D-Branes and Vector Bundles on Calabi-Yau Manifolds: A view from the helix

- SG (IIT Madras)

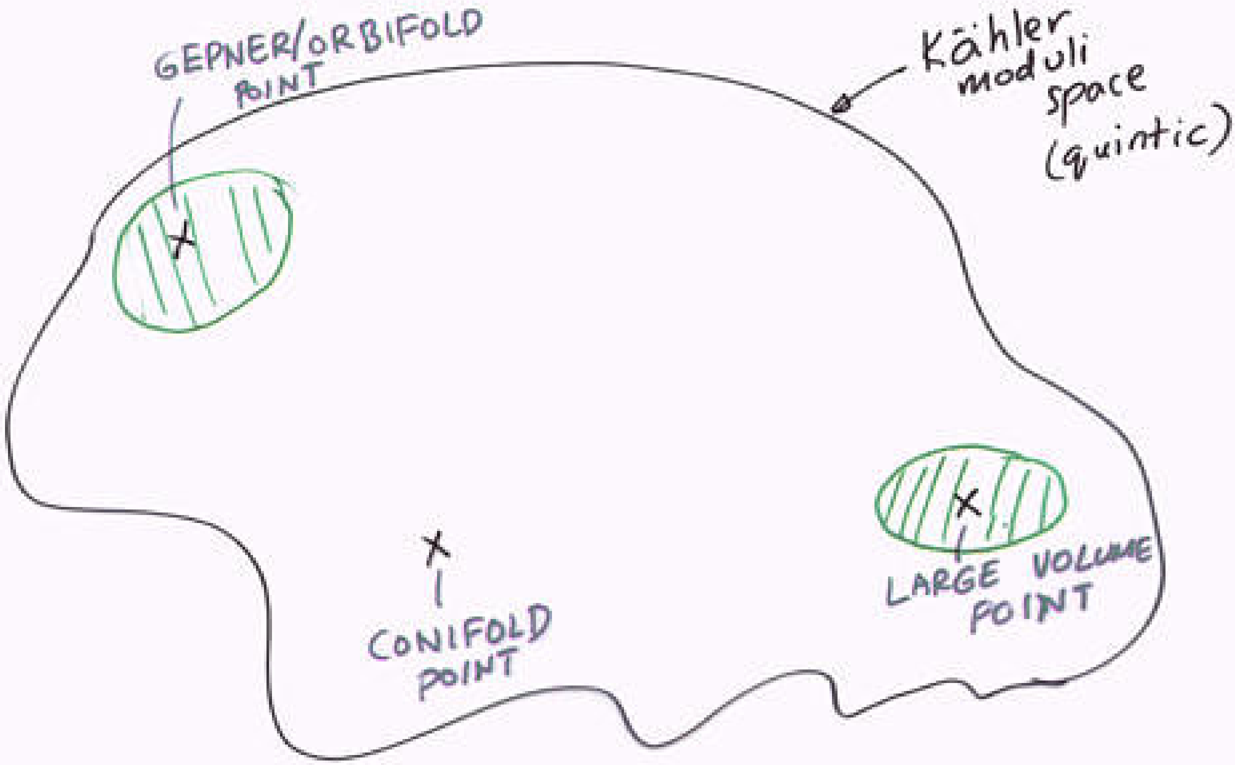
1. SG and T. Jayaraman hep-th/0010196
2. SG, T. Jayaraman, T. Sarkar hep-th/0007075
3. SG and T. Jayaraman (to appear) "Boundary Fermions, Coherent Sheaves and D-branes on CY manifolds"

related work

1. Diaconescu and Gomis hep-th/9906242
2. Diaconescu and Douglas hep-th/0006224
3. Tomasiello hep-th/0010217
4. Mayr hep-th/0010223
5. Hori - Strings 2001
6. Kraus - Strings 2001

Credits: Chennai Telephones!

D-branes on Calabi-Yau manifolds



LARGE VOLUME:
- A-branes wrap special Lagrangian 3-cycles
- B-branes wrap holomorphic cycles ($\leftrightarrow$ coherent sheaves)

we will only consider these in this talk.

