# D-branes on Calabi-Yau Manifolds

**Michael R. Douglas**

**Strings 2001**

Based on

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- 0002037 and 0003263 with B. Fiol and C. Römelsberger.
- 0006189 by B. Fiol and M. Mariño, and 0012079 by B. Fiol.
- 0006224 with D.-E. Diaconescu.
- 0010196 by S. Govindarajan and T. Jayaraman, 0010223 by P. Mayr and 0010217 by A. Tomasiello.

The study of duality has shown that many (most? all?) problems of string/M theory compactification can be solved in terms of **geometric** models.

For $d = 4$, $N = 2$, the broadest class of models is **type II** compactification on **Calabi-Yau threefolds**.

- The IIb prepotential and BPS central charges are given by classical geometrical quantities: $Z = \int_{\Sigma_3} \Omega$.
- Mirror symmetry allows summing IIa world-sheet instantons.
- Heterotic-type II duality sums space-time instantons.
- Local theory $\leftrightarrow$ QFT; branches.
- Global theory $\leftrightarrow$ space of all $N = 2$ theories. Is it "finite" ?

What is an analogous good source of $d = 4$, $N = 1$ models, exhibiting calculable non-trivial superpotentials and moduli spaces of vacua ?

The traditional (large volume) answer: Kaluza-Klein compactification on $\mathrm{CY}_3$ of $E_8 \times E_8$ or $SO(32)$ gauge theory derived from heterotic string theory.

Compared to type II, the extra structure is a choice of vector bundle. This choice is not well understood – no complete systematic constructions were known, and the possibilities are not classified, even qualitatively.

Furthermore, the results get stringy corrections from world-sheet and space-time instantons.

More recent candidate answers:

- Type I and orientifold compactifications on $CY_3$. Find configurations of space-filling branes in type II on $CY_3$, then project.
- F theory on elliptically fibered fourfolds. Also requires adding space-filling branes.

Both start off looking different from heterotic/vector bundle constructions, but under deeper study turn out to be closely related (weak-weak dualities).

The central problem in combining these pictures into a unified picture of $N = 1$ compactification is to translate between brane constructions and the geometric language of vector bundles.

Over the last year, we have developed a unified description of a large class of type II $N = 1$ supersymmetric D-brane configurations on CY, as bound states of a finite generating set of BPS branes, and we have to a large extent understood how to translate this description to and from the algebraic geometry of vector bundles.

In this language, one can get a description of the set of BPS branes on **string-scale CY's** much like the solution of $N = 2$ compactification provided by mirror symmetry and duality, purely in terms of **large volume results**:

A B-type BPS brane on the CY $M$ is a Π-stable object in the derived category of coherent sheaves on $M$.

This rather abstract phrase summarizes a great deal of physical and mathematical information about the branes, as well as concrete and relatively simple techniques for deriving and working with their world-volume theories.

The physical starting point:

Non-renormalization theorems, which constrain the dependence of D-brane world-volume theory on Calabi-Yau moduli: for B branes, holomorphic structure and superpotential is independent of Kähler moduli, while D-flatness conditions and associated lines of marginal stability are independent of complex structure moduli.

F - flatness

$$\frac{\partial W}{\partial \varphi^i} = 0 \quad / \quad G_{\mathbb{C}}$$

holomorphic structure
depends on complex str of CY

D - flatness

$$\sum_i Q_i^{\,i} \varphi_i \varphi_i^{+} = 1 \cdot \zeta_i \quad / G$$

$\Updownarrow$

stability conditions

$$\mu' < \mu$$

depends on Kähler structure
(cpx structure for A branes)

