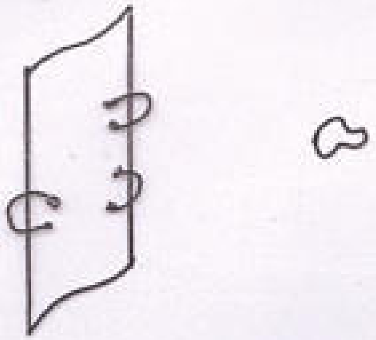
BULK COUPLINGS TO NONCOMMUTATIVE BRANES

S.R.D. and S. Trivedi, hep-th/0011131
S.R.D. and S.J. Rey, hep-th/0008042

Related papers

D. Gross, A. Hashimoto and I. Itzhaki, hep-th/0008075
H. Liu, hep-th/0011125
Y. Okawa and H. Ooguri, hep-th/0012218
H. Liu and J. Michelson, hep-th/0101016. The

study of the coupling of bulk modes with usual D-branes has played a crucial role in our understanding of black holes, holography and many other features of string theory.



How do branes with constant B fields couple a bulk mode?

This should couple to a gauge invariant operator in the noncommutative gauge theory.

However, in such theories **translations** are special **gauge transformations**

$$e^{ib_i x^i} \star \phi(x) \star e^{-ib_i x^i} = \phi(x^i + \theta^{ij} b_j)$$

If $\phi(x)$ is adjoint, this is a gauge transformation.

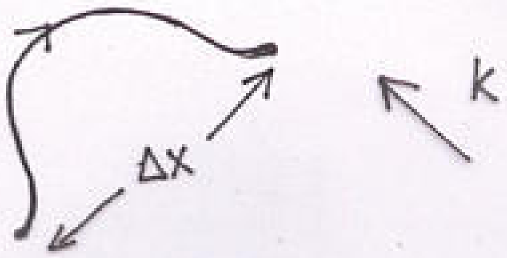
No gauge invariant operators in position space

There are of course gauge invariant operators with definite **momentum** (*Ishibashi,Iso,Kawai and Kitazawa*)

In NCYM in $d+1 = p+2n+1$ dimensions with noncommutativity

$$\begin{aligned} \theta^{AB} &= \theta^{ij} \qquad (A,B) = (i,j) = 1, \cdots 2n \\ \theta^{AB} &= 0 \qquad \text{otherwise} \end{aligned}$$

one may choose an *open* curve C ; $\eta^i(\lambda)$



$$\begin{aligned} Q(k_i, k_\mu) &= \int d^{d+1}x \; e^{ik\cdot x} W(x^i, x^\mu; C) \\ W(x^i, x^\mu; C) &= \exp[i \int_0^1 d\lambda \; \frac{d\eta^j(\lambda)}{d\lambda} \; A_j(x+\eta(\lambda))] \end{aligned}$$

This is gauge invariant if

$$\Delta x^A = k_B \theta^{BA}$$

Size **proportional to the momentum**, but in a direction transverse to it along the noncommmutative directions.

Closed in the commutative directions

These operators are relevant to supergravity couplings and holography

(*SRD and S.J. Rey; Gross, Hashimoto and Itzhaki*)

In particular : operators with **straight** Wilson lines : $\eta^A = k_B \theta^{BA} \lambda$ - **Wilson tails**
(*Gross, Hashimoto and Itzhaki*)

$$\tilde{\mathcal{O}}(k) = \int d^{d+1}x \ \text{tr} \ \mathcal{O}(x + k\cdot\theta) \star W(x,k) \star e^{ik\cdot x}$$
$$W(x,k) = P_\star \exp[i \int_0^1 d\lambda \ k_A \theta^{AB} A_B(x + k\cdot\theta \ \lambda)]$$

In commutative limit these reduce to fourier transforms of local gauge invariant operators.

Several other possibilities (*Dhar and Wadia*)

Correlations of Wilson tails have been studied and leads to interesting **universal** high energy behavior
(*Gross, Hashimoto and Itzhaki; Dhar and Kitazawa; Rozali and van Raamsdonk*)

What kind of Wilson lines are involved in (linearized) coupling to bulk modes ?

