Some Comments on Branes, G-flux, and K-theory

G. Moore, Strings 2000

Based on

1. E. Diaconescu, G. Moore, & E. Witten, hep-th/0005090,91
2. Recent Discussions with E. Diaconescu, J. Harvey, and G. Segal

We will discuss 3 ways in which string theory and M-theory lead to K-theory.

1. K-theory theta functions from M-theory partition functions (with E. Diaconescu and E. Witten)

2. Sewing constraints and topological field theory (with G. Segal)

3. Noncommutative tachyons (with J. Harvey)

Part I: Partition Functions, from M to K

Consider the partition function Z_{IIA} of IIA theory on a smooth 10-manifold X to the partition function Z_M of M-theory on $Y = X \times S^1$.

Goal is to show that $Z_{IIA} = Z_M$

Limit: $g_{\mu\nu} = t\, g_{\mu\nu}{}^{0}$, $t \to +\infty$, $g_{string} \to 0$

Both partition functions reduce to

$$Z = (\text{simple factors}) \times \Sigma_{\,G\text{-flux}}\, e^{-S(G)}$$

But formulating precisely the sum over classical G-fluxes is subtle.

IIA Sum on G-flux

$G = G_0 + G_2 + \ldots + G_{10}$; *Two basic inputs:*

Selfduality $G = * G$ & Quantization: $x \in K^0(X)$

$[G] = ch(x + \frac{1}{2}\theta)\,(A(TX))^{1/2}$ (θ is a quantum shift)

After a long story…

$$\Theta_{IIA} = \Sigma\, e^{-KE(G)}\, e^{i\Phi}$$

Sum over all G_0, G_2, G_4 consistent with K-theoretic quantization x

$KE = t^5 \, ||G_0||^2 + t^3 \, ||G_2||^2 + t \, ||G_4||^2$, standard sugra

Phase: Extremely subtle! Requires ``quantization of the K-theory torus'' (Witten 99)

Description of the phase

$$e^{i\Phi} = \Omega(x) \exp[\, 2\pi i \int (- G_2^5/15 + G_2^3 G_4/6 + \dots)\,]$$

Terms in green are new topological phases in sugra

$\Omega(x) = \pm 1$ based on a mod two index,

$$= (-1)^{N(x)}$$

$N(x)$ = # R FZM's on IIB brane of charge x

NB! *There is no local formula for the mod 2 index!*

M-Theory Partition Function

Now we define precisely the M-theory partition function on an 11-manifold Y.

As with the RR partition function – there is a subtle quantization condition and phase – both were analyzed in (Witten 1996).

Quantization: Choose a *cohomology* class $a \in H^4(Y, \mathbb{Z})$, then $[G(a)] = a - \frac{1}{2}\lambda$, where $\lambda = \frac{1}{2} p_1$

Phase: Roughly $\Omega_M(C) = \exp[2\pi i \int_Y (CGG + C I_8)]$, but if $a \neq 0$ then C is not globally well-defined.

One approach: choose bounding manifold $\partial Z = Y$ and set $\Omega_M(C) = \exp[2\pi i \int_Z G^3 + G I_8]$. *But* (a.) still not manifestly well-defined, and (b.) it is difficult to work with.

M-Theory Phase

Best formulation in terms of E_8 gauge theory! (Witten 1996)

Mathematically, $C \in \widehat{H}^3(Y, \mathbb{R}/\mathbb{Z})$ is a "Cheeger-Simons differential character." E_8 gauge theory allows an alternative, but equivalent definition:

An "M-Theory C-field" is a quadruple (V,A,G,c)

- V an E_8 bundle with connection A
- $G \in \Omega^4(Y, \mathbb{R})$ & $c \in \Omega^3(Y, \mathbb{R})/\Omega^3_{\mathbb{Z}}$
- $G = \alpha\, \mathrm{Tr}(F^2) + \beta\, \mathrm{Tr}(R^2) + dc$
- Equivalence: $c_1 - c_2 + CS(A_1, A_2) = 0$

This definition works because E_8 bundles $V \Leftrightarrow a \in H^4(Y, \mathbb{Z})$,

