OVERVIEW OF K-THEORY APPLIED TO STRINGS

A D-BRANE WRAPPED ON A SUBMANIFOLD OF SPACETIME, S,

MAY CARRY A NONZERO RAMOND-RAMOND CHARGE.

RR (OR RAMOND-RAMOND) FIELDS ARE p-FORMS, AND SUPERFICIALLY IT SEEMS THAT THE CONSERVED CHARGE SHOULD BE MEASURED

BY THE COHOMOLOGY CLASS OF THE RR FORM OR MORE SIMPLY BY THE CLASS

$$[S]$$

HOWEVER, D-BRANES CARRY GAUGE FIELDS; AND GAUGE FIELDS ARE NOT NATURAL IN (CO)-HOMOLOGY THEORY; THEY ARE NATURAL IN "K-THEORY."

IF X IS SPACETIME AND $\mathcal{A}(X)$ IS THE ALGEBRA OF FUNCTIONS ON X, THEN K-THEORY IS DEFINED IN TERMS OF REPRESENTATIONS OF $\mathcal{A}(X)$; A REPRESENTATION OF A RING IS USUALLY CALLED A MODULE.

HERE ARE SOME EXAMPLES OF $\mathcal{A}(X)$-MODULES: $\qquad \mathcal{A}(X) = \mathcal{A}$

$\mathcal{A}$ ITSELF

$$f \in \mathcal{A}(X) \qquad g \in \mathcal{A} = \mathcal{A}(X)$$

$$f(g) = fg$$

$$f_1 f_2 (g) = f_1(f_2(g)) = f_1 f_2 g$$

MORE GENERALLY, CONSIDER ANY Dp-BRANE WRAPPED ON A SUBMANIFOLD S OF X

WITH ANY CHAN-PATON GAUGE FIELD ON THE D-BRANE

LET $a(S)$ = THE SPACE OF SECTIONS OF THE GAUGE BUNDLE ON S (=, ROUGHLY, THE SPACE OF CHARGED PARTICLE STATES ON S).

THEN $a(S)$ IS AN $a(X)$-MODULE

$f \in a(X) \qquad g \in a(S)$

$f(g) = fg$

SO IN, SAY, Type IIB, (5)

A COLLECTION OF D9-BRANES DEFINES A REPRESENTATION OR MODULE

E OF $a(X)$.

THE $\overline{D9}$-BRANES DEFINE ANOTHER MODULE F.

SO ANY CONFIGURATION DETERMINES A PAIR (E, F)

TO CLASSIFY D-BRANE CHARGE WE WANT TO CLASSIFY PAIRS (E, F) MODULO PHYSICAL PROCESSES

AN IMPORTANT PROCESS (Sen ....)
IS BRANE-ANTIBRANE CREATION
OR ANNIHILATION —

CREATION OF A SET OF D9's
AND $\overline{D9}$'s EACH BEARING THE
SAME GAUGE BUNDLE G (G, G)

THIS AMOUNTS TO

$$(E,F) \longleftrightarrow (E\oplus G, F\oplus G)$$

IF WE CLASSIFY PAIRS (E,F)
WITH EQUIVALENCE RELATION

$$(E,F) \simeq (E\oplus G, F\oplus G)$$

THE EQUIVALENCE CLASSES MAKE UP A GROUP CALLED $K(X)$. THE GROUP LAW IS JUST

$$(E,F) + (E',F') = (E \oplus E', F \oplus F')$$

D-BRANES OF Type IIB CARRY CONSERVED CHARGES THAT TAKE VALUES IN

$K(X)$.

I'VE BEEN A BIT VAGUE ABOUT WHETHER WE SHOULD USE ALL Dp-BRANES OR JUST D9-BRANES.

IN FACT, WE CAN CLASSIFY D-BRANE CHARGE JUST VIA $D9 + \overline{D9}$'s AND BUILD THE Dp-BRANES OF $p<9$ VIA PAIRS

$$(E, F) \quad \text{WITH A}$$

SUITABLE TACHYON CONDENSATE.

