NON-COMMUTATIVE SOLITONS - I

RAJESH GOPAKUMAR

Non Commutativity In Field Theory

→ Arises As Limits Of String Theory

(Also S-Dual To Stringy Non-commutativity)

→ Displays A Non-Locality Which Is Stringy, But More Controllable.

→ Sensible Quantum Field Theories

(Renormalisable, Nevertheless Has UV/IR Mixing)

⇒ $\boxed{\text{Toy Model For Quantum Gravity ?}}$

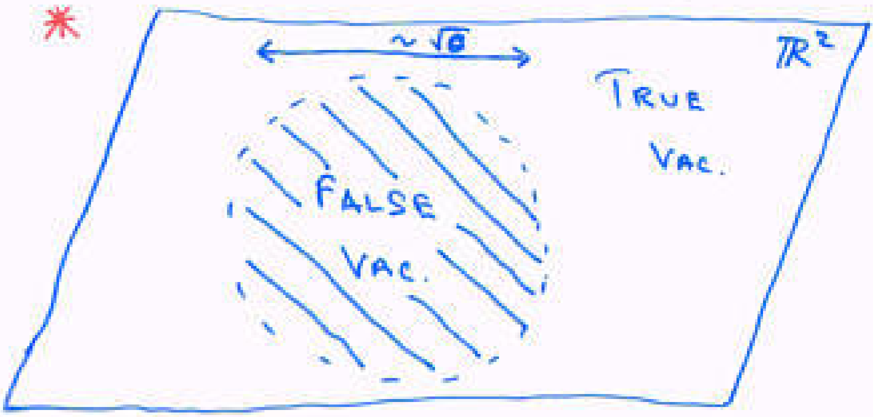
→ Possesses Novel Classical Solutions – Useful Probes Of Stringy Phenomena.

(Tachyon Condensation, Non-Commutative Myers Effect (?))

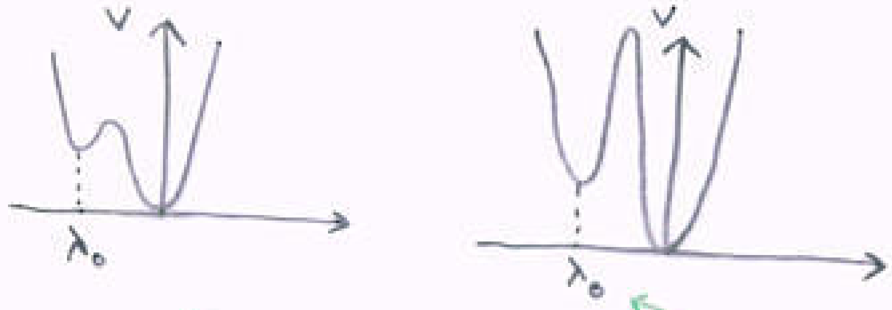
- Non-Commutative Scalar Lumps
  - The Basics (R.G., S. Minwalla, A. Strominger)
    Mar. '00
  - Multi Solitons / Periodic Arrays / Quotient Sp.
    (R.G, M. Headrick - in progress)
- Non-Commutative Flux Lumps
  (M. Aganagic, R.G., S. Minwalla, A. Strominger)
  Sept. '00

→ Onto Shiraz' Talk

HIGHLIGHTS

* 

- EVADES DERRICKS THM.
- NOT NECC. TOPOLOGICAL

* 

SOLUTION DEPENDS ONLY ON

$E \propto V(\lambda_0)$

(CAN EVEN BE ZERO!)

IND. OF DETAILS OF PROFILE OF V (e.g. BARRIER HEIGHT)

— SIGNATURE OF NONLOCALITY

* LARGE $U(\infty)$ GLOBAL SYMM (SCALARS) (APPROX.)

→ GAUGED IN GAUGE THEORIES

⇒ NON ABELIAN FLUCTUATION SPECTRUM ARISES NATURALLY — LIKE D-BRANES

[DASGUPTA, MUKHI, RAJESH
HARVEY, KRAUS, LARSEN, MARTINEC

* Bogomolnyi Like Bound $\Rightarrow$ Approximate No Force Condition On Separated Solitons

(Even For Scalars) $\Rightarrow$ Construct Periodic Arrays

- Solitons On Cylinder, Torus, Other Quotient Spaces

* Non BPS Solutions, Exact For All Θ, With Localised Integer Flux. $(m>0)$

[Polychronakos
Bak
AGMS

* Smooth Instantons Even At Singularities In Moduli Space.