We will use the construction of noncommutative gauge theories in $(p+2n+1)$ dimensions from large-N ordinary gauge theories in $(p+1)$-dimension.

Matrix Theory, Twisted Eguchi-Kawai models, operator formalism $\cdots$

We will heavily use methods developed in

(*Aoki, Ishibashi, Iso, Kawai and Kitazawa; Ambjorn, Makeenko, Nishimura and Szabo; Seiberg*)

NONCOMMUTATIVE BRANES FROM MANY ORDINARY BRANES

Bosonic fields on N p-branes : $U(N)$ gauge theory

$$\mathbf{A}_\mu(\xi) \quad, \mu = 1, \cdots p+1$$
$$\mathbf{X}^I(\xi), \; I = 1, \cdots (9-p) \qquad \text{(adjoint)}$$

Action contains X^I through $D_\mu X^I$ and $[X^I, X^J]$. Thus there is a classical solution for $N = \infty$

$$\begin{aligned} \mathbf{X}^i(\xi) &= \mathbf{x}^i & \qquad i &= 1, \cdots 2n \\ \mathbf{X}^a &= 0 & a &= 2n+1 \cdots 9-p \\ \mathbf{A}_\mu &= 0 \end{aligned}$$

where the constant matrices $\mathbf{x}^i$ satisfy

$$\boxed{[\mathbf{x}^i, \mathbf{x}^j] = i\theta^{ij}\mathbf{I}} \qquad (i, j = 1 \cdots 2n)$$

Expand around this solution

$$\boxed{\begin{aligned} \mathbf{C}_i &= B_{ij}\mathbf{X}^j = \mathbf{p}_i + \mathbf{A}_i \\ \mathbf{X}^a &= \phi^a \\ \mathbf{A}_\mu &= \mathbf{A}_\mu \end{aligned}}$$

where

$$B_{ij}\theta^{jk} = \delta_i^k \qquad \mathbf{p}_i = B_{ij}\mathbf{x}^j$$

Any $\infty \times \infty$ matrix $\mathbf{O}(\xi)$ may be represented as a function $O(x,\xi)$ and then

$$\begin{array}{rcl} \mathbf{O}(\xi) & \rightarrow & O(x,\xi) \\ [\mathbf{p}_i, \mathbf{O}(\xi)] & \rightarrow & i\partial_i O(x,\xi) \\ \mathrm{Tr}\mathbf{O}(\xi) & \rightarrow & \frac{1}{(2\pi)^n}[\mathrm{Pf}\ B]\int d^{2n}x \ \ O(x,\xi) \\ \mathbf{O}_1(\xi)\mathbf{O}_2(\xi) & \rightarrow & O_1(x,\xi)\star O_1(x,\xi) \end{array}$$

This gives a $(p+2n+1)$ dimensional $U(1)$ noncommutative gauge theory

$$\begin{array}{rcl} \mathbf{F}_{\mu\nu} & \rightarrow & \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star \equiv F_{\mu\nu}(x,\xi) \\ D_\mu \mathbf{X}^i & \rightarrow & \theta^{ij}(\partial_\mu A_j - \partial_j A_\mu - i[A_\mu, A_j]_\star) \\ D_\mu \mathbf{X}^a & \rightarrow & \partial_\mu \phi^a - i[A_\mu, \phi^a]_\star \equiv D_\mu \phi^a(x,\xi) \end{array}$$

$$[\mathbf{X}^i, \mathbf{X}^j] \rightarrow i\theta^{ik}\theta^{jl}(F_{kl}(x,\xi) - B_{kl})$$

$$[\mathbf{X}^i, \mathbf{X}^a] \rightarrow i\theta^{ij}(\partial_j \phi^a - i[A_j, \phi^a]_\star) \equiv i\theta^{ij} D_j \phi^a(x,\xi)$$

