NONCOMMUTATIVE FIELD THEORIES:

UNITARITY AND DECOUPLING

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QFT ON $R_\theta^{1,d}$

$R_\theta^{1,d}$ SPECIFIED BY A NC ALGEBRA

$$[x^\mu, x^\nu] = i\,\theta^{\mu\nu} \qquad [\theta] = L^2$$

θ DEFINES A NONCOMMUTATIVITY SCALE

WHAT IS QFT ON $R_\theta^{1,d}$?

VIA WEYL-MOYAL CORRESPONDENCE

$$\underset{\substack{\uparrow\\ \text{OPERATORS}}}{\text{QFT}} \text{ ON } R_\theta^{1,d} \Longleftrightarrow \underset{\substack{\uparrow\\ \text{FUNCTIONS}}}{(\text{QFT})_\theta} \text{ ON } R^{1,d}$$

$$O_f \in R_\theta^{1,d} \longleftrightarrow f \in C^\infty(R^{1,d})$$

$$O_h = O_f \cdot O_g \longleftrightarrow h = f * g$$

WHERE

$$(f*g)(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\beta^\nu}} f(x+\alpha)\, g(x+\beta)\Big|_{\alpha=\beta=0} \simeq$$

$$\simeq fg + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f\, \partial_\nu g + O(\theta^2)$$

$*$-PRODUCT, UNIQUE ASSOCIATIVE DEFORMATION OF $C^\infty(R^{1,d})$

⇒ REPLACE PRODUCTS OF FIELDS BY $*$-PRODUCT

e.g: NONCOMMUTATIVE SCALAR ϕ^n

$$S = \int d^Dx \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\, \phi * \phi * \cdots * \phi \right)$$

SINCE: $\int d^Dx\, f*g = \int d^Dx\, fg$

⇒ NON-LOCAL FIELD THEORIES

$$S_{int} = -\frac{i\lambda}{n!} \int \prod_{j=1}^{n} d^Dp_j\, \phi(p_j)\, e^{-\frac{i}{2}\sum_{k<\ell} p_\mu^{(k)} \theta^{\mu\nu} p_\nu^{(\ell)}}$$

CLASSICALLY, $E \ll \theta^{-1/2}$ ⇒ CONVENTIONAL QFT

QUANTUM THEORY, $E \ll \theta^{-1/2}$ $\not\Rightarrow$ CONVENTIONAL QFT

⇒ LACK OF DECOUPLING OF ENERGY SCALES
(Minwalla, Van Raamsdonk & Seiberg)

HEURISTICALLY,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \Rightarrow \Delta x^\mu \Delta x^\nu \sim i\theta^{\mu\nu}$$
$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \Rightarrow \Delta x^\mu \Delta p^\nu \sim i\eta^{\mu\nu}$$

$$\Rightarrow \quad \underline{\Delta x^\mu \sim \theta^{\mu\nu} \Delta p_\nu}$$

HIGH ENERGY MODES HAVE DRASTIC EFFECTS ON LOW ENERGY PROCESSES

SENSIBLE QFT ¿? (SEE LATER)

- UNITARY?
- RENORMALIZABLE?

NONCOMMUTATIVITY IN STRING THEORY

OPEN STRINGS IN A BACKGROUND $B_{\mu\nu}$

$$\Rightarrow [x^\mu, x^\nu] = i\theta^{\mu\nu}$$

WHERE,

$$\theta^{\mu\nu} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B} \right)_A^{\mu\nu}$$

2 LENGTH SCALES : α' AND θ

D-BRANE EFFECTIVE ACTION OBTAINED FROM S-MATRIX ELEMENTS OF OPEN STRINGS:

$$\langle x^\mu(\tau) x^\nu(\tau') \rangle = -\alpha' G^{\mu\nu} \log(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu} \operatorname{sign}(\tau-\tau')$$

WHERE,

$$G^{\mu\nu} = \left(\frac{1}{g + 2\pi\alpha' B} \right)_S^{\mu\nu}$$

$\Rightarrow \mathcal{L}_{eff}$ OBTAINED FROM $\mathcal{L}_{eff}^{\theta=0}$ BY REPLACING

- FIELDS BY $*$-PRODUCT OF FIELDS

CAN NCQFT DESCRIBE SOME LOW ENERGY LIMIT OF STRING THEORY?

