ignore 2

We have $U(D)$ transf:

$$\tilde{z}^a = M^a{}_b z^b \quad \text{such that}$$

$$M^+ M = 1$$

Gauge fields $\omega_\mu{}^a{}_b$ satisfy:

$$\tilde{\omega}_\mu{}^a{}_b = M^a{}_c \omega_\mu{}^c{}_d M^{-1 d}{}_b - M^a{}_c \partial_\mu M^{-1 c}{}_b$$

$$\omega_\mu{}^a{}_b{}^+ = -\omega_\mu{}^b{}_a$$

The curvature $R_{\mu\nu}{}^a{}_b$ is:

$$R_{\mu\nu}{}^a{}_b = \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c \omega_\mu{}^c{}_b$$

and transforms as:

$$\tilde{R}_{\mu\nu}{}^a{}_b = M^a{}_c R_{\mu\nu}{}^c{}_d M^{-1 d}{}_b$$

Introduce complex vielbein $e_\mu{}^a$ & inverse:

$$e_a{}^\nu e_\mu{}^a = \delta_\mu^\nu, \quad e_a{}^\nu e_\nu{}^b = \delta_a^b$$

These transform as:

$$\tilde{e}_\mu{}^a = M^a{}_b e_\mu{}^b, \quad \tilde{e}_a{}^\mu = e_b{}^\mu M^{-1 b}{}_a$$

Define the complex conjugates:

$$e_{\mu a} \equiv e_\mu^{a\,\dagger}$$
$$e^{\mu a} = e^\mu_a{}^\dagger$$

With this we can write gauge invariant hermitian term:

$$e^\mu_a R_{\mu\nu}{}^a{}_b e^{\nu b}$$

- Transforms into:

$$e^\mu_d M^{-1d}{}_a M^a{}_c R_{\mu\nu}{}^c{}_f M^{-1f}{}_b M^{b}{}_c e^{\nu c}$$
$$= e^\mu_a R_{\mu\nu}{}^a{}_b e^{\nu b}$$

- Hermitian: $(e^\mu_a R_{\mu\nu}{}^a{}_b e^{\nu b})^\dagger$

$$= e^\nu_b (-R_{\mu\nu}{}^b{}_a) e^{\mu a}$$
$$= e^\mu_a R_{\mu\nu}{}^a{}_b e^{\nu b}$$

The measure, however, is not unique.

One can use $(\det e\bar{e})^{\frac{1}{2}}$

or $(\det G)^{\frac{1}{2}}$ where $G_{\mu\nu} = \mathrm{Re}(e_\mu^a e_{\nu a})$

$$I = \int d^Dx \sqrt{G}\, e^\mu_a R_{\mu\nu}{}^a{}_b e^{\nu b}$$

• Notes

• The torsion condition

$$T_{\mu\nu}{}^{a} = \partial_\mu e_\nu{}^{a} - \partial_\nu e_\mu{}^{a} + \omega_\mu{}^{a}{}_{b}\, e_\nu{}^{b} - \omega_\nu{}^{a}{}_{b}\, e_\mu{}^{b} = 0$$

has $D.\frac{(D-1)}{2}.D.2$ conditions $= D^3 - D^2$

$\omega_\mu{}^{a}{}_{b}$ has $D \cdot D^2 = D^3$ variables

cannot be solved for $\omega_\mu{}^{a}{}_{b}$.

• The $\omega_\mu{}^{a}{}_{b}$ appears linearly & quadratically in action except for trace part ($U(1)$)

Trace part appears linearly.

Traceless part can be integrated by solving eqs of motion, while linear part implies a constraint

The traceless part of $\omega_\mu{}^{a}{}_{b}$ has $D^3 - D$ variables

Define the metric

$$g_{\mu\nu} = e_{\mu a}\, e_\nu{}^a$$

Then

$$g_{\mu\nu}^\dagger = e_{\nu a}\, e_\mu{}^a = g_{\nu\mu}$$

Same as in Einstein–Strauss theory

If we write

$$g_{\mu\nu} = G_{\mu\nu} + i\, B_{\mu\nu}$$

then

$$g_{\mu\nu}^\dagger = G_{\mu\nu} - i\, B_{\mu\nu} = G_{\nu\mu} + i\, B_{\nu\mu}$$

$$\therefore\quad G_{\mu\nu} = G_{\nu\mu}\,,\quad B_{\mu\nu} = -B_{\nu\mu}$$

metric combines symm. & antisymm. pieces.