where the noncommutative gauge fields are defined as

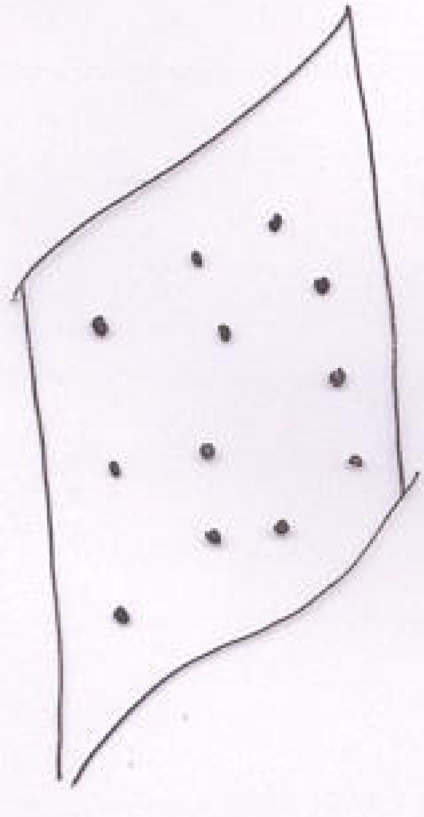
$$F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i, A_j]_\star$$

$U(M)$ noncommutative gauge theory : expand around

$$\mathbf{X}^i(\xi) = \mathbf{x}^i \otimes I_M$$

One can view this a mathematical representation of NCYM

We will take this more literally - as in matrix theory

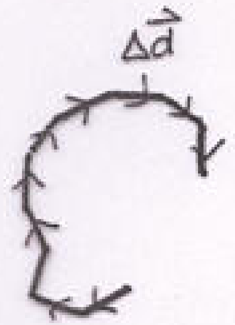


This procedure gives a noncommutative theory in the "$\Phi = -B$" description. Metric which appears is the open string metric

$$G^{ij} = -\theta^{ik} g_{kl} \theta^{lj}$$

The B field on the brane is B_{ij}

$$B_{ij} = (\theta^{-1})_{ij}$$



The open Wilson line was constructed by IIKK using this correspondence

$$W(C,k) = \text{Lim}_{M\to\infty} \int d^{p+1}\xi \,\text{Tr}\, [\prod_{n=1}^{M} U_j]\, e^{ik_\mu \xi^\mu}$$
$$U_j = \exp\, [i\vec{C} \cdot (\vec{\Delta} d)_n]$$

The contour C in the noncommutative space is obtained by sticking together the vectors Δd_n.

$$k_i = B_{ji} d^j \quad \vec{d} = \sum_{n=1}^{M} \vec{\Delta} d$$

follows from construction

Operators of the large-N theory which are of the form

$$\int d^{p+1}\xi \; e^{ik_\mu \xi^\mu} \; \mathrm{Tr}\; [e^{ik_I \mathbf{X}^I} \; \mathcal{O}_\phi(\mathbf{X}, \mathbf{A}, \xi)]$$

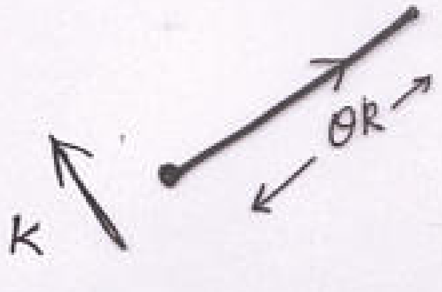
lead to **Wilson tails**

$$\int \frac{d^{p+1}\xi \; d^{2n}x}{(2\pi)^n} \; (\mathrm{Pf}B) \; e^{ik_\mu \xi^\mu + k_i x^i} \; \mathrm{tr}\; [O \cdot W_t]$$

$$O \cdot W_t = [O_\phi(x + k \cdot \theta, \xi) \star P_\star W_t]$$

$$\boxed{W_t = e^{i \int_0^1 d\lambda \; k_i \theta^{ij} A_j(x+\eta(\lambda)) + i \int_0^1 d\lambda \; k_a \phi^a(x+\eta(\lambda))}}$$

Generalization of open Wilson lines to include scalars considered in (*SRD and S.J.Rey*)



To see this, consider the case where the momentum k is along the x^1 direction where only

$$B_{12} = B \neq 0 \qquad \theta = \theta^{12} = -\frac{1}{B}$$

Now write

$$\begin{aligned} k &= NB\epsilon \\ N &\to \infty \qquad \epsilon \to 0 \end{aligned}$$