M-Theory Phase -II

In terms of this data the phase is:

$$\Omega_M(C) = \exp\left[2\pi i\left(\frac{\eta(\not{D}_{V(a)}) + h(\not{D}_{V(a)})}{4} + \frac{\eta(D_{RS}) + h(D_{RS})}{8}\right)\right]$$
$$\cdot \exp\left[2\pi i \int_Y \left(\frac{1}{2}G^2 + \frac{(\lambda^2 - p_2)}{48}\right)c\right]$$

- D_{RS} is the Rarita-Schwinger contribution
- $h(D)$ = number of zeromodes of the operator D on Y.
- $\eta(D)$ is the APS eta invariant:

$$\eta(D) = \sum_{\mathrm{eigenvalues}\,\lambda \neq 0} \frac{\lambda}{|\lambda|}$$

THE DESCRIPTION OF PARTITION FUNCTIONS IS VERY DIFFERENT

FOR GENERAL X, THE KALUZA-KLEIN REDUCTION OF 11-DIMENSIONAL SUGRA ON $X \times S^1$

IS <u>NOT</u>

THE IIA SUGRA ON X ....

NOT ALL $a \in H^4(X,\mathbb{Z})$ ARE OF THE FORM $ch(x) = a + \cdots$ FOR SOME $x \in K^0(X)$ ....

Equality of partition functions

$Y = X \times S^1$, susy spin structure & ''C-field'' pulled back from X

Compare sums at order e^{-t}. DMW: Both sums reduce to

$$\Theta = \sum e^{-\|a - \lambda/2\|^2} (-1)^{f(a)}$$

Sum over torsion in Θ_M projects to a such that $Sq^3(a)=0$, & this is precisely the condition for $ch(x) = a + \ldots$ for some K-theory class x!

2. $f(a)$ = a certain mod-two index

3. Computation extends to nontrivial circle bundles $Y \to X$

Directions for Further Work

1. One-loop determinants
2. Extension to Type I
3. Inclusion of topologically nontrivial B-field
4. S-duality and IIB theory
5. ‘‘Instanton amplitudes’’

But the question for the remainder of this talk is: Can we understand how this subtle topology arises from a more *microscopic* view, i.e. via CFT or SFT ⇒ K-Theory of algebras. (Vancea, Seiberg & Witten, Witten, Leigh et. al., Periwal…)

Part II: Sewing Constraints & D-Branes

(work in progress with Graeme Segal)

- Given a closed string background, what are the possible D-branes?
- Given a closed CFT $\mathcal{C}$, what are the possible boundary states?
- Too hard! But replacing $\mathcal{C}$ by 2D TFT leads to a solvable, yet not entirely trivial problem

Recall Ancient Folktheorem:

2D TFT's are in 1-1 correspondence with commutative Frobenius algebras:

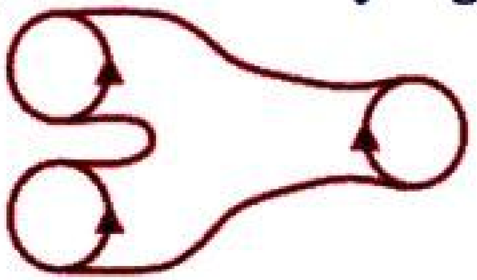
In/out circles ⇒ in/out Hilbert spaces & Surfaces ⇒ linear maps

Closed 2D TFT

Basic in/out circle $\Rightarrow$ vector space $\mathcal{C}$

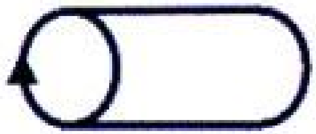
Special surfaces provide the key algebraic data:

Multiplication:



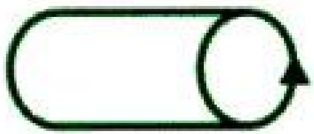
$\Rightarrow \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$

Trace:



$\theta: \mathcal{C} \rightarrow k$

Unit:

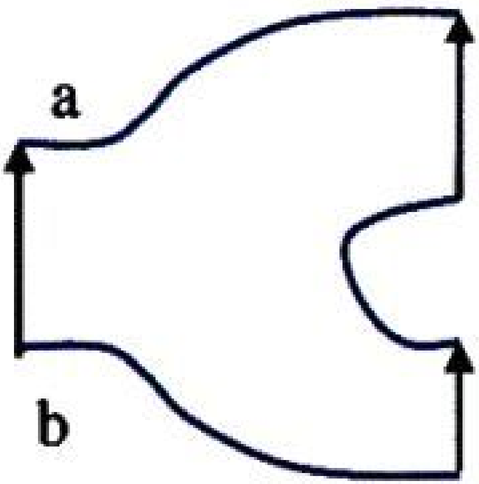


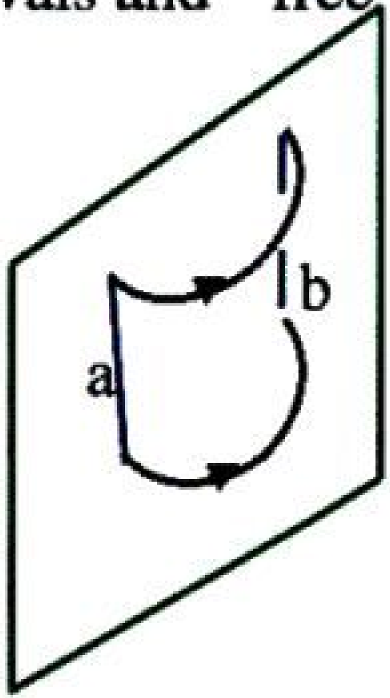
$1 \in \mathcal{C}$

Axioms of commutative Frobenius algebra $\Leftrightarrow$ consistency of sewing

Closed & Open TFT - I

Now allow both open and closed strings ⇒ surfaces have two new boundaries in/out intervals and "free boundaries:

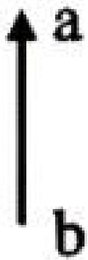




Free boundaries have boundary condition labels a,b,…

2D Open & Closed TFT - II

In/out invervals [0,1] lead to vector spaces



$\Rightarrow \mathcal{O}_{ab}$

•Given $\mathcal{C}$ what are the possible $\mathcal{O}$'s ?

•What algebraic conditions encode open & closed sewing?

•*First focus on a single boundary condition:* $\mathcal{O} = \mathcal{O}_{aa}$

Open & Closed Sewing Constraints

<u>Theorem</u>: To give an open & closed TFT is to give

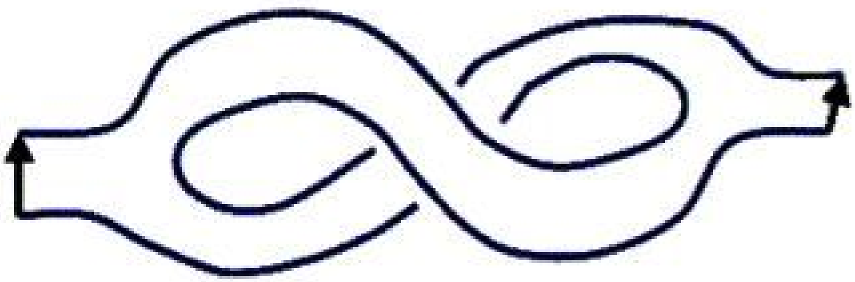
- A commutative Frobenius algebra $(\mathcal{C}, \theta_{\mathcal{C}}, 1)$
- A (non)commutative Frobenius algebra $(\mathcal{O}, \theta_{\mathcal{O}}, 1)$
- A homomorphism $\iota_*: \mathcal{C} \to Z(\mathcal{O})$ such that

a. $\iota_*(1_{\mathcal{O}}) = 1_{\mathcal{C}}$ b. $\pi = \iota_* \iota^*$

where ι^* is adjoint to ι_* : $\theta_{\mathcal{O}}(\psi\, \iota_*(\phi)) = \theta_{\mathcal{C}}(\iota^*(\psi)\phi)$