I WON'T FURTHER EXPLAIN THIS TODAY.

WHAT DO WE GAIN BY KNOWING THAT D-BRANE CHARGE IS CLASSIFIED BY K-THEORY?

FIRST, IT'S THE RIGHT ANSWER:

1. THERE ARE STABLE D-BRANES (LIKE THE NON-SUSY D0-BRANE OF Type I) THAT WOULD NOT EXIST IF D-BRANE CHARGE WERE CLASSIFIED BY COHOMOLOGY

2 IT IS POSSIBLE TO HAVE A Type II D-BRANE WRAPPED ON A HOMOLOGICALLY NON-TRIVIAL CYCLE

WHICH IN FACT IS UNSTABLE...
IT CAN DECAY VIA AN INTERMEDIATE STATE (E, F) OF $D9$'s + $\overline{D9}$'s

3 THE K-THEORY INTERPRETATION ON D-BRANES IS NEEDED (FREED + E.W.) TO CANCEL A GLOBAL WORLDVOLUME ANOMALY

BUT I THINK THERE IS A MORE BASIC REASON THAT IT IS GOOD TO KNOW ABOUT THE K-THEORY INTERPRETATION OF D-BRANES:

IT MIGHT HAVE A NATURAL STRINGY GENERALIZATION

IN FACT (THOUGH I SUPRESSED SOME TECHNICALITIES) I DEFINED $K(X)$ IN TERMS OF REPRESENTATIONS OF THE ALGEBRA $\mathcal{A}(X)$ OF FUNCTIONS ON SPACETIME...

WE CAN SIMILARLY DEFINE $K(\mathcal{A})$ FOR ANY NONCOMMUTATIVE ALGEBRA $\mathcal{A}$.

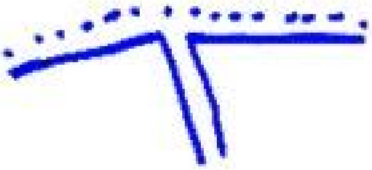
FOR EXAMPLE, TURNING ON A B-FIELD, WE CAN MAKE $\mathcal{A}(X)$ NONCOMMUTATIVE; THE ASSOCIATED $K(\mathcal{A})$ WAS USED BY CONNES, DOUGLAS, AND SCHWARZ.

THIS STILL DOESN'T SEEM FULLY STRINGY ... A REALLY STRINGY VERSION WOULD BE TO USE A NONCOMMUTATIVE ALGEBRA CONSTRUCTED USING ALL THE MODES OF THE STRING ....

NOT JUST THE ZERO MODE OR CENTER OF MASS POSITION.

WHAT IS THE RIGHT NONCOMMUTATIVE ALGEBRA THAT USES ALL THE MODES? WE DON'T KNOW, OF COURSE.

ONE CONCRETE CANDIDATE IS THE * ALGEBRA OF OPEN STRING FIELD THEORY, DEFINED IN TERMS OF GLUING STRINGS TOGETHER ....



IF I CALL THIS ALGEBRA $\mathcal{a}_{st}$, IT SEEMS PLAUSIBLE THAT D-BRANE CHARGE IS REALLY LABELED BY

$$K(\mathcal{a}_{st})$$

I'LL COME BACK TO THIS LATER.

FIRST I WANT TO FINISH OUR SURVEY OF WHAT K-THEORY DOES IN STRING PHYSICS.

I EXPLAINED THAT $K(X)$ CLASSIFIES D-BRANE CHARGE FOR Type IIB

AN EQUALLY FUNDAMENTAL ROLE OF K-THEORY IS THAT $K^1(X)$ CLASSIFIES RAMOND-RAMOND FIELDS,

AGAIN IN Type IIB

NAIVELY SPEAKING, AN RR p-FORM FIELD G_p OBEYS A DIRAC QUANTIZATION LAW

$$\int_U \frac{G_p}{2\pi} = \text{INTEGER} \qquad *$$

FOR ANY p-CYCLE U.