Break up the exponential operator into little bits

$$\begin{aligned} & \text{Tr } [e^{ik\mathbf{X}^1 + ik_a \mathbf{X}^a}] \\ = & \text{ Tr } [(e^{iB\epsilon\mathbf{x}^1} \ e^{-i\epsilon\mathbf{A}_2} \ e^{i\frac{\epsilon}{k\theta} \ k_a\phi^a})^N] \\ = & \text{ Tr } [e^{iB\epsilon\mathbf{x}^1}(e^{-i\epsilon\mathbf{A}_2} \ e^{i\frac{\epsilon}{k\theta} \ k_a\phi^a})e^{-iB\epsilon\mathbf{x}^1} \\ & e^{2iB\epsilon\mathbf{x}^1}(e^{-i\epsilon\mathbf{A}_2} \ e^{i\frac{\epsilon}{k\theta} \ k_a\phi^a})e^{-2iB\epsilon\mathbf{x}^1} \\ & \ldots \\ & e^{NiB\epsilon\mathbf{x}^1}(e^{-i\epsilon\mathbf{A}_2} \ e^{i\frac{\epsilon}{k\theta} \ k_a\phi^a})e^{-iNB\epsilon\mathbf{x}^1} \ \ e^{iNB\epsilon\mathbf{x}^1}] \end{aligned}$$

Using the correspondence this becomes

$$\begin{aligned} & \text{Tr}[e^{ik\mathbf{X}^1 + ik_a\mathbf{X}^a}] \\ & = B\int dx \ [e^{-i\epsilon A_2(x_1,x_2+\epsilon) + i\frac{\epsilon}{k\theta} \ k_a\phi^a(x_1,x_2+\epsilon)} \\ & e^{-i\epsilon A_2(x_1,x_2+2\epsilon) + i\frac{\epsilon}{k\theta} \ k_a\phi^a(x_1,x_2+2\epsilon)} \\ & \ldots \\ & e^{-i\epsilon A_2(x_1,x_2+N\epsilon) + i\frac{\epsilon}{k\theta} \ k_a\phi^a(x_1,x_2+N\epsilon)}] \star e^{ikx^1} \\ & = B\int dx \ e^{ikx^1}[P_\star e^{-i\int d\tau\theta k A_2(x+\theta k\tau) + i\int d\tau k_a\phi^a(x+\theta k} \end{aligned}$$

Quite often the operators which appear in nonabelian brane actions involve symmetric traces of products of operators

$$\int d^{p+1}\xi \; e^{ik_\mu \xi^\mu} \, \mathrm{STr} \, [e^{ik_I \mathbf{X}^I} \prod_{\alpha=1}^{n} \mathcal{O}_\alpha(\mathbf{X}, \mathbf{A}, \xi)]$$

This becomes a further generalization

$$\int \frac{d^{p+1}\xi \; d^{2n}x}{(2\pi)^n} \, (\mathrm{Pf} B) \; e^{ik_\mu \xi^\mu} \; e^{ik_i x^i} \; \mathrm{tr} \; L_*[O_n \cdot W_t]$$

where

$$\boxed{L_*[O_n \cdot W_t] = \int \prod_{\alpha=1}^{n} d\tau_\alpha \; P_*[\prod_{\alpha=1}^{n} O_\alpha(x^i + \theta^{ji} k_j \tau_\alpha) W_t]}$$

The composite operator is now smeared over the Wilson tail.

BULK COUPLINGS

Suppose we know the form of the operator in the theory of infinite number of p branes with $B = 0$, which couples to *linearized* supergravity modes in the bulk *with definite momentum along the brane directions*

Proposal : **Use above dictionary to write down couplings to noncommutative $(p+2n)$- branes.**

Typically a bulk mode $\Phi(k_\mu, k_I)$ with definite momenta will have a coupling of the form

$$\Phi(k_\mu, k_I) \int d^{p+1}\xi \; e^{ik_\mu \xi^\mu} \; \mathrm{STr} \; [e^{ik_I \mathbf{X}^I} \; \mathcal{O}_\phi(\mathbf{X}, \mathbf{A}, \xi)]$$

These would become Wilson tails with operators smeared along it

Gives the couplings in $\Phi = -B$ description

However one must know the operators in the original system of large number of branes with no B field.