Non Comm. Scalar Fields (2+1 dim.)

(Static Config.)

$$E = \frac{1}{g^2}\int d^2z \left(\partial_z \phi \partial_{\bar z} \phi + V(\phi)_* \right)$$

non-local field theories w/ Moyal prod.

$$(\phi * \phi)(z,\bar z) = e^{\frac{\theta}{2}(\partial_z \partial_{\bar z'} - \partial_{z'} \partial_{\bar z})} \phi(z,\bar z)\phi(z',\bar z')\Big|_{z=z'}$$

Analyse By Map To $U(\infty)$ Matrix Model
(Moyal-Weyl Correspondence)

$$\boxed{\begin{array}{ccc} \phi & \longleftrightarrow & \hat\phi \\ \phi * \phi & \longleftrightarrow & \hat\phi^2 \\ \frac{1}{2\pi}\int d^2z\, V(\phi)_* & \longleftrightarrow & \theta\, \mathrm{Tr}_{\mathcal{H}} V(\hat\phi) \\ \partial_z \phi & \longleftrightarrow & -\frac{1}{\sqrt\theta}[a^\dagger, \hat\phi] \end{array}}$$

$[a, a^\dagger] = \mathbb{I}$

$$\Downarrow$$

$$E = \frac{1}{g^2}\mathrm{Tr}_{\mathcal{H}}\left[\underbrace{-[a,\phi][a^\dagger,\phi]}_{\text{K.E.}} + \underbrace{\theta V(\phi)}_{\text{P.E.}} \right]$$

$\mathcal{H}$ = Hilbert Space of D=1 Quantum Mech.

STUDY CLASSICAL SOLUTIONS IN EXPANSION

ABOUT INFINITE Θ $\left(\epsilon = \frac{1}{\Theta m^2} \ll 1 \right)$

⇒ DROP KINETIC TERM IN LEADING ORDER E.O.M.

$$\frac{\partial V}{\partial \phi} = 0 \qquad \text{(OPERATOR EQN.)}$$

SOLN. IN TERMS OF PROJ. OPS. $\phi = \lambda_0 P_0$

$\left[P_0^2 = P_0 \;,\; \left.\frac{\partial V}{\partial \lambda}\right|_{\lambda_0} = 0 \right]$ e.g. $\lambda_0 |0\rangle\langle 0|$

$$E = \frac{2\pi\Theta}{g^2} V(\lambda_0)$$

GENERAL SOLN.

$$\phi_0 = \sum_n \lambda_n P_n \qquad \text{w/} \quad \mathrm{Tr}\, P_n = 1$$

$$P_n P_m = P_m \delta_{mn}$$

$$E = \frac{2\pi\Theta}{g^2} \sum_n V(\lambda_n) \qquad \lambda_n \in \left\{ \left.\frac{\partial V}{\partial \lambda}\right|_{\lambda_n} = 0 \right\}$$

STABLE IF λ_n ARE ALL MINIMA OF V

SPACE OF ZERO MODES GEN. BY $U(\infty)$ SYMM.

$$\phi_0 \to U \phi_0 U^+$$

E.G. AT "LEVEL 1", ALL $\lambda_0 |n\rangle\langle n|$ ARE

DEGENERATE AND CONTINUOUSLY CONNECTED

However, Degeneracy Lifted At Finite Θ

$$\epsilon\left[a,[a^\dagger,\phi]\right] + \frac{1}{m^2}V'(\phi) = 0 \qquad \left(\epsilon = \frac{1}{\Theta m^2}\right)$$

For e.g. at level I, Kinetic Term Corrects Solns. $|n\rangle\langle n|$ By $O(\epsilon)$. (P.E. ~ $O(1)$, K.E. ~ $O(\epsilon)$)

$\Rightarrow$ Correction To Soln. Changes E By $O(\epsilon^2)$

$\Rightarrow$ Leading Correction To E Is K.E. Of ϕ_0

$$\delta E = \frac{1}{g^2}\mathrm{Tr}\left[-[a,\phi_0][a^\dagger,\phi_0]\right] \qquad \left[O(\epsilon) \text{ w.r.t. } E_0 = \frac{\Theta}{g^2}\mathrm{Tr}\,V(\phi_0)\right]$$

Of All $|n\rangle\langle n|$, $|0\rangle\langle 0| \sim e^{-r^2}$ Has Lowest Energy

($\because$ $|n\rangle\langle n| \sim L_n(2r^2)e^{-r^2}$ – Has More Wiggles As $n\uparrow$)

Others Have A Tachyon

$$\boxed{\text{Thus Stable Solution At Level I, For Large But Finite } \Theta\text{, Is Gaussian } (|0\rangle\langle 0|) + \text{Small Corrections}}$$

Multi Solitons

Level 2: (At $\theta = \infty$)

$\phi_0 \propto |n_1\rangle\langle n_1| + |n_2\rangle\langle n_2|$ Is A Soln. (Radially Symm.)