B-Branes at the Gepner point

Use CFT to construct boundary states. [Recknagel-Schomerus States]

Brunner et al.

$$|L_1, L_2, L_3, L_4, L_5 ; M\rangle$$

For the quintic $L_i = 0, 1$
$M = 0, 1, 2, 3, 4$

Case when all $L_i = 0$ is special.
All 5 D branes are rigid
(i.e., have no moduli)

They are given by restrictions of bundles on $\mathbb{P}^4$ to the quintic hypersurface.

Douglas-Fiol-Römelsberger

$$\Omega^p(p)\big|_{\text{quintic}} \qquad p = 0, 1, 2, 3, 4.$$

"large volume analogs of the $L_i = 0$ states"

Ω = cotangent bundle to $\mathbb{P}^4$.

$$\Omega^p(p) = \Lambda^p \Omega \otimes \mathcal{O}(p).$$

Douglas
Fiol
Römelsberger

• All $\sum L_i \neq 0$ RS states are bound states of the $\sum L_i = 0$ states. (easy to see at the level of charges)

The method used in establishing the relationship between the RS boundary states and large-volume bundles makes use of mirror symmetry and is rather cumbersome.

Is there a more straightforward relationship which does not make use of mirror symmetry?

The Gauged Linear Sigma Model (GLSM)

Field Content: Φ_i (2,2) chiral multiplets with charge Q_i

$(\phi_i, \psi_{\pm i})$

P – chiral multiplet with charge

$Q_P = -\sum_i Q_i \qquad (p, \psi_{\pm p})$

V – real vector multiplet (abelian)

Field ~~string~~ strength $\to \Sigma$ twisted chiral multiplet

$(\sigma, \lambda_\pm, D, v_{01})$

Fayet–Iliopoulos D-term with parameter

$t = \frac{\theta}{2\pi} + i r$

Case (i) No superpotential @ low-energies

$r \gg 0$ Total space of a line-bundle $\mathcal{O}(|Q_P|)$ over $\mathbb{P}^{Q_1 \cdots Q_n}$ (non-compact CY)

$r \ll 0$ Orbifold phase $\mathbb{C}^n / \mathbb{Z}_{|Q_P|}$

$\langle p \rangle \neq 0$

Case (ii) $\quad W = P\,G(\Phi)$ of degree $|Q_P|$

$r \gg 0$ — Calabi-Yau manifold given by the hypersurface $G=0$ in $\mathbb{P}^{Q_1 \cdots Q_n}$

CY-phase

$r \ll 0$ — LG orbifold $\longleftrightarrow$ RS states.

LG-phase.

RS states must come from Dirichlet boundary conditions in the LG orbifold.

SG, TJ, T. Sarkar '99 (Warner)

$$\phi_i = 0$$

$\Rightarrow$ these are branes localised at the orbifold singularity!

Ignoring the superpotential, these are the so-called "FRACTIONAL BRANES"

D-branes on an orbifold $\mathbb{C}^n/\Gamma$

$\Updownarrow$

Representations of a quiver (McKay Quiver)

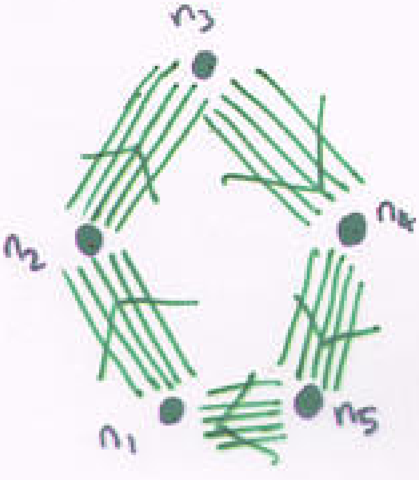
Douglas-Greene-Morrison; Douglas-Moore; Fiol-Mariño; Feng et al; Fiol

$\mathcal{N}=1$ Quiver Gauge Theory

(worldvolume theory)

$\prod_i U(n_i)$

+ bifundamental scalars for every arrow



$\mathbb{C}^5/\mathbb{Z}_5$

vertices of the quiver $\Longleftrightarrow$ Fractional branes on the orbifold.