locus of marginal stability

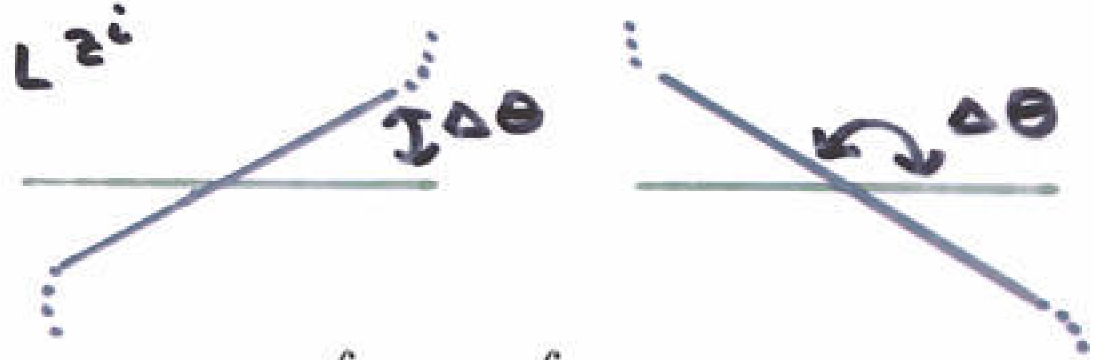
$$\mu' = \mu$$

Thus, brane configurations which solve world-volume F-flatness conditions can be defined in topological open string theory, as "topological D-branes."

There is an important subtlety which enters at this point however: the notion of "brane" and "antibrane" depends on Kähler moduli. To decide whether a pair of branes $A$ and $B$ are both branes, brane and antibrane, or something else, we must look at their BPS central charges. If these are aligned (as complex numbers), $N = 1$ supersymmetry is preserved and the pair can both be considered branes. If they are not aligned, supersymmetry is broken, while if they are antialigned, we have brane and antibrane.

However, BPS central charges vary drastically with Kähler moduli, preventing us from consistently maintaining the distinction between brane and antibrane.

This is particularly clear in the special Lagrangian picture:




$$Z = \int \Omega \sim \int dz^1 \wedge dz^2 \wedge dz^3 .$$

To get a description which is truly independent of Kähler moduli, one must include all brane-antibrane bound states. Thus, B branes are **more general than bundles or coherent sheaves.**

Topological D-branes can be naturally identified with objects in the **derived category**, as foreseen by Kontsevich. This construction starts from a category of "objects" and much like K theory allows combining them with their anti-objects (by forming "complexes"). The new element is that it also keeps track of **all massless fermions** between any pair of branes, not just the index $\mathrm{Tr}\,(-1)^F$.

The derived category keeps much more structure than K theory: for example, every point is a distinct object. Distinct CY manifolds can have the same derived category of coherent sheaves, but only if they are very closely related (e.g. one is the flop of the other).

What is the derived category?

$E \xrightarrow{\alpha} F \qquad \alpha \in \mathrm{Hom}(E, F)$

$E, F$ "branes" = sb manifold
sheaf
⋮

$\alpha$ = oriented open massless fermionic string.
e.g. $H^{0,p}_{\bar\partial}(M, E^* \otimes F)$

- $E^\cdot$ complex

$$
\begin{array}{ccccccccc}
\to E_0 & \xrightarrow{d_0} & E_1 & \xrightarrow{d_1} & E_2 & \xrightarrow{d_2} & E_3 & \cdots & d_E^2 = 0 \\
\varphi_0 \downarrow & & \varphi_1 \downarrow & & \varphi_2 \downarrow & & \varphi_3 \downarrow & & d_F \varphi = \varphi d_E \\
\to F_0 & \xrightarrow{d_0} & F_1 & \xrightarrow{d_1} & F_2 & \xrightarrow{d_2} & F_3 & \cdots & d_F^2 = 0
\end{array}
$$

has K theory class $\sum_n (-)^n [E_n]$,

but particular maps $d$ carry more information.

$$H^n(E) \equiv \ker d_{n+1} / \mathrm{im}\, d_n$$

On the other hand, many different $d$'s can lead to same homology.