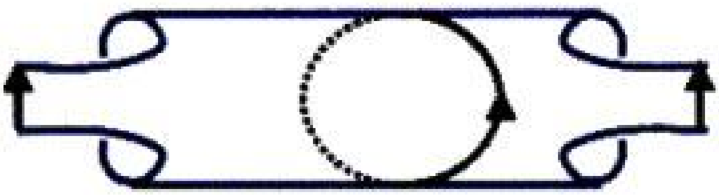
The operator π is defined by the double-twist diagram

Sewing Constraints – II



$\Rightarrow \pi: \mathcal{O} \to \mathcal{O}$

$\pi: \psi \to \Sigma_\mu \psi_\mu \psi \psi^\mu$ ψ_μ : basis for $\mathcal{O}$,



$\pi = \iota_* \iota^*$ is sometimes called the "Cardy condition"

Claim: The above axioms form the complete list of sewing constraints. (Lewellen 1992). Also $\exists$ ``Morse theory proof''

Classification of $\mathcal{O}$'s

If the ``fusion rules'' of $\mathcal{C}$ are diagonalizable,
(i.e. $\mathcal{C}$ is ``semisimple'') then we can classify the $\mathcal{O}$'s:

Theorem (Moore & Segal). If $\mathcal{C}$ is semisimple then $\mathcal{O} = \mathrm{End}_{\mathcal{C}}(M)$ with M = finitely generated projective $\mathcal{C}$-module

Explicitly: Semisimple $\Rightarrow \varepsilon_i = \Sigma_\mu S_0^{\,j} (S^{-1})_j^{\,\mu} \phi_\mu$ satisfy

$\varepsilon_i \varepsilon_i = \delta_{jk} \varepsilon_i$ ``basic idempotents'' $\quad \mathcal{C} = \oplus_i \mathbb{C} \varepsilon_i$,

ε_i correspond to spacetime points: $\chi(\phi) = \theta_{\mathcal{C}}(\varepsilon_i \phi)$

Theorem $\Rightarrow \mathcal{O} = \oplus_i \mathrm{End}(W_i) \Rightarrow$ "Vector bundle over spacetime"

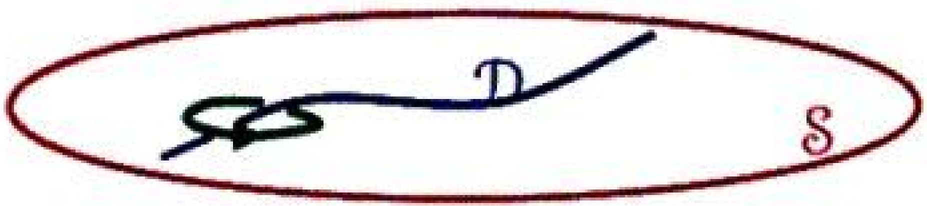
Boundary State

• The ``boundary state,'' which inserts holes:

is given by $B = \iota^{*}(1) = \oplus_i \dim(W_i)\, \varepsilon_i/\sqrt{\theta_i}$

where $\theta_i = \theta_e(\varepsilon_i)$

• Squareroot $\Rightarrow$ in *families* the sign is ambiguous $\Rightarrow$



$\varepsilon_i \rightarrow \varepsilon_{\pi(i)}$ But! $\sqrt{\theta_i} \rightarrow \pm \sqrt{\theta_{\pi(i)}}$

$\Rightarrow$ We must allow +ve and –ve dimensions for W_i

Multiple Boundary Conditions

$\mathcal{O}_{ab}$ is a bimodule for $\mathcal{O}_{aa} \times \mathcal{O}_{bb}$

The Cardy condition generalizes in an obvious way:

$\pi_a{}^b = \iota_a \iota^b$

$\Rightarrow \mathcal{O}_{ab} = \mathrm{Hom}(W_a, W_b) \Rightarrow$ *No new data from mixed boundary conditions.* Together with the previous result we reach the

Conclusion: The boundary conditions labels a are in 1-1 correspondence with $K_0(\mathcal{C})$.