$X =$ the resolution of $\mathbb{C}^n/\Gamma$ given by the GLSM for a Gepner model

e.g. the total space of $\mathcal{O}(5)$ over $\mathbb{P}^4$ for the quintic.

Diaconescu - Douglas propose the following dual relationship between

tautological bundles

fractional branes

$\{R_a\}$ and $\{S^a\}$

basis for $K(X)$

basis for $K_c(X)$

bundles with compact support i.e., those that live on the exceptional divisors. corresponding to the resolution

- Kronheimer-Nakajima
- Ito-Nakajima

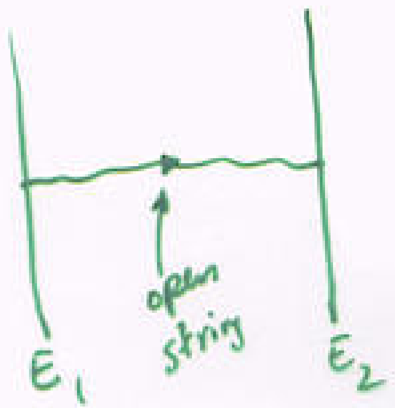
Duality

$$\langle R_a, S^b \rangle \equiv \chi(R_a, S^b) = \delta_a{}^b$$

$$= \int_X ch(R_a^*)\, ch(S^b)\, Td\, X$$

$\chi(E_1, E_2)$ = "no." of Ramond ground states of an open-string connecting D-branes associated with E_1 and E_2.

Euler form.



$$= \sum_i (-)^i \{\text{no. of Ramond ground with charge } i\}$$

$$= \sum_i (-)^i \dim\left[\underset{\overset{\parallel}{H^i_0(X,\, E_1^* \otimes E_2)}}{Ext^i(E_1, E_2)}\right]$$

if E_1 is a vector bundle.

What are the R_a? Given X, can we give a simple recipe to obtain them?

Douglas & Diaconescu propose an inverse toric construction that works when X is non-singular. Method is tedious (however).

A crucial observation

The $\{R_a\}$ form the foundation of a helix.

{ SG + TJ
Tomasiello
Mayr }

What is a helix?

- Rudakov

A few definitions first

- A vector bundle/coherent sheaf E is called exceptional if

$$Ext^i(E,E) = 0 \qquad i \geq 1$$
$$Ext^0(E,E) = \mathbb{C}$$

- An ordered collection of exceptional sheaves $(E_1 \ldots E_K)$ is called strongly exceptional if for all $a < b$

$$Ext^i(E_b, E_a) = 0 \qquad i \geq 0$$
$$Ext^i(E_a, E_b) = 0 \qquad i \neq 0$$

$\Rightarrow$ Ramond ground states exist only in the zero-charge sector.

$\Rightarrow$ $\chi(E_a, E_b)$ is an UPPER-TRIANGULAR matrix with ones on the diagonal.

- New exceptional sheaves from old

Given an exceptional pair (E_a, E_{a+1}), there is an operation called a LEFT MUTATION which produces a new exceptional pair

$$(E_a, E_{a+1}) \xrightarrow[\text{MUTATION}]{\text{LEFT}} (L_{E_a}(E_{a+1}), E_a)$$

Zaslow
Hori, Iqbal, Vafa
(Brane creation in the mirror)

where the new sheaf $L_{E_a}(E_{a+1})$ is defined by the exact sequence

$$0 \longrightarrow \mathrm{Ext}^0(E_a, E_{a+1}) \otimes E_a \to E_{a+1} \to L_{E_a}(E_{a+1}) \to 0$$

if the map

$$\mathrm{Ext}^0(E_a, E_{a+1}) \otimes E_a \to E_{a+1}$$

is injective.