K theory class is $(E,F)/\sim$
with $(E,F) \sim (E+X,\ F+X)$

Similarly, we want to identify all complexes obtained by adding $B\bar{B}$ pairs:

$$
\begin{array}{ccccccc}
\to & E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \to \cdots \\
\oplus & B_0 & \xrightarrow{1} & B_1 & & & \\
 & \varphi \downarrow & & \varphi \downarrow & & \varphi \downarrow & \\
\to & F_0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \to \cdots
\end{array}
$$

$Q_{(E,F)} = Q^{(0)} + \partial_E - \partial_F$

so $H^0_Q = \dfrac{\ker \partial_E - \partial_F}{\operatorname{im} \partial_E - \partial_F} = \operatorname{Hom}(E,F) / Q_E$

Extra maps $B \to F$ cancel out of $Q$ cohomology: pair $B_0 \to F$ with $B_1 \to F$.

Identify all complexes which are related by
- adding $B\bar{B}$ pairs
- morphisms $1 + Q_E$
- complex g.t.

$\Rightarrow$ derived category.

This construction allows representing all sheaves in terms of complexes whose terms have no internal structure and no moduli ("free resolution")

$$E \downarrow \quad E_0 \to E_1 \to E_2 \to E_3 \to 0$$

$F \cdots\cdots \nearrow E_2$

$F[-2] \cdots\cdots \uparrow E_2$

$$H^2(X, F^* \otimes E) \cong \mathrm{Ext}^2(F, E) \quad \mathrm{Hom}(F, E[2])$$

Bound state formation through tachyon condensation will produce composite objects, with tachyon vevs encoded in $d$.

$$d_X = \begin{pmatrix} \partial_E & 0 \\ T & d_F \end{pmatrix}$$

$$\begin{array}{ccccccc} E_0 & \to & E_1 & \to & E_2 & \cdots & \\ & \searrow^{T} & & \searrow & & \searrow & \cdots \\ & & F_0 & \to & F_1 & \to & F_2 \to \cdots \end{array}$$

$$\to E \xrightarrow{T} F \to X \to E[1] \to \cdots$$

ex: $0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_\Sigma$

All topological branes can be formed as bound states starting from a finite generating set. By starting with rigid branes (with no moduli), all moduli appear explicitly in the bound state construction.

The same category of topological branes will arise no matter what point in Kähler moduli space one studies. In particular, orbifolds and Landau-Ginzburg orbifold theories can be related to their large volume limits by using the generalized McKay correspondence. Their quiver gauge theories provide a more concrete construction of the same derived category of topological D-branes.

B-type D-branes in very general orbifold theories can be understood as "fractional branes." Their geometric interpretation is implicit in the quiver gauge theory framework; they correspond to particular cycles and bundles in the exceptional divisor. Their K theory classes are Poincaré dual to a natural basis of "tautological bundles" on the total space of the resolved orbifold; in examples this determines their geometric interpretation uniquely.

One can test this by comparing the spectrum of massless fermions between each pair of branes at orbifold and large volume limits, finding precise agreement.

While topological open string theory describes massless fermions, the masses of their bosonic partners are controlled by Kähler moduli.

A BPS D-brane should be thought of as Dirichlet (or Neumann) in the bosonized $U(1)$ of the world-sheet $(2,2)$ superconformal algebra; its position is the phase $\varphi$ of its BPS central charge.

$$\varphi(B) = \frac{1}{\pi}\mathrm{Im}\ \log Z(B)$$

Varying Kähler moduli varies these positions, the winding number ($U(1)$ charge) of stretched strings, and thus their mass. Fermions remain massless, and there is a simple formula for the masses of partner bosons.

The mass of a boson in a chiral multiplet of strings $A \to B$ is related to $U(1)$ charge $Q$, or equivalently the degree $q$ of an element of $H^q_{\bar\partial}(M, A^* \otimes B)$, as

$$m^2_{A\to B} = \frac{1}{2}\left(Q_{A\to B} - 1\right).$$

Under variations of Kähler moduli, the charge $Q$ and thus the mass $m^2$ will flow with the gradings (phases) $\varphi$ of BPS central charges as

$$\Delta Q_{A\to B} = \Delta\varphi_A - \Delta\varphi_B.$$

When an $A \to B$ boson goes tachyonic, a bound state of $A$ and $B$ becomes stable.