Operators with Wilson tails

$$\tilde{\mathcal{O}}(k) = \int d^{d+1}x \ \text{tr} \ \mathcal{O}(x + k\cdot\theta) \star W_t(x,k) \star e^{ik\cdot x}$$

$$W(x,k) = P_\star \exp[i\int_0^1 d\lambda \ k_A\theta^{AB} A_B(x + k\cdot\theta \ \lambda)]$$

may be expanded in powers of A_i as

$$\mathcal{O}(k) = \int d^{p+1}\xi \frac{d^{2n}x}{(2\pi)^n} e^{ik_\mu\xi^\mu} \ e^{ik_ix^i} (\text{PfB})(\mathcal{O}\cdot W)$$

where

$$\begin{aligned}(\mathcal{O}\cdot W) = \text{tr} \ [\mathcal{O}(x,\xi) \ &+\theta^{ij}\partial_j(\mathcal{O}\star' A_i)\\ &+\frac{1}{2}\theta^{ij}\theta^{kl}\partial_j\partial_l[\mathcal{O}A_iA_k]_{*3}]\\ &+O(A^3)\end{aligned}$$

(*Mehen and Wise; Liu*)

These expressions involve **commutative** but **non-associative** products

$$f(x) \star' g(x) = \frac{\sin(\frac{\partial_1 \wedge \partial_2}{2})}{\frac{\partial_1 \wedge \partial_2}{2}} f(x_1) g(x_2)|_{x_1 = x_2 = x}$$

and

$$[f\ g\ h](x)_{\star 3} = [\frac{\sin(\frac{\partial_2 \wedge \partial_3}{2}) \sin(\frac{\partial_1 \wedge (\partial_2 + \partial_3)}{2})}{\frac{(\partial_1 + \partial_2) \wedge \partial_3}{2} \frac{\partial_1 \wedge (\partial_2 + \partial_3)}{2}} + (1 \leftrightarrow 2)]\ f(x_1) g(x_2) h(x_3)|_{x_i = x}$$

where

$$\partial_1 \wedge \partial_2 = \theta^{ij} \frac{\partial}{\partial x_1^i} \frac{\partial}{\partial x_2^j}$$

Similar products defined for higher products

$$[f_1(x) f_2(x) \cdots f_n(x)]_{\star n}$$

Such generalized $\star$ products appeared in one loop calculations in NCYM
(*Liu and Michelson; Zanon; Ardalan and Sadoghi; Santambrogio and Zanon*)
and in Seiberg-Witten maps relating noncommutative gauge fields and ordinary gauge fields

$$f_{ab} = F_{ab} + \theta^{kl}(A_k \star' \partial_l F_{ab} - F_{ak} \star' F_{bl}) + O(A^3)$$

with higher products appearing at higher orders.
(*Garousi; Mehen and Wise; Liu*)

Since bulk modes couple to Wilson tails, this coupling is naturally expressed in terms of these products.

- They appear in loop diagrams in gauge theory since they should be expressible in terms of tree level exchange of bulk modes.

- The fact that they appear in the Seiberg-Witten map is consistent wth the fact that coupling to bulk modes can be written in terms of ordinary gauge fields as well.

SLOWLY VARYING FIELDS - DBI LIMIT
Dilaton and graviton couplings

Our proposal is in principle valid without any approximation once the operators in the lower dimensional (ordinary) brane theory are known.

One test of our prescription is check whether one gets the right operator in the limit of slowly varying fields - the DBI approximation.

- Start with some nonabelian form of the DBI action, and obtain the operator as above

- Perform a Seiberg-Witten map to "ordinary" gauge fields

- Compare with DBI form of coupling written in terms of "ordinary" gauge fields, which is known.