One can see that

$$ch(L_{E_a}(E_{a+1})) = ch(E_{a+1}) - \chi(E_a, E_{a+1})\, ch(E_a)$$

- An exceptional collection $\{E_i, i \in \mathbb{Z}\}$ is called a HELIX OF PERIOD p if

$$L^{p-1}(E_s) = E_{s-p}.$$

i.e., a sequence of $(p-1)$ left mutations gives one an element of the helix modulo a shift of p.

- Any collection of p-consecutive elements of a helix are called a foundation of the helix.

The collection of line bundles

$$\{\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)\}$$

is the foundation of a helix of period 'n' on $\mathbb{P}^n$.

Two Conjectures

Conjecture 1 The large-volume monodromy $(t \to t+1)$ on $\mathcal{O}$ [i.e., the bundle which wraps all of X] produces an exceptional collection which is a helix with foundation

$$\{R_1 = \mathcal{O},\ R_2, \ldots, R_p\}$$

Remarks 1. The period p reflects the quantum $\mathbb{Z}_p$-symmetry !

2. Procedure seems to work for cases when X is <u>not</u> fully resolved.

3. In the cases when there are more than Kähler moduli, the $t_i \to t_i + 1$ generates $_{i=1,\ldots,d}$ a lattice of dim-d of which the helix is a one-dim. sublattice (Mayr)

Conjecture 1 has been verified in several examples.

Conjecture 2 All exceptional bundles on X are generated by mutations.

Remarks

1. There exists a mutated helix with foundation
$$S = (S^{p}_{p}, \ldots, S^{1}_{1} = \mathcal{O})$$
with
$$S^a = L^{a-1}(R_a).$$
This helix satisfies.
$$\boxed{\chi(R_a, S^b) = \delta_a^{\,b}.}$$

2. The restriction of S^a to the Calabi-Yau hypersurface gives the $l_i = 0$ RS states in all examples that we have considered.

3. For $X = \mathbb{P}^n$, it is known that $\{R_a\}$ generate all sheaves on $\mathbb{P}^n$. This follows from Beilinson' theorem. $\Rightarrow$ such a generalisation exists for weighted projective spaces as well.

Gorodentsev
Rudakov
Drézet

Quivers from Helices: The Beilinson Quiver

Zaslow

Given a helix $\{R_1 \ldots R_p\}$, one can construct a quiver as follows:

1. Consider a quiver whose i-th vertex is associated with R_i
2. Draw $\dim[\mathrm{Hom}(R_i, R_j)]$ arrows beginning at vertex i and ending at vertex j.
3. The relations of the quiver are the obvious ones.

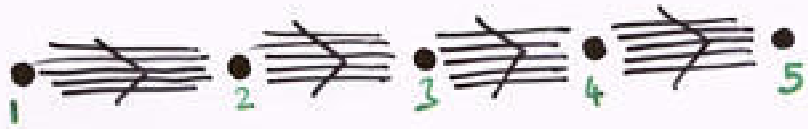
Consider $X = \mathbb{P}^4$. $R = (\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(4))$

- $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i+1))$ = multiplication by ϕ_i $i = 1, \ldots, 5$.

- Relations: $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(2))$

$$\phi_1 \phi_2 = \phi_2 \phi_1$$

BEILINSON QUIVER



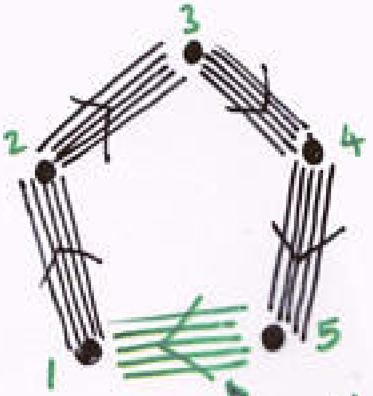
reps. of this quiver $\longleftrightarrow$ vector bundles on $\mathbb{P}^4$.