$$\begin{pmatrix} \mathcal{A}_B & \mathcal{T}_{A\to B} \\ \mathcal{T}_{B\to A} & \mathcal{A}_A \end{pmatrix} \quad 0 \to B \to C \to A \to 0$$

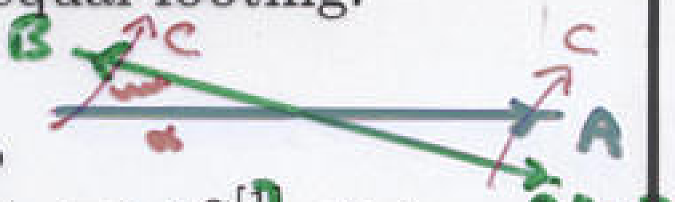
**Conjecture:** All lines of marginal stability have this origin.

Since

- Massless fermions are topological;
- The masses of partner bosons are determined by BPS central charges, computable using mirror symmetry;

this conjecture implies that the spectrum of BPS branes at any point in Kähler moduli space and all lines of marginal stability can be determined purely using large volume results.

Making this precise requires one more mathematical ingredient: a generalized notion of exact sequence which can describe arbitrary bound states of branes and antibranes. As we discussed, the derived category is already set up to do this; the necessary construction is called the "distinguished triangle" and can describe the process of forming a bound state $C$ of objects $A$ and $B$, treating all three objects on an equal footing.



$$\longrightarrow B \xrightarrow{\alpha} C \xrightarrow{\beta} A \xrightarrow{\gamma} B[1] \xrightarrow{\alpha[1]} C[1] \longrightarrow \cdots$$

$$Q_\alpha + Q_\beta + Q_\gamma = 1$$

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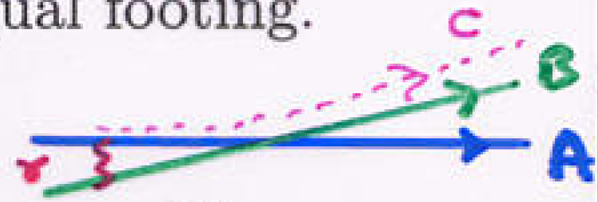
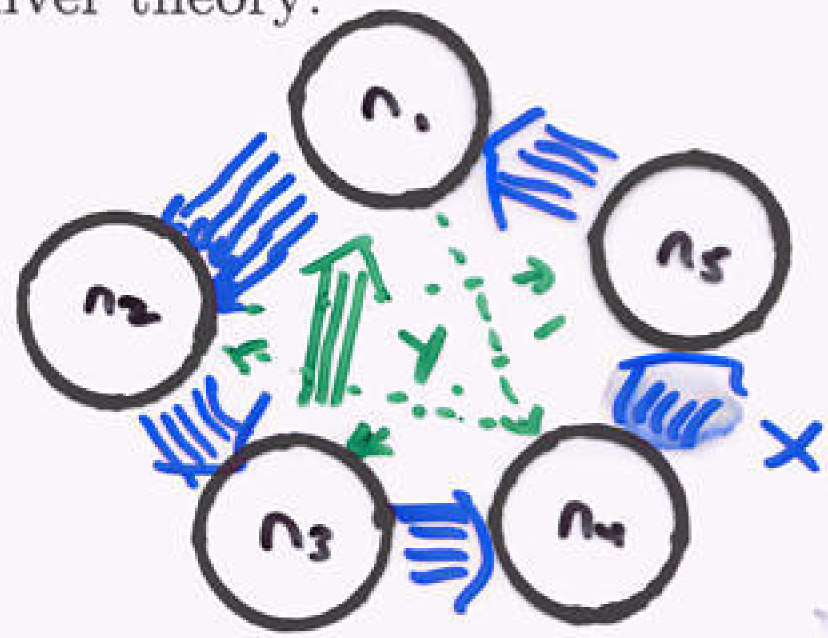



$$\longrightarrow B \xrightarrow{\alpha} C \xrightarrow{\beta} A \xrightarrow{\gamma} B[1] \xrightarrow{\alpha[1]} C[1] \longrightarrow \cdots$$

$$Q_\alpha + Q_\beta + Q_\gamma = 1$$

Let us discuss the example of the quintic from this point of view, and see how the rational boundary states of Recknagel and Schomerus can be understood in this language. A similar discussion can be made for any $\mathbb{C}^3$ orbifold or Gepner model.

