Brane Resolution on Ricci Flat Spaces

- Motivation: resolution via transgression
- Examples:
  - Illustration: Resolved self-dual string
  - Summary of resolved D3 and M2-branes & other examples
- Mathematical aspects: explicit harmonic forms on Ricci flat spaces
- Summary, open avenues

- w/ H. Lü and C. Pope, hep-th/0011023
- w/ G. Gibbons, H. Lü and C. Pope, hep-th/0012011
- (w/ K. Behrndt, hep-th/0101007)
- & work in progress

I. Motivation

AdS_{D+1}/CFT_D correspondence: new insights into strongly coupled superconformal gauge theories in D-dim.
Prototype: D=4; D3-brane Type IIB SG:

$$\begin{aligned} ds^2_{10} &= H^{-1/2}\, dx \cdot dx + H^{1/2}\,(dr^2 + r^2\, d\Omega_5^2)\,, \\ F_{(5)} &= d^4x \wedge dH^{-1} + \hat{*}(d^4x \wedge dH^{-1})\,, \\ \Box H &= 0 \Rightarrow H = 1 + \frac{R^4}{r^4}\ . \end{aligned}$$

Decoupling:

$$H = 1 + \frac{R^4}{r^4} \xrightarrow[\text{limit}]{\text{decoupling}} \frac{R^4}{r^4}\ :$$

$$AdS_5 \times S^5:\ \ ds^2_{10} \to \frac{r^2}{R^2}\, dx \cdot dx + R^2\, \frac{dr^2}{r^2} + R^2\, d\Omega_5^2\,.$$

string th. on $AdS_5 \times S^5 \Leftrightarrow D = 4, N = 4$ SYM

Goal: Elucidate QCD, or at least N=1 D=4 SYM
$\Rightarrow$
viable (non-singular) supergravity duals w/ less SUSY

Comment

Progress within D=5 N=2 gauged supergravity:

- Gauging of vector-multiplets: generically singular solutions.

  w/ Behrndt, Kallosh/Linde, Ceresole/Dall'Agata ⋯

- Gauging of hyper-multiplets; novel gravity solutions:

  w/ Behrndt hep-th/0101007

  - smooth conformal solutions in IR and UV;
  - solutions that are flat/supersymmetric in IR;
  - (c-theorem violating solution/potential to trap gravity)

Thus, examples of viable gravity duals of N=1 D=4 FTs

⇒

Higher-dimensional embedding and interpretation

Supergravity solutions with less supersymmetry

Flat transverse 6-dim. space $ds_6^2 = dr^2 + r^2\, d\Omega_5^2$ replaced by (non-compact) Ricci-flat space w/ fewer Killing spinors. Still:

$$\Box H = 0\,.$$

- \+ Two birds with one stone: solution with reduced supersymmetry & broken conformal invariance.
- – H - singular $\Rightarrow$ solution singular!

Resolution of Singularity

Turning on additional fluxes ("fractional" branes)
In the D3-brane context:

Chern-Simons Term ⇒ Modified eqs.:

$$dF_{(5)} = d{*}F_{(5)} = F^{\rm NS}_{(3)} \wedge F^{\rm RR}_{(3)} = \frac{1}{2i} F_{(3)} \wedge \bar{F}_{(3)} \, ,$$

$$F_{(3)} \equiv F^{\rm RR}_{(3)} + {\rm i}\, F^{\rm NS}_{(3)} = m L_{(3)} .$$

$L_{(3)}$- harmonic self-dual three-forms on 6-dim. Ricci-flat space. ⇒ A possibility for a smooth solution.

Employed by Klebanov/Strassler [hep-th/0007191]
[also Klebanov/Tseytlin [hep-th/0002159], Pandos-Zayas/Tseytlin [hep-th/0010088]]

Related/Follow up work: Graña/Polchinski, Maldacena/Nuñez, Gubser, Becker/Becker, Herzog/Klebanov, ···

Earlier work: Klebanov/Witten, Klebanov/Nekrasov, Klebanov/Tseytlin

General Context: Resolution via Transgression!

w/Lü and Pope [hep-th/0012011]

Chern-Simons type modifications- "transgressions" $\Rightarrow$ modify Bianchi identities or eqs. of motion when additional fluxes turned on.

p-brane configurations w/ $(n+1)$-transverse dim. ("magnetic" flux $F_{(n)}$) and additional fluxes $F_{(p,q)}$. Transgression implies:

$$dF_{(n)} = F_{(p)} \wedge F_{(q)}; \quad (p+q = n+1).$$

If the $(n+1)$-dim transverse Ricci-flat space admits harmonic p-form $L_{(p)}$:

$$F_{(p)} \sim mL_{(p)}, \quad F_{(q)} \sim \mu * L_{(p)}.$$

Depending on L^2 normalizability properties of $L_{(p)}$ $\Rightarrow$ resolved (non-singular) solution

Illustration first for a somewhat simpler example than $D3$-brane.