SUCH A CONDITION MEANS RR-FIELDS ARE CLASSIFIED BY COHOMOLOGY, BUT THAT ISN'T THE RIGHT ANSWER, BECAUSE THE ACTUAL QUANTIZATION CONDITION ON RR FLUX IS MUCH MORE COMPLICATED THAN *

THE CORRECTIONS INVOLVE
THE FACT THAT A Dp-BRANE
MAY CARRY Dq-BRANE
CHARGE FOR $q < p$, AS WELL
AS SELF-DUALITY AND GLOBAL
ANOMALIES.

ANYWAY, THE ANSWER TURNS
OUT TO BE THAT RR
FIELDS ARE CLASSIFIED BY
$K^1(X)$

$$= \{\text{maps}: X \to U(N)\}$$

FOR (ANY) LARGE N.
TOPOLOGICALLY, THERE IS A MAP

$$U: X \to U(N)$$

AND

$$G_p = \frac{1}{(2\pi)^{(p+1)/2}\, c_p}\, \mathrm{Tr}\, \underbrace{U^{-1}dU \wedge U^{-1}dU \wedge \cdots U^{-1}dU}_{p}$$

THE PHYSICAL MEANING OF U ISN'T CLEAR.

(NOTE: THERE IS AN ANALOGOUS, MYSTERIOUS

$$U : X \to E_8$$

IN M-THEORY)

IN HINDSIGHT, ONCE IT WAS KNOWN THAT RR CHARGES ARE CLASSIFIED BY K-THEORY, WE SHOULD HAVE GUESSED THAT THE RR FIELDS WOULD BE CLASSIFIED THIS WAY.

AFTER ALL, RR CHARGES PRODUCE RR FIELDS!

SO THE MATH USED TO CLASSIFY RR CHARGES MUST BE SIMILAR TO THE MATH USED TO CLASSIFY RR FIELDS.

WE CAN DEFINE K^1 FOR ANY NONCOMMUTATIVE ALGEBRA:

GIVEN $\mathcal{A}$, WE SET

$\mathcal{A}_N$ = the group of invertible $N \times N$ matrices whose matrix elements are in $\mathcal{A}$

THEN

$K^1(\mathcal{A})$ = the group of components of $\mathcal{A}_N$, for large N.

FOR $\mathcal{A} = \{\text{functions on } X\}$

$K^1(\mathcal{A}) = \{\text{maps}: X \to U(N)\}$

NOW IN THE REST OF THIS TALK, I WANT TO DESCRIBE THINGS THAT REALLY DEPEND ON TAKING $N \to \infty$.

LET US RECALL THE ROLE OF LARGE N IN PHYSICS:

conjectured link
gauge theory $\Longleftrightarrow$ strings

old matrix models $\Longleftrightarrow$ soluble examples of string theory

matrix model of M-theory

AdS/CFT correspondence
$\frac{1}{N} \Longleftrightarrow g_{string}$

MY THEME WILL BE THAT WE SHOULD SOMEHOW WORK WITH INFINITELY MANY D-BRANES

TO MOTIVATE THIS, I'LL CONSIDER TWO CONCRETE QUESTIONS THAT REQUIRE TAKING THE NUMBER OF D-BRANES TO BE INFINITE:

1. Type IIA & K-theory

2. $H_{NS} \neq 0$ & K-theory

① Type IIA

FOR Type IIB, WE HAD

RR CHARGE $\longleftrightarrow$ $K(X)$

RR FIELDS $\longleftrightarrow$ $K^1(X)$

THE T-DUAL STATEMENT SHOULD BE

FOR Type IIA

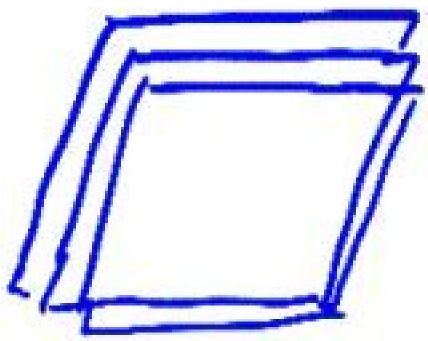
RR CHARGE $\longleftrightarrow$ $K^1(X)$

RR FIELDS $\longleftrightarrow$ $K(X)$

THE MOST CONCRETE AND NATURAL ATTEMPT TO EXPLAIN

Type IIA RR CHARGE $\longleftrightarrow K^1(X)$

IS (HORAVA) TO LOOK AT N UNSTABLE D9-BRANES OF Type IIA

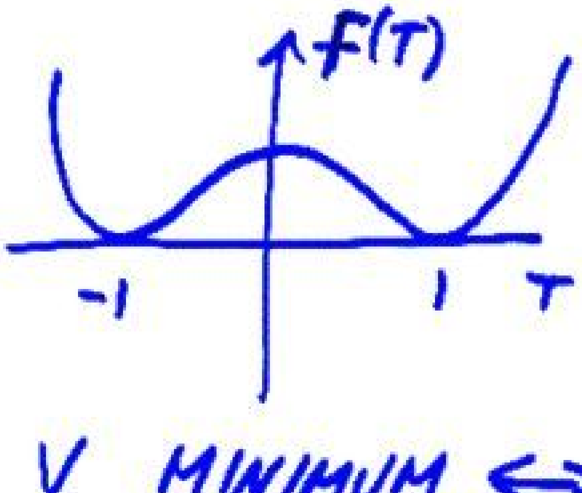


THERE IS A $U(N)$ GAUGE FIELD AND A TACHYON FIELD T IN THE ADJOINT REPRESENTATION OF $U(N)$

AFTER SUITABLY RESCALING T, IT IS BELIEVED THAT THERE IS AN EFFECTIVE POTENTIAL

$$V(T) = \frac{1}{g_{st}} \operatorname{Tr} f(T)$$

WHERE QUALITATIVELY

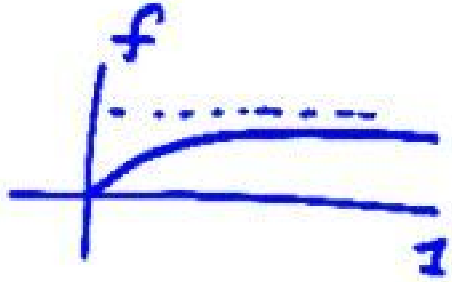


SO V MINIMUM $\Leftrightarrow$ EACH EIGENVALUE OF T IS ± 1.

HORAVA NOTED THAT WE CAN TRY TO MAKE SUPERSYMMETRIC Dp-BRANES (p = EVEN) AS SOLITONS OF T.

FOR EXAMPLE, TO MAKE A D6-BRANE, SET $N=2$ AND LETTING $\vec{x}$ = THE THREE COORDINATES NORMAL TO THE D6-BRANE, SET

$$T = \frac{\vec{\sigma}\cdot\vec{x}}{|\vec{x}|} f(|x|)$$



FOR $|x| \to \infty$, EIGENVALUES OF T ARE EVERYWHERE ± 1

NEAR $x \simeq 0$, TOPOLOGICAL KNOT $\Leftrightarrow$ D6-BRANE

IN FLAT $\mathbb{R}^{10}$, ONE CAN SIMILARLY MAKE Dp-BRANES FOR OTHER EVEN p ...

BUT ON A GENERAL SPACETIME, [redacted] THIS DOESN'T WORK FOR ARBITRARY Dp-BRANES UNLESS WE SET $N = \infty$.