The quiver theory:




$$W = \sum_n \mathrm{Tr}\, X^i_{(n+2,n+1)} X^j_{(n+1,n)} Y^{(n,n+2)}_{[ij]}$$

$$+ \ldots$$

The McKay correspondence on $\mathbb{C}^5/\mathbb{Z}_5$ tells us that the $n$'th fractional brane is related to a bundle whose sections are degree zero $5-n$-forms on $\mathbb{P}^4$.

$$\psi^{(n)}_{[i_1 \ldots i_{5-n}]} dz^{i_1} \ldots dz^{i_{5-n}}; \qquad z^i \psi_{i\ldots} = 0.$$

These are rigid on the quintic.

The cohomology groups tell us the massless fermion spectrum at large volume and the masses of partner bosons. For example,

$$B_4 \longrightarrow B_5$$

are the maps

$$\psi^{(5)} = X^i_{(5,4)} \psi^{(4)}_i .$$

which are elements of $H^0(M, B_4^* \otimes B_5)$.

Finally, the McKay correspondence also tells us that the fractional branes $B_{2k+1}$ are "branes" at large volume ($Q_6 > 0$), while $B_{2k}$ are "antibranes" ($Q_6 < 0$). Thus the elements of $H^0(M, B_n^* \otimes B_{n+1})$ we just discussed are massless fermions whose bosonic partners are tachyons of $m^2 = -1/2$. Each pair of branes $B_n \oplus B_{n+1}$ for $1 \leq n \leq 4$ can thus form a bound state at large volume, while $B_5 \oplus B_1$ (which are both branes) cannot.

These bound states are simply described by tachyon condensation: in $N = 1$ effective theory, tachyon masses can only come from D terms, and always come from a bounded below potential. (At large volume, consider vortex equations: see Oz, Pantev and Waldram.)

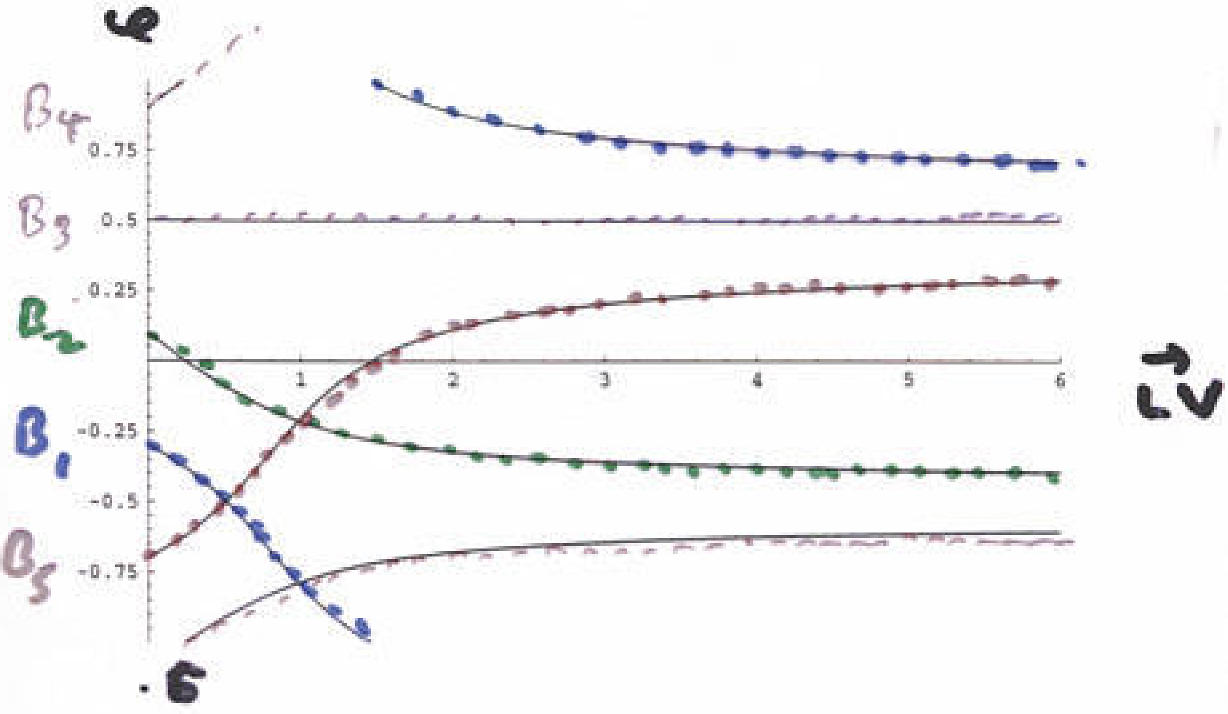
Let us compare this with the spectrum of rational boundary states at the Gepner point. The bound states $B_n \oplus B_{n+1}$ exist with the expected properties (moduli space dimension 4), consistent with the fact that these bosons are still tachyonic (they now have $Q = 3/5$ and can be regarded as elements of "$H^{3/5}$").