II. Examples

IIa.Self-dual string

The self-dual string in $D = 6$, $(1,0)$ supergravity theory:

$$\mathcal{L} = \sqrt{g}(R - \tfrac{1}{12}F^2_{(3)})\,, \quad F_{(3)} = *F_{(3)}\,.$$

Flat transverse space solution:

$$\begin{aligned} ds^2_6 &= H^{-1}(-dt^2 + dx^2) + H\,(dr^2 + r^2\,d\Omega^2_3)\,. \\ F_{(3)} &= dt \wedge dx \wedge dH^{-1} + \text{dual}\,. \\ \Box H &= 0 \;\Rightarrow\; H = 1 + \frac{R^2}{r^2}\,. \end{aligned}$$

Decoupling:

$$H = 1 + \frac{R^2}{r^2} \xrightarrow[\text{limit}]{\text{decoupling}} \frac{R^2}{r^2} : AdS_3 \times S^3$$

Self-dual string with less supersymmetry:

Replace the 4-dim. flat space with Eguchi-Hanson:

$ds_4^2 = W^{-1}\,dr^2 + \frac{1}{4}r^2\,W\,(d\psi + \cos\theta\,d\phi)^2 + \frac{1}{4}r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2)\,,$

$W = 1 - \frac{a^4}{r^4}$, $r = \{a, \infty\}$.
[Vielbeine:

$$e^0 = W^{-1/2}\,dt\,, \qquad e^3 = \tfrac{1}{2}r\,W^{1/2}\,(d\psi + \cos\theta\,d\phi)\,,$$
$$e^1 = \tfrac{1}{2}r\,d\theta\,, \qquad e^2 = \tfrac{1}{2}r\,\sin\theta\,d\phi\,.$$

]

Solution:

$$\Box H = 0 \;\Rightarrow$$
$$(r^3 W H')' = 0 \;\Rightarrow$$
$$H = 1 + c_1\,\log\big(\frac{r^2 + a^2}{r^2 - a^2}\big)\,.$$

As $r \to \infty$, $H \sim 1/r^2$, however it is singular as $r \to a$.

Resolved self-dual string

Modification of Bianchi identity via transgression [D=6 (1,0) SG as K3 reduction of heterotic string]:

$$dF_{(3)} = d{*}F_{(3)} = F^i_{(2)} \wedge F^i_{(2)} \,.$$

(F^i_2 ($i = 1, \cdots 16$) are gauge-field strengths in Cartan subalgebra of $E_8 \times E_8$ or $Spin(32)$.)

Modified Ansatz (single U(1)-gauge field):

$$\begin{aligned} ds_6^2 &= H^{-1}(-dt^2 + dx^2) + H\, ds_4^2 \,, \\ F_{(3)} &= dt \wedge dx \wedge dH^{-1} + \text{dual} \,, \\ F_{(2)} &= mL_{(2)} \,, \\ \Box H &= -\tfrac{1}{2} L^2_{(2)} \,. \end{aligned}$$

$L_{(2)}$-harmonic, self-dual 2-form:

$$L_{(2)} = \tfrac{1}{r^4}\,(e^0 \wedge e^3 + e^0 \wedge e^2) \,.$$

(e^i-Vielbeine of Eguchi-Hanson metric, $L_{(2)}$-L^2-normalizable)

A solution:

$$H = 1 + \frac{m^2 + a^4 b}{4a^6} \log(\frac{r^2 - a^2}{r^2 + a^2}) + \frac{m^2}{2a^4 r^2},$$

w/ integration constant choice: $b = -m^2/a^4$:

$$H = 1 + \frac{m^2}{2a^4 r^2}.$$

Supersymmetric, **completely regular** solution [due to L^2 normalizability of $L_{(2)}$].

In the proper distance coordinate ρ ($W^{-1/2}\, dr = d\rho$), $\rho \to \infty$:

$$H \sim 1 + \frac{Q}{\rho^2} - \frac{c_1}{\rho^6} + \cdots .$$

Field Theory implications

Reduction on ψ, θ and ϕ coordinates of Eguchi-Hanson & the decoupling limit ($H \to R^2/r^2$);
$D = 3$ domain wall solution:

$$ds_3^2 = \frac{r^2}{R^2} W \left(-dt^2 + dx^2\right) + \frac{R^2\, dr^2}{r^2}\,.$$

Non-singular, asymptotically AdS_3 (conformal) gravity dual of 2-dim. CFT theory with 1/2 of maximal SUSY.