- YES, NCQFT BETTER MAKE SENSE

- NO, TRUNCATION TO NCQFT MUST BE INCONSISTENT

- WHICH BACKGROUNDS HAVE A LIMIT S.T:

  - $G_{\mu\nu}$, $\theta_{\mu\nu}$ FINITE, AND

  - MASSIVE OPEN STRINGS AND CLOSED STRINGS DECOUPLE

$$\Downarrow$$

NC GAUGE THEORY

<u>D3-BRANE</u>

$$I_1 = \frac{1}{2} B_{\mu\nu} B^{\mu\nu} = \vec{B}^2 - \vec{E}^2$$

$$I_2 = \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma} = \vec{E} \cdot \vec{B}$$

CAN BOOST S.T. EITHER E∥B, −E∥B OR E⊥B (9 POSSIBLE CASES)

NCQFT LIMIT REQUIRES:

- $\Theta^{\mu\nu}$ AND $G^{\mu\nu}$ FIXED

- $\alpha' \to 0$

(Seiberg & Witten)

(Connes, Douglas & Schwarz)

(Douglas & Hull)

WHICH BACKGROUNDS HAVE A SENSIBLE LIMIT?

$g_{\mu\nu} = (-g, \dots, g)$

$$\Rightarrow \quad \mathcal{L}_{DBI} = -T_3 \sqrt{-\det(g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} =$$

$$= -T_3 \sqrt{g^4 + (2\pi\alpha')^2 g^2 (\vec{B}^2 - \vec{E}^2) - (2\pi\alpha')^4 (\vec{E}\cdot\vec{B})^2}$$

$I_2 = \vec{E}\cdot\vec{B} \neq 0 \Rightarrow$ INSTABILITY FOR $|E| > E_c = \frac{g}{2\pi\alpha'}$

$\Rightarrow$ SENSIBLE LIMIT IF $\frac{|E|}{E_c} < 1$, BUT

$\frac{|E|}{E_c} \to \infty$ IN DECOUPLING LIMIT!!

$\Rightarrow$ $I_2 \neq 0$ BACKGROUNDS DO NOT HAVE A SENSIBLE NCQFT LIMIT

$\Rightarrow$ WHAT ABOUT CASES WITH $I_2 = 0$?

THREE CASES

$(I_2 = 0)$

- $I_1 > 0$, PURELY MAGNETIC: eg B_{23} $\Rightarrow$
θ^{23} SPACE-LIKE NONCOMMUTATIVITY
$\Rightarrow$ $\exists$ DECOUPLING FOR SPACE-LIKE NCQFT

- $I_1 < 0$, PURELY ELECTRIC: eg B_{01} $\Rightarrow$
θ^{01} TIME-LIKE NONCOMMUTATIVITY
$\Rightarrow$ $\nexists$ DECOUPLING LIMIT FOR TIME-LIKE NCQFT

Seiberg, Susskind & Toumbas ; Gopakumar, Maldacena, Minwalla & Strominger; Barbón & Rabinovici

- $I_1 = 0$, LIGHT-LIKE FIELD: e.g $B_{02} = B_{12}$ or B_{2+}
$\Rightarrow$ θ^{2-} LIGHT-LIKE NONCOMMUTATIVITY, NO E_{cr}
$\Rightarrow$ $\exists$ DECOUPLING FOR LIGHT-LIKE NCQFT

<u>SUMMARY</u>: ONLY PURELY MAGNETIC AND LIGHT-LIKE BACKGROUNDS ADMIT A DECOUPLED NCQFT LIMIT

OTHER BACKGROUNDS CAN HAVE FINITE θ AT THE EXPENSE OF HAVING A FINITE α'

IN FACT, (FOR AN ELECTRIC BACKGROUND)

$$\theta \simeq \alpha'$$

$\Rightarrow$

MASSIVE OPEN STRINGS DO NOT DECOUPLE.