Consider e.g. large number of D-instantons. In a DBI approximation interaction with a definite momentum dilaton field $D(k)$ given by (*Myers, Taylor and van Raamsdonk*)

$$S_I = \frac{D(k)}{g_s} \text{STr}\, e^{ik\cdot \mathbf{X}} \sqrt{\det(\delta^I_J + i[\mathbf{X}^I, \mathbf{X}^K] g_{KJ})}$$

Expand around

$$\mathbf{X}^i = \mathbf{x}^i \; (i = 1 \cdots 2n) \quad \mathbf{X}^a = 0 \; (a = (2n+1) + \cdots d)$$

Simplify further : take $k_a = 0$

$$\frac{D(k)|PfB|}{g_s} \int d^{2n}x \, e^{ikx} \, L_*[W_t \sqrt{\det(I - \theta(F - B)\theta g)}]$$

W_t is the **straight Wilson tail**.

In terms of the open string metric and open string coupling

$$S_I = \frac{D(k)}{G_s} \int d^{2n}x \; e^{ikx} \; L_*[W_t \sqrt{\det(G + F - B)}]$$

Same coupling may be written in terms of commutative gauge fields f

$$\tilde{S}_I = \frac{D(k)}{g_s} \int d^{2n}x \; e^{ikx} \sqrt{\det(g + f + B)}$$

These two should be equal if F and f are related by the Seiberg-Witten map.

We will verify this upto terms of $O(A^2)$, but to all orders in θ

Terms involving ∂F may be ignored **but not if it comes multiplied by an A**

Seiberg and Witten showed that

$$\begin{aligned} \frac{1}{g_s}\sqrt{\det(g+f+B)} &= \frac{1}{G_s}\sqrt{\det(G-B+F)} \\ &\quad + O(\partial F) \\ &\quad + \text{total derivatives} \end{aligned}$$

We will show (to this order) :
these total derivatives are precisely the terms which comes from the expansion of the Wilson tail.

To $O(A)$ the operator in terms of f is

$$\frac{\sqrt{\det(g+B)}}{g_s}\int d^{2n}x\; e^{ikx}\;[1+\frac{1}{2}(\frac{1}{(g+B)})^{ij}f_{ji}]$$

- Express this in terms of open string metric G and open string coupling G_s using

$$\boxed{\begin{aligned}\frac{1}{G-B} &= -\theta+\frac{1}{g+B}\\ G_s &= g_s(\frac{\det(G-B)}{\det(g+B)})^{\frac{1}{2}}\end{aligned}}$$

- Use Seiberg-Witten map to write in terms of nocommutative gauge fields F_{ij} and potential A_i.

To this order the Seiberg-Witten map is trivial

$$\frac{\sqrt{\det(G-B)}}{G_s}\int d^{2n}x\; e^{ikx}\;[1+\frac{1}{2}(\frac{1}{(G-B)}+\theta)^{ij}F_{ji}]$$

In terms of the noncommutative F use expansion of the Wilson tail in powers of A

$$\boxed{P(x) + \theta^{ij}\partial_j(P(x) \star' A_i) + O(A^2)}$$

where

$$P(x) = \sqrt{\det(G + F - B)}$$

To $O(A)$ first term gives

$$\frac{\sqrt{\det(G-B)}}{G_s} \int d^{2n}x \; e^{ikx} \; [1 + \frac{1}{2}(\frac{1}{(G-B)})^{ij} F_{ji}]$$

while the second term gives

$$\frac{\sqrt{\det(G-B)}}{G_s} \int d^{2n}x \; e^{ikx} \; [\frac{1}{2}(\theta)^{ij} F_{ji}]$$

They add up to

$$\boxed{\frac{\sqrt{\det(G-B)}}{G_s} \int d^{2n}x \; e^{ikx} \; [1 + \frac{1}{2}(\frac{1}{(G-B)} + \theta)^{ij} F_{ji}]}$$

- To $O(A^2)$ the nontriviality of Seiberg-Witten map becomes important : calculation straight-forward but long.
  The result continues to hold.

- $O(A^3)$ and higher : the symmetrized trace starts making a difference - no explicit check yet.

A similar comparison can be carried out for the coupling of graviton to the noncommutative brane

$$g_{ij} = \bar{g}_{ij} + h_{ij}(x)$$

In the $\Phi = -B$ description the coupling is

$$\boxed{L_*[W_t \sqrt{\det(\bar{G} - B + F)}\,(\frac{1}{1-\theta(F-B)\theta\bar{g}}\theta(F-B))^{ij}]}$$

The agreement with commutative description has been verified following similar steps.