However the Gepner point has a $\mathbb{Z}_5$ symmetry which is reflected in the spectrum: the bound state $X \equiv B_5 \oplus B_1$ also exists with properties identical to the others. This bound state is not a coherent sheaf.

What is it?

The chiral multiplets $B_5 \longrightarrow B_1$ must also correspond to large volume cohomology: in fact they are in $H^3(M, B_5^* \otimes B_1)$. Their bosonic partners have string scale masses at large volume.

As one approaches the Gepner point, the gradings and bosonic masses flow:



Using $\Delta m^2 = \frac{1}{2}(\Delta\varphi_A - \Delta\varphi_B)$, we now know the mass of each boson, everywhere along the flow. The $B_n \longrightarrow B_{n+1}$ tachyons go up from $m^2 = -1/2$ to $m^2 = -1/5$, while the $B_5 \longrightarrow B_1$ bosons go down from $m^2 = 1$ to $m^2 = -1/5$.

On the line in moduli space where the $B_5 \longrightarrow B_1$ bosons go tachyonic, the bound state $X$ becomes stable. Thus we have identified it as a specific object in the derived category,

$$B_5[-3] \longrightarrow B_1 \longrightarrow X$$

which only becomes stable in the stringy regime.

Many new phenomena are accessible using these techniques:

- Objects have computable moduli spaces with branches of different dimension, as is generic in $N = 1$ supersymmetry.
- Flops and the "phase" structure of Kähler moduli space can be defined precisely in terms of preferred lines of marginal stability.
- The spectrum of stable BPS branes can change drastically as a function of Kähler moduli, but appears never to reduce to a finite set.

## Conclusions

- We have a precise definition of BPS D-branes in weakly coupled type II strings on a general CY manifold.
- It is modeled after the formal structure of $N = 1$ supersymmetry, with F- and D-flatness conditions. F-flatness conditions are exact at large volume: solutions are objects in the derived category of coherent sheaves, or of quiver analogs. D-flatness conditions are replaced by Π-stability (generalizing DUY).
- Practical computations can be made, by combining techniques from algebraic geometry, quiver gauge theory and mirror symmetry.

Important directions for future work

- Add orientifolding and anomaly constraints, and apply to $N = 1$ duality.
- Find exact superpotentials for $N = 1$ string compactifications. Very likely, this is possible within **classical** frameworks (obstruction theory; RR field strengths).
- The **entire** set of string configurations (of a certain type) can admit a simple general characterization. Can we describe **all** $N = 2$ or $N = 1$ compactifications in such a way ?