NCQFT BECOMES RELEVANT WHENEVER MASSIVE OPEN STRING STATES CANNOT BE IGNORED

<u>INTUITIVE GUESS</u>:

- NCQFT WITH $\theta^{\mu\nu}$ ARISING FROM BACKGROUNDS WITH $I_1 \not\gtrless 0$ AND $I_2 \neq 0$ <u>ARE NOT</u> UNITARY

- NCQFT WITH $\theta^{\mu\nu}$ ARISING FROM BACKGROUNDS WITH $I_1 \gtrsim 0$ AND $I_2 = 0$ <u>ARE</u> UNITARY

CONFIRM WITH A NCQFT COMPUTATION

UNITARITY OF NCQFT

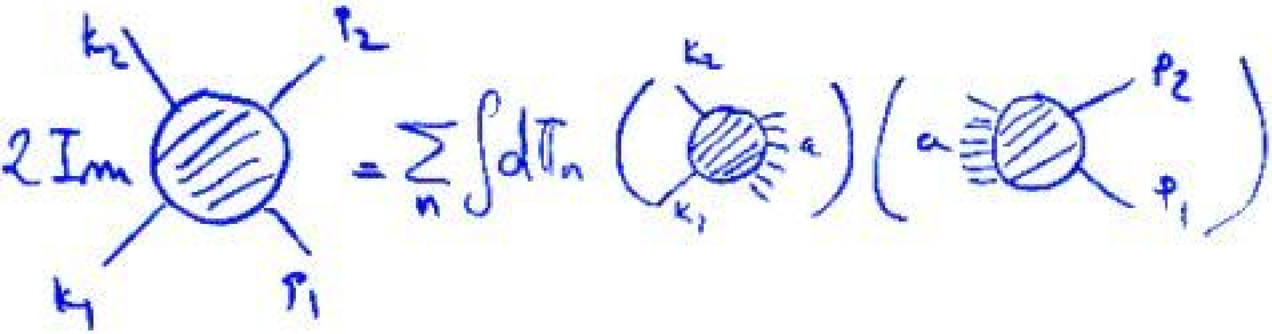
M_{ab}: TRANSITION MATRIX BETWEEN STATES a AND b.

$$S = 1 + iT$$

$$SS^{\dagger} = 1 \implies$$

$$2\,\mathrm{Im}\, M_{ab} = \sum_n M_{an} M_{nb} \qquad (A)$$

EXAMPLE:



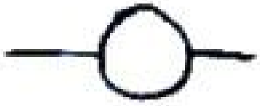
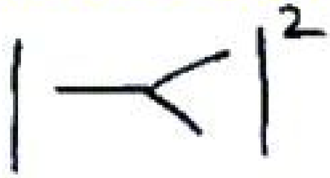
(A) ALSO SATISFIED BY GREEN'S FUNCTIONS (GENERALIZED UNITARITY)

(B) EXACT RELATION. IN PERTURBATION THEORY VERIFY ORDER BY ORDER IN λ

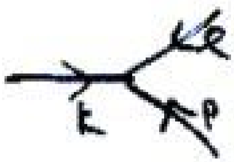
$$M = \sum_n c_n \lambda^n$$

TOY MODEL: NONCOMMUTATIVE ϕ^3

VERIFY CUTTING RULES FOR TWO-POINT FUNCTION

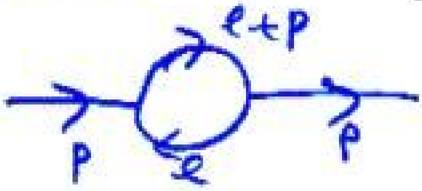
$2\,\mathrm{Im}$  $=$ 

FEYNMAN RULES:

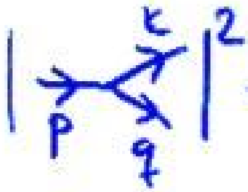
 $\dfrac{i}{p^2-m^2+i\epsilon}$ 

$-i\lambda\cos\left(\dfrac{k\wedge p}{2}\right)$

WHERE $k\wedge p = k_\mu \Theta^{\mu\nu} p_\nu$



$$iM = \frac{\lambda^2}{2}\int \frac{d^D l}{(2\pi)^D}\,\frac{1+\cos(p\wedge l)}{2}\,\frac{1}{l^2-m^2}\,\frac{1}{(l+p)^2-m^2} = iM_p + iM_{np}$$



$$= \frac{\lambda^2}{2}\,\frac{1}{(2\pi)^{D-2}}\int \frac{d^{D-1}k}{2k_0}\,\frac{d^{D-1}q}{2q_0}\,\delta^D(p-k-q)\,\frac{1+\cos(p\wedge k)}{2}$$

PLANAR PIECE SATISFIES (A) $\simeq$ CONVENTIONAL ϕ^3

WHAT ABOUT NON-PLANAR PIECE?

THE NON-PLANAR DIAGRAM EVALUATES TO:

$$M_{D=4} = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\, K_0\left(\sqrt{p\circ p\,(m^2 - p^2 x(1-x) - i\epsilon)}\right)$$

WHERE:

$$p\circ p = -p_\mu \theta^{\mu\rho} G_{\rho\sigma}\theta^{\sigma\nu}p_\nu$$

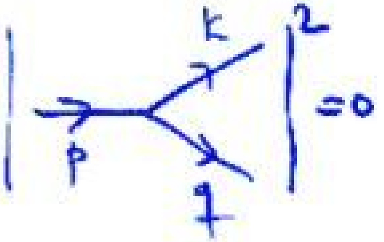
THERE ARE ESSENTIALLY TWO EXPRESSIONS:

- $\theta^{01}=\theta_E$ $\theta^{23}=\theta_B$; $p\circ p = \theta_E^2(p_0^2-p_1^2)+\theta_B^2(p_2^2+p_3^2)$

- $\theta^{2-}=\theta$; $p\circ p=\theta^2(p_0-p_1)^2$

CUTTING RULES MUST BE SATISFIED BY ANY CONFIGURATION OF EXTERNAL MOMENTUM:

- $p^2<0$; p SPACE-LIKE



$\delta(p-k-q)=0$ SINCE

$k^2=m^2$

$q^2=m^2$

IS $\mathrm{Im}\, M_{D=4} = 0$?

ANSWER: NO, IF $p\circ p < 0$, BECAUSE

$$M_{D=4} = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\, K_0\left(\sqrt{p\circ p\,(m^2 - p^2 x(1-x) - i\epsilon)}\right)$$

AND THEREFORE:

$$\mathrm{Im}\, M_{D=4} = \frac{\lambda^2}{64\pi}\int_0^1 dx\, J_0\left(\sqrt{|p\circ p|\,(m^2 + |p|^2 x(1-x))}\right)$$

$\Rightarrow$ UNITARITY RELATION SATISFIED ONLY FOR

- SPACE-LIKE NONCOMMUTATIVITY $\theta^{23} = \theta$
- LIGHT-LIKE NONCOMMUTATIVITY $\theta^{2-} = \theta$

$\Rightarrow$ UNITARITY CUTTING RULES SATISFIED ONLY BY TYPES OF NCQFT WHICH CAN BE OBTAINED FROM A DECOUPLED LIMIT OF STRING THEORY ☺

MOREOVER, ONE CAN VERIFY THAT FOR AN S-MATRIX ELEMENT, THAT THE OPTICAL THEOREM IS SATISFIED IFF

$$p \circ p = -p^{\mu}\,\theta^{\mu\rho}\,G_{\rho\sigma}\,\theta^{\sigma\nu}\,p_{\nu} \geq 0$$

HEURISTIC EXPLANATION:

$e^{p \circ p}$ REGULATES INTEGRAL, IF $p \circ p < 0$ A BRANCH CUT AS A FUNCTION OF p EMERGES WHEN ANALYTICALLY CONTINUING BACK TO MINKOWSKI SPACE