THIS IS MOST OBVIOUS IF
X IS COMPACT. THEN

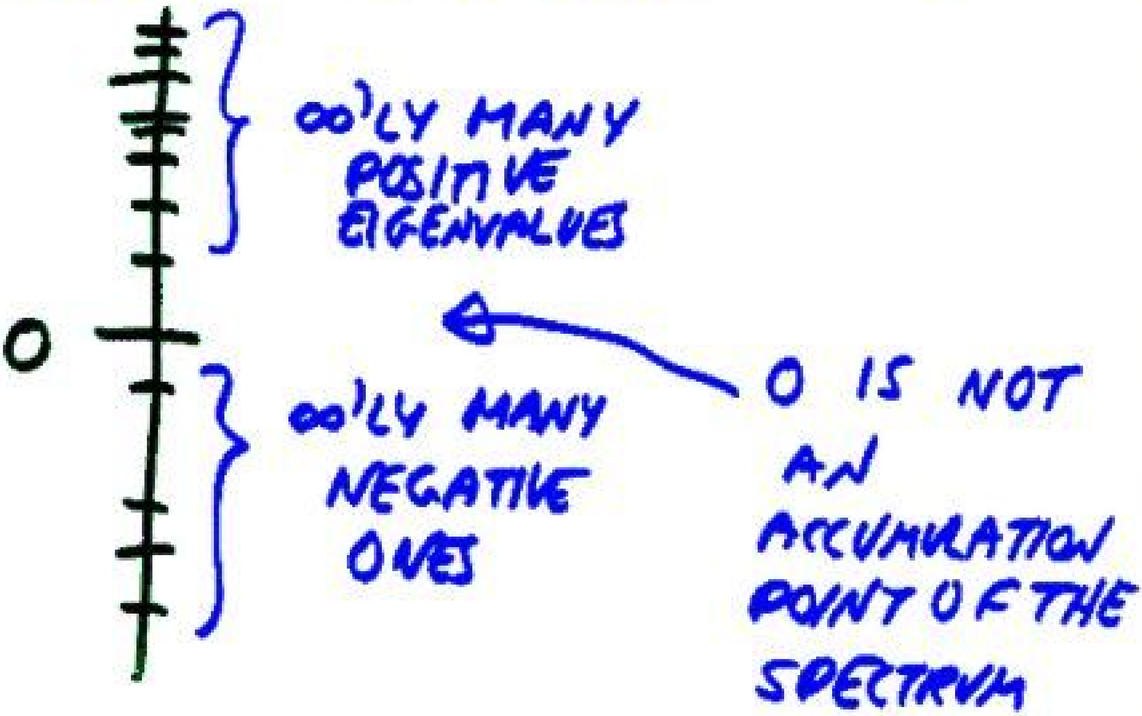
$T: X \to$ Lie algebra of $U(N)$

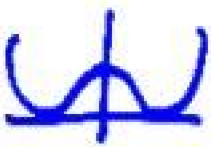
HAS NO TOPOLOGY
(SINCE THE LIE ALGEBRA IS
CONTRACTIBLE!) AND DOESN'T
LABEL AN ELEMENT OF $K^1(X)$.
WE NEED THE GROUP, NOT THE
LIE ALGEBRA:

$U: X \to U(N)$ DOES THE
JOB, AS I SAID BEFORE.

AMAZINGLY, AS ATIYAH AND SINGER SHOWED, WE GET BACK THE RIGHT TOPOLOGY FROM THE LIE ALGEBRA IF WE SET $N=\infty$!

WE HAVE TO INTERPRET LIE ALGEBRA $(U(\infty))$ TO CONSIST OF SELF-ADJOINT OPERATORS OF SPECTRUM THAT LOOKS LIKE





PHYSICALLY, IF THE ENERGY AND D8-BRANE CHARGE ARE FINITE, THOSE CONDITIONS ARE OBEYED.