- In any approximation scheme our procedure yields the operators in the $\Phi = -B$ description

- Our procedure would be most useful for slowly varying **coherent** brane waves with large amplitudes.

  Coupling to (nonconstant) RR fields, extending results of (*Mukhi and Suryanarayana*) for constant fields.

- In principle, one could use Seiberg-Witten maps to obtain the operators in any other description. In practice this could be rather cumbersome.

Another way to obtain couplings to noncommutative branes is to perform a worldsheet calculation - get in $\Phi = 0$
(*Hyun, Kiem, Lee and Lee; Garousi*
Okawa and Ooguri; Liu and Michelson)

The straight Wislon tail appears in these computations

In the Seiberg-Witten low energy limit, the $\Phi = 0$ description becomes the $\Phi = -B$ description to leading order

Tachyon/Dilaton : agreement
Graviton: agreement for superstring

HOLOGRAPHY

So far we have looked at couplings with bulk modes at a linearized level

Another place where gauge invariant operators appear is in holography involving such branes (*Hashimoto and Itzhaki; Maldacena and Russo*)

We expect that our techniques will be useful to extract these operators as well

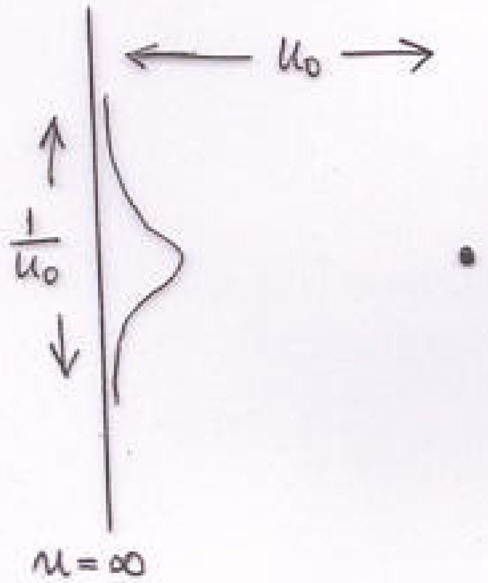
In particular one should be able to use the fact that the asymptotic geometry of supergravity duals of noncommutative $(p+2)$ branes is identical to the supergravity dual of smeared ordinary p-branes
(*Lu and Roy; Cai and Ohta; Alishalia, Oz and Sheikh-Jabbari*)

The main problem here is that the supergravity modes are badly coupled

Nevertheless we can draw one general lesson - the IR/UV relation
(*SRD and S.J. Rey*)

In the usual AdS/CFT correspondence, the size of the hologram of a source in the bulk **descreases** monotonically as the source approaches the boundary - the standard IR/UV connection

$$ds^2 = u^2[-dt^2 + dx_1^2 + \cdots dx_3^2] + \frac{du^2}{u^2} + u^2 d\Omega_5^2$$



$$\Delta x \sim \frac{1}{u_0}$$

With $B_{23} \neq= 0$ and $B_{12} = 0$

$$ds^2 = u^2(-dt^2 + dx_1^2) + \frac{u^2}{1 + a^4u^4}(dx_2^2 + dx_3^2) + \frac{du^2}{u^2} + u^2 d$$

where asymptotic value of B_{23}, hence θ related to a

$$\theta = \frac{a^2}{\sqrt{4\pi g_s N}}$$

On gravity side, momentum space one point functions (in the presence of a source) may be calculated using an operator form of the correspondence
(*SRD and B. Ghosh;*
Danielsson, Guijosa, Kruczenski and Sundborg)

In the interior the IR/UV relationship is the same

Near the boundary this is however *inverted*

$$\boxed{\Delta x \sim \theta \, \sqrt{g_s N} \, u_0}$$

If u_0 is identified with the momentum scale in the NCYM, this is exactly what one expects from the size of the Wilson tail, apart from the factor $\sqrt{g_s N}$

This should be a universal feature of the IR/UV connection.