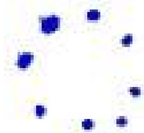
Examples

1. Any CFT has a semisimple Verlinde algebra $\mathcal{C}$
2. Landau-Ginzburg models: e.g. $\mathcal{C} = \mathbb{C}[x]/dW(x)$ semisimple $\Leftrightarrow$ critical points of W are Morse,
   ($\Rightarrow$ Classification of D-branes for strings in <1 dimensions)
   (Related remarks: Iqbal,Hori,Vafa).
3. $\mathcal{C} = H^*(X, \mathbb{C})$, X compact, orientable. Note $\mathcal{C} \otimes Mat_N(\mathbb{C})$ does *not* satisfy Cardy condition. But,

a. If $\mathcal{C} = H^*_q(X, \mathbb{C})$, X Fano, then $\mathcal{C}$ is semisimple

b. X of dimension 4k, $Y \rightarrow X$ is of ½ dimension and nonzero selfintersection then we can take $\mathcal{O} = H^*(Y, \mathbb{C}) \otimes Mat_N(\mathbb{C})$

Orbifolds



- Suppose G acts on $\mathcal{C}$. X= Spec($\mathcal{C}$) = {basic idempotents}
- The orbifold spacetime X/G has a B-field with $h \in H^3_G(X,\mathbb{Z})$.
- One can formulate open & closed G-equivariant TFT.

Conjecture: Boundary labels are classified by $a \in K^0_{G,h}(X)$…
``al~~mo~~st proved''

Part III: Noncommutative Tachyons and K-Theory

∃ Nice recent progress in understanding Sen's conjecture using noncommutative geometry (GMS,HKLM,DMR,Witten):

D25 fills $X_{24} \times \mathbb{R}^2{}_B$

Tachyon field= NC soliton on $\mathbb{R}^2{}_B \Longrightarrow$ D23 brane on X_{24} !

<u>We'll assume this generalizes to topologically nontrivial X_{24}</u>

$B \neq 0 \Longrightarrow T: X_{24} \to \mathcal{B}$; GMS $\Longrightarrow T = \lambda P, P^2 = P$

P = rank n projection operator, can vary along X_{24}

T varies slowly $\Longrightarrow T \in \mathrm{Map}[X_{24}, BU(n)] \Longrightarrow$

Homotopy classes of rank n tachyons = $Vect_n(X_{24})$!

IIB Branes

Transverse space: $\mathbb{R}^{2p_B}$ with $\{z_i, \bar{z}_j\} = \theta_i \delta_{ij}$

$T = f(r)\,\Gamma_i\, x^i : \mathcal{H}_{Barg} \otimes S^- \rightarrow \mathcal{H}_{Barg} \otimes S^+$ has Index(T)=1

$\mathcal{H}_{barg} = \{ \psi(z^1, \ldots, z^p) \}$ i.e. coherent state quantization.

? How can we restrict the tachyon field to the sphere:

$$\Sigma_i |z^i|^2 = R^2$$

An answer: Consider the Hardy subspace $H_{hardy} \subset L^2(S^{2p-1})$ of boundary values of holomorphic wavefunctions.

Commutative algebra of multiplication operators by functions f becomes the noncommutative algebra of *Toeplitz operators*: $T_f = P\, M_f$, where P is the projector $L^2(S^{2p-1}) \rightarrow H_{hardy}$

Analytic K-homology

The Tachyon field is a matrix-valued Toeplitz operator,

The algebra of Toeplitz operators defines a nontrivial extension of $C(S^{2p-1})$ by compact operators: Such extensions define "analytic K-homology" $K_{1,a}(S^{2p-1})$ (Brown-Douglas-Filmore) $\Longrightarrow$ A B-field naturally defines an element of K-homology.

Moreover:

Index(T) = Winding #(ABS) = Winding#(Sen-Witten tachyon)

is a nontrivial mathematical fact known as the index theorem of Boutet de Monvel

Conclusion

We've seen how the perspectives of

1. 11-dimensional M-theory
2. Worldsheet sewing and boundary TFT
3. SFT and noncommutative spacetime

All lead naturally to connections between D-branes and K-theory.

I think there is probably a lot more to say…