Construct Separated Solitons:

$U(z_i)|n_i\rangle \equiv |n_i, z_i\rangle$ $\left\{ U(z) = e^{a^\dagger z - a\bar{z}} \right.$ – gen. of translation$\}$

$|n_i, z_i\rangle\langle n_i, z_i|$ Is A Soliton Localised At z_i

But, $|n_i, z_i\rangle$ Are Not Orthonormal. $(i = 1, \ldots)$

Form An Orthonormal Basis $|(\vec{n}, \vec{z})_j\rangle$

Then $\sum_j |(\vec{n}, \vec{z})_j\rangle\langle(\vec{n}, \vec{z})_j|$ Is A Proj. Op.

Corresponds To Solitons $|n_i\rangle\langle n_i|$ Localised At z_i Only When $|z_i - z_j| \gg 1$.

$\because$ It Is A Proj. Op. The Energy (P.E.) Is Independent Of z_i.

E.G. $|z\rangle \equiv U(z)|0\rangle$ — Coherent state $(U(z) = e^{a^\dagger z - a\bar{z}})$

$$|z_\pm\rangle = \frac{|z\rangle \pm |-z\rangle}{\sqrt{2(1 \pm e^{-2|z|^2})}}$$ ARE ORTHONORMAL

$$|z_+\rangle\langle z_+| + |z_-\rangle\langle z_-| \xrightarrow{|z| \gg 1} |z\rangle\langle z| + |-z\rangle\langle -z|$$

(GAUSSIANS AT $\pm z$)

$$\xrightarrow{z \to 0} |0\rangle\langle 0| + |1\rangle\langle 1|$$

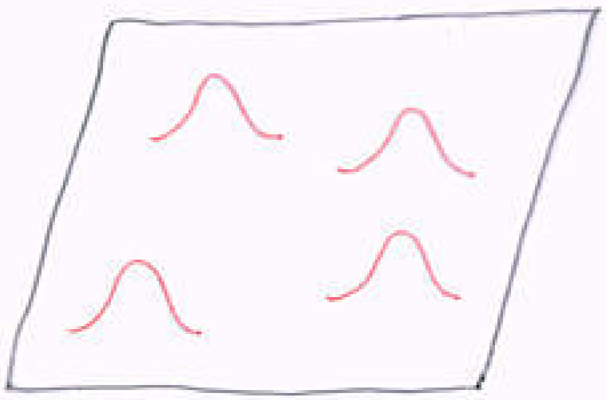
LEVEL 2 PROJECTION OPERATOR $\Rightarrow$ P.E. ind. of z

AT FINITE θ, MIGHT EXPECT $|0\rangle\langle 0| + |1\rangle\langle 1|$ TO HAVE LOWEST KINETIC ENERGY.

IN FACT, K.E. OF $|z_+\rangle\langle z_+| + |z_-\rangle\langle z_-|$ IS IND. OF z!

$\Rightarrow$ EVEN AT $O(\epsilon)$, SEPARATED GAUSSIAN SOLITONS FEEL NO FORCE.

TRUE FOR ANY NUMBER OF SEPARATED GAUSSIAN SOLITONS



— THERE IS A MODULI SPACE FOR ENERGIES $\sim O(\epsilon)$

WHY?