Implications: e.g., bound-state spectra in IR (small r) regime $\Rightarrow$
discrete spectrum, indicating *confinement*

IIb. Summary of Resolved D3 and M2-branes

D3-branes

Bianchi identity modification:

$$dF_{(5)} = d{*}F_{(5)} = F_{(3)} \wedge F_{(3)}\,,$$

Modified D3-brane Ansatz (w/ additional fluxes):

$$\begin{aligned} d\hat{s}_{10}^2 &= H^{-1/2}\, dx^\mu\, dx^\nu\, \eta_{\mu\nu} + H^{1/2}\, ds_6^2\,, \\ F_{(5)} &= d^4x \wedge dH^{-1} + \hat{*}dH\,, \\ F_{(3)} &= F_{(3)}^{\rm RR} + {\rm i}\, F_{(3)}^{\rm NS} = m\, L_{(3)}\,, \end{aligned}$$

$L_{(3)}$ complex, harmonic self-dual three-form of 6-dim. Ricci-flat space &

$$\Box H = -\tfrac{1}{12} m^2\, |L_{(3)}|^2\,.$$

[C.f., Klebanov/Strassler [hep-th/0007191] and
Pando-Zayas/Tseytlin [hep-th/0010088]
two-different Ricci-flat resolutions of the 6-dim. conifold.]

M2-brane

Modified Bianchi identity:

$$d{*}F_{(4)} = \tfrac{1}{2}F_{(4)} \wedge F_{(4)}\,,$$

Modified M2-brane Ansatz:

$$\begin{aligned} d\hat{s}_{11}^2 &= H^{-2/3}\, dx^\mu\, dx^\nu\, \eta_{\mu\nu} + H^{1/3}\, ds_8^2\,,\\ F_{(4)} &= d^3x \wedge dH^{-1} + m\, L_{(4)}\,, \end{aligned}$$

$L_{(4)}$- harmonic, self-dual 4-form of 8-dim. Ricci-flat transverse space &

$$\Box H = -\tfrac{1}{48} m^2\, L_{(4)}^2\,.$$

[Related work Hawking/Taylor-Robinson [hep-th/9711042], Duff et al. [hep-th/9706124].]

IIIc. Other Examples

Bianchi identity:

$$dF_{(n)} = F_{(p)} \wedge F_{(q)}\,, \quad p + q = n + 1\,.$$

Branes with additional with $(n+1)$-transverse dimensions:

- D0-brane: $d{*}F_{(2)} = {*}F_{(4)} \wedge F_{(3)}$
- D1-brane: $d{*}F^{\rm RR}_{(3)} = F_{(5)} \wedge F^{\rm NS}_{(3)}$
- → D2-brane: $d{*}F_{(4)} = F_{(4)} \wedge F_{(3)}$
- D3-brane: $dF_{(5)} = F_{(3)} \wedge \bar{F}_{(3)}$
- D4-brane: $dF_{(4)} = F_{(3)} \wedge F_{(2)}$
- → IIA string: $d{*}F_{(3)} = F_{(4)} \wedge F_{(4)}$
- IIB string: $d{*}F^{\rm NS}_{(3)} = F_{(5)} \wedge F^{\rm RR}_{(3)}$
- → Het 5-brane: $dF_{(3)} = F^i_{(2)} \wedge F^i_{(2)}$

Construction of explicit harmonic forms on non-compact Ricci-flat spaces.

$\Rightarrow$

Explicit solutions of resolved p-brane solutions.

III. Mathematical developments

w/Gibbons, Lü and Pope [hep-th/0012011]

- Explicit construction of (p,q)-forms in the middle dimension $(p+q=n+1)$ for $2(n+1)$-dim. Stenzel metric.
- Explicit construction of a class of Kähler metrics on C^k bundles and their middle-dimension harmonic forms in $2(n+1)$-dim.
- Construction of harmonic forms on 7-dim. Ricci-flat spaces with G_2 holonomy, as well as 8-dim. spaces with $Spin(7)$ holonomy.

IIIa. Stenzel metric

- Ricci-flat metric in $2(n+1)$ dim. w/ asymptotically ($r \to \infty$) conical space.
- Level surfaces (at fixed r) are $SO(n+2)/SO(n)$ coset space.
- As $r \to 0$:

$$ds^2 \sim dr^2 + r^2\tilde{\sigma}_i^2 + \sigma_i^2 + \nu^2$$

  Topology: $R^{n+1} \times S^{n+1}$ (S^{n+1}-"bolt").
- Examples:
  - $n = 1$: Eguchi-Hanson metric
  - $n = 2$: deformed conifold [Candelas/de la Ossa]
  - $n = 3$: new example of eight-dimensional Ricci-flat space
  - $n > 3$: dimension ≥ 10.