Examine gauge transf.:

$$\delta e_\mu{}^a = \Lambda^a{}_b\, e_\mu{}^b \qquad \Lambda^a{}_b{}^\dagger = -\Lambda^b{}_a$$

Write: $e_\mu{}^a = e_{0\mu}{}^a + i\, e_{1\mu}{}^a$

$$\Lambda^a{}_b = \Lambda_0{}^a{}_b + i\,\Lambda_1{}^a{}_b$$

then: $\Lambda_0{}^a{}_b{}^T = -\Lambda_0{}^a{}_b\,, \quad \Lambda_1{}^a{}_b{}^T = \Lambda_1{}^a{}_b$

· Notes

- The torsion condition

$$T_{\mu\nu}{}^{a} = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b = 0$$

has $\frac{D(D-1)}{2} \cdot D \cdot 2$ conditions $= D^3 - D^2$

$\omega_\mu{}^a{}_b$ has $D \cdot D^2 = D^3$ variables

cannot be solved for $\omega_\mu{}^a{}_b$.

- The $\omega_\mu{}^a{}_b$ appears linearly & quadratically in action except for trace part ($U(1)$)

Trace part appears linearly.

Traceless part can be integrated by solving eqs of motion, while linear part implies a constraint

The traceless part of $\omega_\mu{}^a{}_b$ has $D^3 - D$ variables

Second order formulation

The ω-equation could be solved, perturbatively, for traceless part, while the trace part imposes a constraint:

$$\frac{1}{\sqrt{G}}\partial_\nu\left(\sqrt{G}\left(e_a{}^\nu e^{\mu a} - e_a{}^\mu e^{\nu a}\right)\right) = 0$$

Solution of ω-eq is very complicated and must be expressed as a perturbative expansion in function of $B_{\mu\nu}$.

Examine theory in the limit $G_{\mu\nu} = \delta_{\mu\nu}$:

$$e_\mu{}^a = \delta_\mu{}^a - \frac{i}{2} B_{\mu a} \quad , \quad e^{\mu a} = \delta^{\mu a} + \frac{i}{2} B_{\mu a} + \cdots$$

The constraint becomes:

$$\partial_\nu B^{\mu\nu} = 0$$

and the Lagrangian reduces to

$$L = \frac{1}{12} H_{\mu\nu\rho}^2 \quad , \quad H_{\mu\nu\rho} = 3\,\partial_{[\mu} B_{\nu\rho]}$$

It was argued by Damour & Deser that NGT have ghosts because of the presence of the linear term : $\sim \omega_\mu \partial_\nu B^{\mu\nu}$ which effectively makes ω_μ to propagate. These could be saved by adding a cosmological term:

$$e^{\mu a} e^{\nu b} (e_{\mu a} e_{\nu b} - e_{\nu a} e_{\mu b})$$

$$(\sim m^2 B_{\mu\nu} B^{\mu\nu})$$

This problem is related to the problem of the lack of gauge invariance associated with $B_{\mu\nu}$ to protect it.

One can ask the question whether the action in second order form has an extended diffeomorphism invariance following from $\delta e_\mu^a = D_\mu \zeta^a$

$$\zeta^a = e^a_\nu \zeta^\nu$$

where ζ^a is complex. This will generate

$$\omega_{\alpha\beta\gamma} = N_{\alpha\beta\gamma}^{\sigma\rho\kappa}\ Y_{\sigma\rho\kappa}$$