Because These Solitons Saturate A Kind Of Bogomolnyi Bound

For Any Projection Operator P, The $O(\theta)$ Term (K.E.)

$$\delta E(P) = -\frac{2\pi}{g^2}\,\mathrm{Tr}\{[a,P][a^\dagger,P]\}$$

$$= \frac{2\pi}{g^2}\left[\mathrm{Tr}\,P + 2\sum_{\substack{|\alpha\rangle\in\mathcal{H}_\perp\\ |i\rangle\in\mathcal{H}_{\parallel}}} |\langle\alpha|a|i\rangle|^2\right]$$

$$\geq \boxed{\frac{2\pi}{g^2}\,\mathrm{Tr}\,P} = \frac{2\pi}{g^2}\dim\mathcal{H}_{\parallel}$$

Equality Holds Iff $a\mathcal{H}_{\parallel}\subset\mathcal{H}_{\parallel}$

[$\mathcal{H}_{\parallel}$, $\mathcal{H}_\perp$ Are The Subspaces Of $\mathcal{H}$ s.t.

$$P\big|_{\mathcal{H}_{\parallel}} = 1 \qquad P\big|_{\mathcal{H}_\perp} = 0$$]

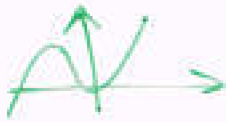
In Particular, For $P = \sum_{i=0}^{N-1}|i\rangle\langle i|$, $\mathcal{H}_{\parallel} = \{|i\rangle, i=0\ldots N-1\}$ And $a\mathcal{H}_{\parallel}\subset\mathcal{H}_{\parallel}$. Not The Case For Other Sets $\{|i\rangle\}$.

But Also True For $\mathcal{H}_{\parallel} = \{|z_i\rangle\}$

$(\because\ a|z_i\rangle = z_i|z_i\rangle,\ a\mathcal{H}_{\parallel}\subset\mathcal{H}_{\parallel})$

At $O(\epsilon^2)$, There Is An Attractive Force — From Including The Corrections To The Solution.

The Radially Symm. Solution ($r \to 0$) Is Stable

[Curiously, For A Potential  w/ $M_0^2 = -M_1^2$, There Is No Force Even At $O(\epsilon^2)$]

Exploit This "Bogomolnyi Property" To Construct Stable Non-Comm. Lumps At Finite θ On Quotient Spaces.

For Example, A Periodic Array Of Solitons. In Covering Space, Find Projection Operators Invariant Under Some Discrete Translations

$$U(2\pi n R)\, P\, U^{+}(2\pi n R) = P.$$

$$|\phi_R\rangle \equiv \sum_{n=-\infty}^{\infty} c_n U^n |\phi_0\rangle$$

$|\phi_0\rangle$ arbitrary

$U^n \equiv U(2\pi n R)$

Choose c_n s.t. $\{ U^n |\phi_R\rangle \}$ Is An Orthonormal Set.

$\Rightarrow$ $$P = \sum_{n=-\infty}^{\infty} U^n |\phi_R\rangle\langle\phi_R| U^{-n}$$

Is A Soliton At $\theta = \infty$ On Cylinder

All Solitons (w/ Different $|\phi_0\rangle$) Are Degenerate In Energy At Order 1.

AT FINITE θ,

ANALOGUE OF GAUSSIAN (STABLE, LOWEST ENERGY)

CONSTRUCTED FROM $|\phi_0\rangle = |0\rangle$.

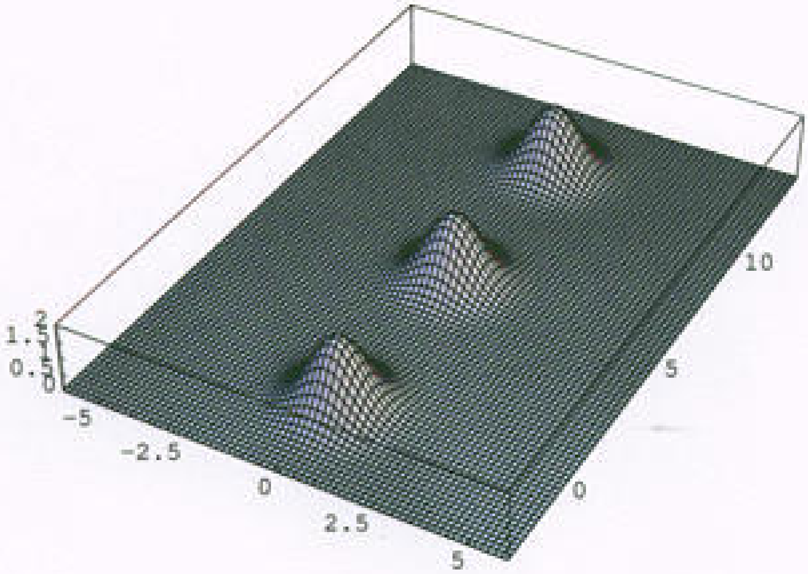
($\because$ $\mathcal{H}_\alpha = \{U^n|0\rangle\}$ is invariant under a $\Rightarrow$ Corresp. Proj. op. saturates Bogomolnyi bound for kinetic energy)