Metric construction

L_{AB}-left-invariant one-forms on $SO(n+2)$ group manifold w/ split: $A = \{1, 2, i\}$, $i = 1, \cdots, n$.
One-forms on $SO(n+2)/SO(n)$ coset:

$$\sigma_i \equiv L_{1i}\,, \qquad \tilde{\sigma}_i \equiv L_{2i}\,, \qquad \nu \equiv L_{12}\,.$$

Metric:

$$ds^2 = h^2 dr^2 + a^2 \sigma_i^2 + b^2\, \tilde{\sigma}_i^2 + c^2\, \nu^2$$

$\{h, a, b, c\}$- functions of r, only and their explicit form obtained by solving the first order diff. eqs. $\Rightarrow$ Ricci-flat space.

[Vielbeine:

$$e^0 = h\, dr\,, \qquad e^i = a\, \sigma_i\,, \qquad e^{\tilde{i}} = b\, \tilde{\sigma}_i\,, \qquad e^{\tilde{0}} = c\, \nu\,.$$

A holomorphic basis of complex one-forms:

$$\epsilon^0 \equiv -e^0 + \mathrm{i}\, e^{\tilde{0}}\,, \qquad \epsilon^i = e^i + \mathrm{i}\, e^{\tilde{i}}\,.$$

Kähler form:

$$J = \tfrac{\mathrm{i}}{2}\, \epsilon^\alpha \wedge \bar{\epsilon}^{\bar{\alpha}}\,.$$

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Harmonic middle dimension (p, q) forms

The Ansatz:

$$L_{(p,q)} = f_1 \, \epsilon_{i_1 \cdots i_{q-1} j_1 \cdots j_p} \epsilon^0 \wedge \bar{\epsilon}^{i_1} \wedge \cdots \wedge \bar{\epsilon}^{i_{q-1}} \wedge \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_p} + f_2 \, \epsilon_{i_1 \cdots i_{p-1} j_1 \cdots j_q} \epsilon^0 \wedge \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_{p-1}} \wedge \bar{\epsilon}^{j_1} \wedge \cdots \wedge \bar{\epsilon}^{j_q} \, ,$$

w/ f_1, f_2 functions of r, only.

Harmonicity condition:

$$dL_{(p,q)} = 0$$

(since $*L_{(p,q)} = \mathrm{i}^{p-q} \, L_{(p,q)}$).

f_1, f_2-solutions of coupled first-order, homogeneous differential equations.

Solution (finite as $r \to 0$):

$$f_1 = q \, {}_2F_1 \left[\tfrac{1}{2}p, \tfrac{1}{2}(q+1), \tfrac{1}{2}(p+q)+1; -(\sinh 2r)^2 \right]$$

$$f_2 = -p \, {}_2F_1 \left[\tfrac{1}{2}q, \tfrac{1}{2}(p+1), \tfrac{1}{2}(p+q)+1; -(\sinh 2r)^2 \right]$$

Specific (p, q): elementary functions of r!

Special cases:

- (p,p)-forms in $4p$-dim: $f_1 = -f_2 = \frac{p}{(\cosh r)^{2p}}$;

$$|L_{(p,p)}|^2 = \frac{const.}{(\cosh r)^{4p}}$$

  $r \to 0$, finite; $r \to \infty$ falls-off fast enough. $\Rightarrow$
  The only L^2 normalizable form!

- $(N+1,N)$-forms in $(4N+2)$-dim. As $r \to \infty$:

$$|L_{(N+1,N)}|^2 \sim \frac{1}{[\sinh{(2r)}]^{2N}}$$

  Marginally L^2-non-normalizable.

- degree of non-normalizability grows with $|p-q|$.

Applications

- $2(n+1) = 6$: $L_{(2,1)}$-form; resolution of $D3$-brane
  Klebanov/Strassler
  $L_{(2,1)}$- marginally non-normalizable as $r \to \infty$,
  but pure $(2,1)$-type. $\Rightarrow$ supersymmetric! Gubser
- $2(n+1) = 8$: $L_{(2,2)}$-form: new resolved $M2$-brane.
  Since $|L_{(2,2)}|^2 = \frac{const.}{\cosh r^8}$ normalizable!
  Decoupling: $M2$- asymptotically AdS_4;
  Field theory: Herzog/Klebanov hep-th/0101020
  3-dim. CFT perturbed by relevant operators.