SOME COMMENTS ON LIGHT-LIKE NCQFT

D-BRANES IN PRESENCE OF

$$B_{02} = B_{12} \Rightarrow B_{2+}$$

TAKE:

$$\begin{array}{l} g_{\mu\nu} = \eta_{\mu\nu} \\ B_{2+} \end{array} \longrightarrow \begin{array}{l} G^{+-} = -G^{22} = -1 \\ \text{(AFTER COORD. TRANSF.)} \\ \theta^{2-} = (2\pi\alpha')^2 B_{2+} \end{array}$$

$\theta^{2-} \neq 0 \Rightarrow \theta^{02} = -\theta^{12}$; $\theta^{02} \neq 0$ WHY SENSIBLE?

- NON-LOCAL IN TIME, BUT
- X^+ TAKEN AS TIME COORDINATE, SUCH THAT $H = P_+$ (THEORY IS LOCAL IN X^+)

DECOUPLING LIMITS:

$$G_S = g_s \sqrt{\frac{\det(g + 2\pi\alpha' B)}{\det(g)}} = g_s$$

SUCH THAT

$$g_{YM}^2 = (2\pi)^{p-2} g_s (\alpha')^{\frac{p-3}{2}}$$

$\Rightarrow$ USUAL DECOUPLING LIMIT (g_{YM}^2 = FIXED)

$p < 3$ $\alpha' \to 0$ $g_s \to 0$

$p = 3$ $\alpha' \to 0$ g_s ARBITRARY

$p > 3$ $\alpha' \to 0$ $g_s \to \infty$

EXAMPLES:

- D3-BRANE, NCQFT WITH (θ^{-2}, g^2_{YM})

WHAT IS THE STRONG COUPLING DUAL?

USING S-DUALITY OF TYPE IIB

$$\Rightarrow \quad g_s \to \frac{1}{g_s} \Rightarrow g^2_{YM} \to \frac{1}{g^2_{YM}}$$

MOREOVER, $F_{\mu\nu} \longrightarrow * F_{\mu\nu} \Rightarrow \theta^{-i} \to \epsilon^{ij}\,\theta^{-j}$

$$\Rightarrow \mathcal{N}=4 \text{ SYM } \theta^{-2} \xleftrightarrow{\text{S-DUAL}} \mathcal{N}=4 \text{ SYM } \theta^{-3}$$

- D4-BRANE,

$$g^2_{YM} = g_s\sqrt{\alpha'} \quad ; \quad \alpha' \to 0 \Rightarrow g_s \to \infty \quad \Rightarrow$$

USE M-THEORY PICTURE

$$D4 \to M5 \text{ ON } S^1_R$$

$$M_P^3 = \frac{M_s^3}{g_s} \sim \frac{1}{\alpha'} \to \infty$$

$$R = g_s\sqrt{\alpha'} = g^2_{YM} = \text{FIXED}$$

$\Rightarrow$ GET DECOUPLED THEORY ON M5 WITH $C_{2(11)}$ +

IN $R \to \infty$ LEADS TO A NONCOMMUTATIVE DEFORMATION OF THE (2,0) SCFT!

SUMMARY

- NCQFT'S GIVE A GENERALIZATION TO THE FRAMEWORK OF LOCAL QFT CONSISTENT WITH QUANTUM MECHANICS

  ISSUES:

  - STRUCTURE?
  - WHAT ARE THE IR SAFE OBSERVABLES?
  - IS THERE A REFORMULATION OF THE LOW ENERGY THEORY WHICH IS LOCAL?

- IS SPACE-TIME NONCOMMUTATIVITY A GENERIC FEATURE OF STRING THEORY?

  - FUNDAMENTAL? (Yoneya)

- IS THERE A NONCOMMUTATIVE STRUCTURE IN M-THEORY?

  - FRAMEWORK?

D4 ON F_{2+} IS LEADING TO SUCH A POSSIBILITY ...