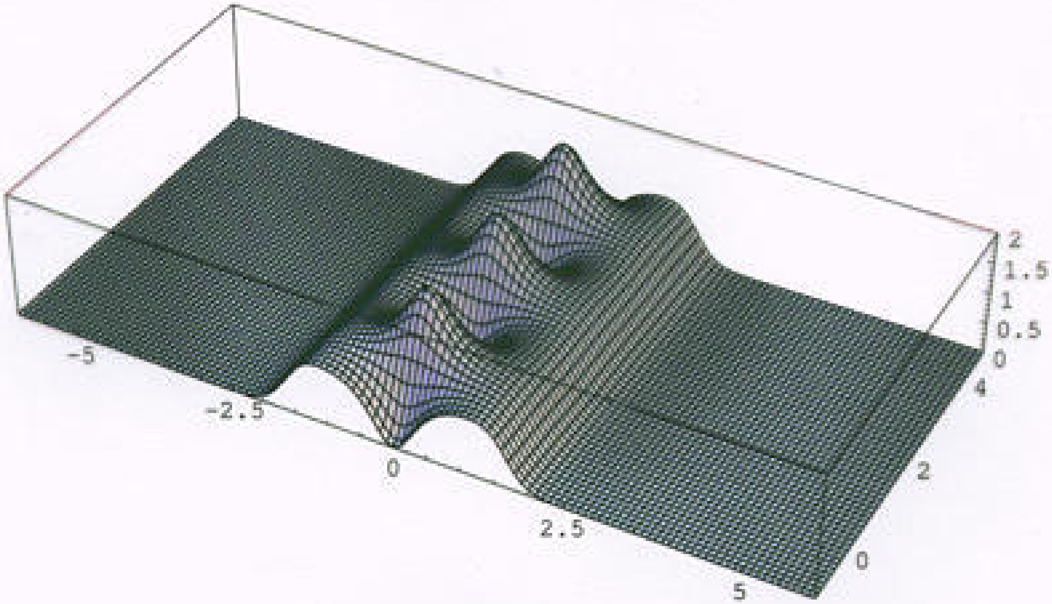
$$P = \sum_{n=-\infty}^{\infty} U^n |0_R\rangle\langle 0_R| U^{-n}$$

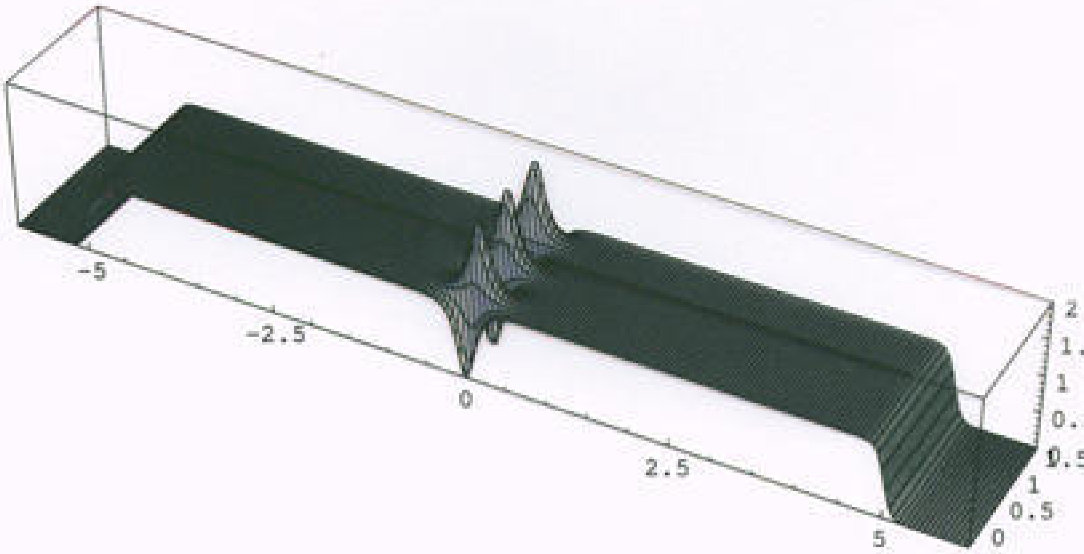
$$\longleftrightarrow \frac{\sum_n e^{-2inx/R} e^{-y^2-n^2/R^2}}{\sum_n e^{-(y+n/R)^2}} + \frac{\sum_n e^{-2i(n+\frac{1}{2})x/R} e^{-y^2-(n+\frac{1}{2})^2/R^2}}{\sum_n e^{-(y+(n+1/2)/R)^2}}$$

$$= \frac{\Theta_{00}(x/\pi R, i/\pi R^2)}{\Theta_{00}(iy/\pi R, i/\pi R^2)} + \frac{\Theta_{10}(x/\pi R, i/\pi R^2)}{\Theta_{10}(iy/\pi R, i/\pi R^2)}$$