SO IT SEEMS THAT ONE CAN STUDY Type IIA D-BRANES VIA D9-BRANE CONDENSATION, PROVIDED ONE STARTS WITH INFINITELY MANY D9-BRANES (WHICH UNDERGO TACHYON CONDENSATION DOWN TO A FINITE SET)

(2)

ANOTHER SUCH PROBLEM IS TO CONSIDER D-BRANES WHEN THE NEVEU-SCHARZ THREE-FORM FIELD H IS TOPOLOGICALLY NONTRIVIAL.

SAY IN Type IIB....

ONE WOULD LIKE TO CONSIDER D9 - $\overline{D9}$ PAIRS AND FORM A SUITABLE K-GROUP OF PAIRS (E,F) TO CLASSIFY D-BRANE CHARGE WITH $H \neq 0$

THERE IS A PROBLEM IN HAVING A D9-BRANE WHEN H IS TOPOLOGICALLY NONTRIVIAL:

MIXING WITH THE U(1) ON THE D9-BRANE

$$dF = H$$

U(1) FIELD STRENGTH

A SINGLE D9-BRANE IS ONLY ALLOWED IF H IS TOPOLOGICALLY TRIVIAL

IF H IS TORSION,

i.e. $MH = 0$ (TOPOLOGICALLY)

FOR SOME INTEGER M

THEN D9-BRANES CARRY

$U(N)/\mathbb{Z}_M$ "GAUGE BUNDLES"

$N_{branes} = mM$

(OR $U(mN)/\mathbb{Z}_M$ FOR $m = 1, 2, 3, \ldots$)

AND IN PARTICULAR THE NUMBER OF D9 OR $\overline{D9}$ BRANES IS A MULTIPLE OF M, $N = mM$

D-BRANE CHARGE WITH TORSION H IS CLASSIFIED BY PAIRS

(E, F) (MOD $(E, F) \simeq (E \oplus G, F \oplus G)$)

THE PAIRS (E,F) MAKE A GROUP

$K_H(X)$ THAT CLASSIFIES D-BRANE

CHARGE FOR $H \neq 0$ (AND M FINITE).

IF ONE WANTS TO INTERPRET

$K_H(X)$ AS THE K-THEORY OF

REPRESENTATIONS OF AN ALGEBRA,

THEN ONE MUST PICK A

TWISTED BUNDLE E AND REPLACE

THE COMMUTATIVE ALGEBRA

$\mathcal{a}(X)$ OF FUNCTIONS ON X

BY A NONCOMMUTATIVE ALGEBRA

$\mathcal{a}_E(X)$ OF "GAUGE TRANSFORMATIONS OF E"

THERE IS NO CANONICAL CHOICE OF

$\mathcal{a}_E(X)$, SINCE THERE IS NO PREFERRED CHOICE OF E

BUT THE K-THEORY ONE GETS DOESN'T DEPEND ON THE CHOICE OF E, AS FOR ANY E, E'

$$\boxed{\text{$\mathcal{a}_E(X)$ AND $\mathcal{a}_{E'}(X)$ ARE "MORITA EQUIVALENT"}}$$

IN MOST EXAMPLES OF INTEREST, H IS NOT TORSION, THERE IS NO $M \in \mathbb{Z}$ WITH

$$MH = 0 \quad \ldots$$

AS SHOWN BY

BOUWKNEGT-MATHAI
ATIYAH-SEGAL

ONE MUST THEN TAKE THE LARGE M LIMIT OF $U(M)/\mathbb{Z}_M$

AND USE

$$PU(\mathcal{H}) = \underbrace{U(\mathcal{H})}_{\substack{\text{UNITARY} \\ \text{OPERATORS ON } \infty\text{-DIM'L} \\ \text{HILBERT SPACE}}} / U(1)$$

THIS MEANS THAT, WHEN H IS NOT TORSION, ONE CANNOT HAVE A FINITE SET OF D9-BRANES, BUT ONE CAN HAVE AN INFINITE SET, WITH A SUITABLE TWISTED GAUGE BUNDLE E OR F.