IIIb. Ricci-flat Kähler metrics on C^k bundles:

- level surfaces-$U(1)$ bundles over a product of N-Einstein-Kähler base spaces $\mathcal{M}$. Focus on

$$\mathcal{M} = \prod_{i=1}^{N} CP^{m_i} .$$

- As $r \to 0$, a particular base space factor CP^m singled out.
- Metric construction:
  - One factor: CP^m [m-odd, $m = 1$-Eguchi Hanson].
  - Two factors: $CP^{m_1} \times CP^{m_2}$ [$m_1 = m_2 = 1$-conifold resolution of Candelas/de la Ossa; $m_1 = 1$, $m_2 = 2$– novel 8-dim. dimensional space].
  - examples of multi-factor base spaces and/or dim. ≥ 10.
- Construction of middle dimension forms.

Applications

- 6-dim: resolved conifold & fractional $D3$-branes supported by $L_{(3)}$: Pandos-Zayas/Tseytlin

  – Not L^2 normaliz., neither at $r \to \infty$ nor $r \to 0$.

  – Mixture of $(1,2)$ and $(2,1)$ form $\Rightarrow$ non-supersymmetric!

- A number of new examples of resolved $M2$-branes: supersymmetric, non-singular.

IIIc. Forms on G_2-holonomy and $Spin(7)$ spaces

Techniques to construct harmonic forms developed

Gibbons, Page& Pope ('90)

- Explicit harmonic three-forms on G_2 holonomy spaces $\Rightarrow$ resolved $D2$-branes: [redacted]singular, supersymmetric examples?
- Explicit harmonic four-forms on $Spin(7)$ holonomy spaces $\Rightarrow$ another resolved $M2$-brane: non-singular, supersymmetric (c.f., also M. Becker).

IV. Summary of solution properties

- Supersymmetry depends on the structure of the harmonic form employed.
- Singularity structure depends on normalizability of forms:
  - Finiteness at *small distance* ensures no singularity there.
  - At *large distance*:

    $$\rho \to \infty : \quad H = 1 + \frac{Q}{\rho^d}(1 - \frac{c_1}{\rho^\gamma} + \cdots).$$

    For L^2 normalizable forms: $\gamma > 0$: decoupling limit to AdS (conformal) space.

    For L^2 non-normalizable, $\gamma < 0$, no decoupling limit to AdS :no conformal symmetry

- Field theory of resolved branes: related to Higgs branch (to be contrasted with distributed branes-Coulomb branch)

  $\gamma > 0$-CFT pert. by relevant operators; $\gamma < 0$ ("fractional" branes).

Herzog/Klebanov (c.f., I. Klebanov's talk)

Summary of explicit examples

- D3-branes (3-form of 6-dim.space, not L^2 normalizable): $H = 1 + (Q + c_1 \log(\rho))/\rho^4$, fractional $D3$-brane.
  - deformed conifold (KS) non-singular as $r \to 0$; supersymmetric.
  - resolved conifold (PT) singular as $r \to 0$; non-supersymmetric.

- M2-branes ($(2,2)$-forms of 8-dim. space):

  $\gamma = 4/3, 8/3, 4, 8$; supersymmetric, 3-dim. CFT perturbed by relevant opers.

  & $\gamma = -4/3$; non-supersymmetric, "fractional" brane .

- D2-branes(3-forms of G_2 holonomy space):

  $\gamma \lesssim 0$; [redacted]singular, supersymmetric

- Heterotic NS-NS 5-brane (2-forms of 4-dim. space); Large non-singular, supersymmetric.

- 6-dim. Dyonic string (2-forms of 4-dim. space): non-singular, including "tensionless" string $\Rightarrow$ complementary mechanism to that of Johnson, Polchinski & Peet, resolving "repulson"-type singularity (c.f., Behrndt).

Conclusions

- *Brane Resolution:* General mechanism to find regular p-brane solutions with less supersymmetry:

  Introduction of non-compact Ricci-flat transverse space and additional fluxes, supported by harmonic-forms that modify the original p-brane solution via Chern-Simons (transgression) terms.

- *Mathematical Developments:* Explicit middle-dimension harmonic forms for Stenzel metrics in $2N$-dimensions and for a class Kähler metrics on $\mathcal{C}^k$ bundles. [Only $(\frac{N}{2}, \frac{N}{2})$-forms L^2 normalizable.]

Open Avenues

- Possible dualities among various solutions; unifying treatment; dual field theory

- Construction of harmonic forms for other Ricci-flat spaces, such as hyper-Kähler, and those in other than middle dimensions.