P.E. + K.E. ARE SEPARATELY INDEPENDENT OF RADIUS R.


2
1.5
1
0.5
0
-5
-2.5
0
2.5
5
0
5
10


-5
-2.5
0
2.5
5
0
2
4
0
0.5
1
1.5
2


-5
-2.5
0
2.5
5
0
0.5
1
1.5
2
0.5
1
1.5

Non Comm. Gauge Fields (2+1 dim.)

$$E = -\frac{1}{4g_{YM}^2}\int d^2z\, F_{z\bar z}^2$$

(Gauge $A_0 = 0$, Static Config.)

$$F_{z\bar z} = \partial_z A_{\bar z} - \partial_{\bar z} A_z + i(A_z * A_{\bar z} - A_{\bar z} * A_z)$$

Map To Operators

$$\boxed{\begin{array}{lcl} D_z & \longleftrightarrow & \frac{a^+}{\sqrt{\theta}} - i\hat{A}_z \equiv C \\ D_{\bar z} & \longleftrightarrow & \frac{a}{\sqrt{\theta}} + i\hat{A}_{\bar z} \equiv \bar{C} \\ F_{z\bar z} & \longleftrightarrow & i[C,\bar{C}] + \frac{i}{\theta} \end{array}}$$

$[a, a^+] = 1$

$$\Downarrow$$

$$E = \frac{2\pi\theta}{g_{YM}^2}\,\mathrm{Tr}_{\mathcal{H}}\left([C,\bar{C}] + \frac{1}{\theta}\right)^2$$

$$\Downarrow$$

E.O.M. $\quad [C,[C,\bar{C}]] = 0$

Flux Lumps:

$$[c,[c,\bar{c}]] = 0 \qquad \left[\begin{array}{l} c = \frac{a^\dagger}{\sqrt{\theta}} - iA_z \\ \bar{c} = \frac{a}{\sqrt{\theta}} + iA_{\bar{z}} \end{array}\right.$$

Vac. Soln.

$$c = \frac{a^\dagger}{\sqrt{\theta}}, \quad \bar{c} = \frac{a}{\sqrt{\theta}} \qquad (\text{i.e. } A_z = A_{\bar{z}} = 0)$$

Flux Solns.

[Polychronakos, Bak
Agms

$$c^{(1)} = S^\dagger \frac{a^\dagger}{\sqrt{\theta}} S, \qquad \bar{c}^{(1)} = S^\dagger \frac{a}{\sqrt{\theta}} S$$

$$S = \sum_{i=0}^{\infty} |i\rangle\langle i+1| \; ; \quad SS^\dagger = 1, \quad S^\dagger S = 1 - |0\rangle\langle 0|, \quad \underset{=0}{S|0\rangle}$$

Non-BPS Solns. w/

$$\text{Integer Flux} = -\frac{i}{2\pi}\int d^2z\, F_{z\bar{z}} = \mathrm{Tr}\,(1 + \theta[c,\bar{c}]) = 1$$

Unstable $\because$ Energy $= \frac{\pi}{g_{YM}^2\theta}$. Can Be Lowered

By Spreading The Flux "Thinly."

Generally,

$$C^{(m)} = (S^{+})^{m} \frac{a^{+}}{\sqrt{\theta}} S^{m} + \sum_{a=0}^{m-1} c^{a} |a\rangle\langle a|$$

w/ $F_{z\bar{z}} \propto \sum_{i=0}^{m-1} |i\rangle\langle i|$ $\Rightarrow$ m (>0) Units Of Flux

Not So Obvious From Gauge Theory. Easy To See In String Theory

$$E = \frac{m\pi}{g_{YM}^{2}\theta}$$

Fluctuation Analysis $\Rightarrow$ $M_T^2 = -\frac{1}{\theta}$