D-BRANE CHARGE IS CLASSIFIED BY

$K_H(X)$ = THE K-THEORY OF PAIRS (E,F) MOD THE USUAL RELATIONS

THUS WHEN H IS NOT TORSION, TO DESCRIBE ORDINARY D-BRANE PHYSICS WITH FINITELY MANY Dp-BRANES ($p<9$) IN A UNIFIED WAY VIA $D9-\overline{D9}$ PAIRS AND TACHYON CONDENSATION, WE MUST START WITH INFINITELY MANY $D9$'S AND $\overline{D9}$'S.

THIS CONSTRUCTION HAS A BEAUTIFUL PROPERTY (POINTED OUT BY BOUWKNEGT & MATHAI)

THE NONCOMMUTATIVE ALGEBRA WHOSE K-THEORY IS K_H IS UNIQUE, INDEPENDENT OF ANY ARBITRARY CHOICE OF TWISTED BUNDLE E OR E'.

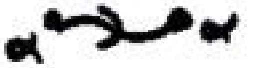
THIS REALLY DEPENDS ON THE NUMBER OF $D9$ AND $\overline{D9}$ BRANES BEING INFINITE.

THERE IS A CONJECTURAL STRINGY GENERALIZATION OF THIS:

START WITH A CLOSED STRING BACKGROUND

LET US TRY TO FORM AN OPEN STRING ALGEBRA.

WE PICK AN OPEN STRING BOUNDARY CONDITION α

THE  OPEN STRINGS FORM AN ALGEBRA $\mathcal{a}_\alpha$

IT ISN'T UNIQUE; WE COULD PICK ANOTHER BOUNDARY CONDITION β AND DEFINE ANOTHER ALGEBRA $\mathcal{A}_\beta$

THE EXAMPLE OF STRINGS IN A BACKGROUND H-FIELD (AND ALGEBRAS $\mathcal{A}_E$, $\mathcal{A}_F$) STRONGLY SUGGESTS THE NATURE OF THE RELATION BETWEEN

$\mathcal{A}_\alpha$ AND $\mathcal{A}_\beta$:

THEY ARE DIFFERENT ALGEBRAS, BUT ARE MORITA-EQUIVALENT AND HENCE HAVE THE SAME K-THEORY.

THE FACT THAT

$$a_\alpha \neq a_\beta$$

IS TROUBLESOME; IT IS ONE MAJOR ASPECT OF THE LACK OF MANIFEST BACKGROUND INDEPENDENCE OF OPEN STRING FIELD THEORY

THE EXAMPLE OF OPEN STRINGS IN AN H-FIELD SUGGESTS A CURE:

TAKE THE NUMBER OF D9, $\overline{D9}$- BRANES TO BE INFINITE

TAKE m COPIES OF α OR β:

$$a_{\alpha m} \neq a_{\beta m}$$

BUT CONJECTURE

$$a_{m\alpha} = a_{m\beta}$$

FOR $m = \infty$

$$\boxed{a_\alpha \otimes \mathcal{K} = a_\beta \otimes \mathcal{K}}$$

MORE PRECISELY, THE CONJECTURE IS

$$\mathcal{A}_\alpha \otimes K = \mathcal{A}_\beta \otimes K$$

K = THE ALGEBRA OF COMPACT OPERATORS IN HILBERT SPACE

THE IDEA THEN WOULD BE TO START WITH ∞'LY MANY D9-BRANES TO DO STRING FIELD THEORY, WITH GREATER BACKGROUND INDEPENDENCE

RELYING ON TACHYON CONDENSATION TO GET US DOWN TO SOMETHING REASONABLE.

FOR THIS TO BE USEFUL, WE'D NEED A SIMPLER DESCRIPTION OF

$$a_\alpha \otimes K$$

I AM AFRAID I CAN'T OFFER ONE TODAY!