$$N_{\alpha\beta\gamma}^{\sigma\rho\kappa} = \frac{1}{2}\left(\delta_\gamma^\sigma \delta_\beta^\rho + \delta_\beta^\sigma \delta_\alpha^\rho \delta_\gamma^\kappa - \delta_\alpha^\sigma \delta_\gamma^\rho \delta_\beta^\kappa\right)$$
$$-\frac{1}{4}\left(\delta_\beta^\kappa \delta_\alpha^\sigma L_\gamma^\rho + \delta_\alpha^\kappa \delta_\gamma^\sigma L_\beta^\rho - \delta_\gamma^\kappa \delta_\beta^\sigma L_\alpha^\rho\right)$$
$$+\frac{1}{4}\left(L_\gamma^\kappa \delta_\beta^\sigma \delta_\alpha^\rho + L_\beta^\kappa \delta_\alpha^\sigma \delta_\gamma^\rho - L_\alpha^\kappa \delta_\gamma^\sigma \delta_\beta^\rho\right)$$
$$-\frac{1}{4}\left(\delta_\alpha^\kappa L_\gamma^\sigma \delta_\beta^\rho + \delta_\gamma^\kappa L_\beta^\sigma \delta_\alpha^\rho - \delta_\beta^\kappa L_\alpha^\sigma \delta_\gamma^\rho\right) + O(L^2)$$

$$L^\mu{}_\nu = i\, G^{\mu\rho} B_{\rho\nu} - 2\, G^{\mu\rho} B_{\rho\sigma} G^{\sigma\lambda} B_{\lambda\nu} + O(B^3)$$

$$Y^\mu{}_{\rho\kappa} = -X^\mu{}_{\rho\kappa} + \frac{1}{8}\delta^\mu_\kappa\left(3X^\lambda{}_{\rho\lambda} - X^\lambda{}_{\lambda\rho}\right) + \frac{1}{8}\delta^\mu_\rho\left(-X^\nu{}_{\kappa\nu} + 3X^\nu{}_{\nu\kappa}\right)$$

$$X^\mu{}_{\rho\kappa} = e_{\rho c}\, e^a_\kappa \left(\frac{1}{\sqrt{G}}\right)\partial_\nu\left(\sqrt{G}\left(e^\nu_a e^{\mu c} - e^\mu_a e^{\nu c}\right)\right)$$

diffeomorphism for $G_{\mu\nu}$ & antisymm transf for $B_{\mu\nu}$.

(It is difficult to prove that this follows because the ω-eq could not be solved in closed form)

Generalization to NC spaces with * product:

This is straightforward. All expressions were written in such a way that the commutativity of the transfor. matrices $M^a{}_b$ were not used:

$$R_{\mu\nu}{}^a{}_b = \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c * \omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c * \omega_\mu{}^c{}_b$$

$$I = \int d^Dx \sqrt{G}\; e^\mu{}_a * R_{\mu\nu}{}^a{}_b * e^{\nu b}$$

It is quite possible that there exists a non linear transformation for $G_{\mu\nu}$ & $B_{\mu\nu}$ that will absorb the * product making this action equivalent to the commutative one.

The spectral triple

In NC geometry one takes the faithful representation:

$$\pi : H \longrightarrow B(H)$$

such that

$$\pi(a_0 \, d a_1 \ldots d a_n) = a_0 [D, a_1] \ldots [D, a_n]$$

A one form is represented by

$$A = \sum a^i d b^i \longrightarrow$$

$$\pi(A) = \sum a^i [D, b^i]$$

To deform this with $*$ product one cannot take

$$\sum a^i * [D, * b^i]$$

because

$$[D, * b^i] \quad , \quad D = \gamma^a e_a{}^\mu \nabla_\mu$$

contains : $\gamma^a (e^a_\mu * b^i - b^i * e^a_\mu) \nabla_\mu$

One must take $\sum a^i * [D, b^i]$ by complications

Open problems:

- Is the complex gravity action equivalent to Einstein + Rosen NGT?

  Geometrically there are 2 possibilities to take, but only one for the gauged case

- Is there is a hidden "complex" diffeomorphism invariance combining diffeom parameter ξ^μ and $B_{\mu\nu}$ abelian gauge parameter Λ_μ?

  Does it arise from translational invariance $\delta e_\mu^a = D_\mu \xi^a$ ξ^a: complex

  This will guarantee consistency of theory although $B_{\mu\nu}$ would appear in action. (This would be similar to $G_{\mu\nu}$ appearance protected by diffeomorphism)

- Is the action with $*$ product equivalent to the one without $*$ product by a non-linear field redefinition?

- How to define a spectral triple for NC spaces with arbitrary curved backgrounds?

  Information about Dirac operator.