RETURN OF THE P-FIELD

BEILINSON QUIVER ⟶ MCKAY QUIVER

$\gamma \gg 0$ $\qquad\qquad$ $\gamma \ll 0$

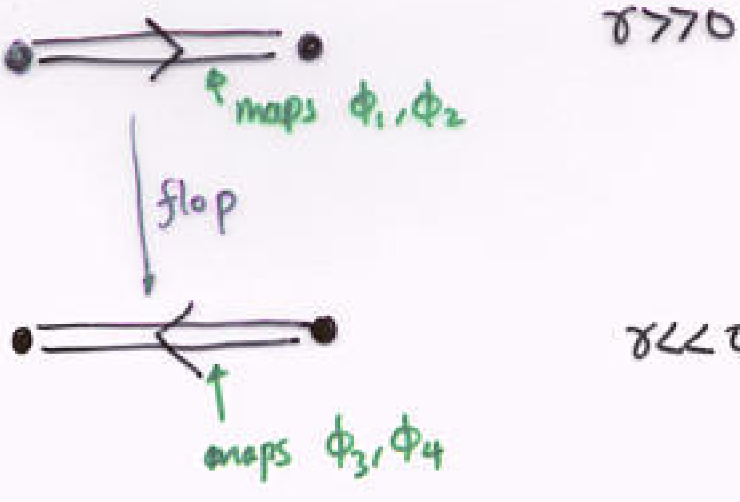
$\mathbb{P}^4$:



these are maps of the form $\underline{P\phi_i}$

these are possible in the LG phase.

<u>Flop</u>: GLSM has four fields $(\phi_1 \cdots \phi_4)$ of charge $(+1, +1, -1, -1)$.



flop transition }

"change of t-structure" in D^b(mod-A)

(Parthasarathy)

VECTOR BUNDLES IN THE GLSM WITH BOUNDARY

Holomorphic Vector Bundles from Monads:

Kachru Strings 2000

$$0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0$$

A,B,C : vector bundles

a : injective map

b : surjective.

The holomorphic bundle E

$$E = \ker b / \operatorname{Im} a$$

is the cohomology of the monad.

Field Theoretic Realisation:

Witten

fermions $\rightarrow \Pi_a \qquad a = 1, \ldots, \mathrm{rk}\, B$

The map 'a' is realised as a gauge invariance

$$\Pi_a \sim \Pi_a + E_a^i(\phi)\, K_i$$

K_i: section of A, $i = 1, \ldots, \mathrm{rk}\, A$

Gauge Fix: $\bar{E}_a^i \Pi_a = 0$

The map 'b' imposes the hol. constraint

$$J_m^a(\phi)\, \Pi_a = 0 \qquad m = 1, \ldots, \mathrm{rk}\, C$$

Vector bundles in the GLSM

BULK		BOUNDARY.
(2,2) chiral multiplet Φ	$\longrightarrow$	Φ' ← scalar Ξ ← fermi multiplet.
(2,2) twisted chiral multiplet Σ	$\longrightarrow$	complex unconstrained multiplet.

$$\tilde{v}_0 \equiv v_0 + \eta\left(\frac{\sigma+\bar{\sigma}}{\sqrt{2}}\right)$$

boundary gauge field.

Boundary Fermi Multiplets:

$$\bar{\mathcal{D}}\Pi_a = \sqrt{2}\,\Sigma'\,E_a(\Phi')$$

boundary chiral multiplet.

Similarly, the holomorphic constraints $J^a(\Phi')$ require the introduction of an additional boundary chiral multiplet. $-P'$

So far story seems similar to (0,2) constructions for the heterotic string.

Distler
Greene
Kachru

New Ingredient

Large volume monodromy of vector bundles must be implemented.

E.g. For the quintic, $\theta \to \theta + 2\pi$

$$E \to E \otimes \mathcal{O}(-1)$$

In the monad construction, this corresponds to a shift in $U(1)$ charge of the fermion.

This requires the addition of boundary contact terms — ~~some~~ part of it can be fixed in the NLSM limit ($e^2 \to \infty$, $\gamma \to \infty$).

Boundary conditions in the GLSM must have a ~~suitable~~ proper NLSM limit

This is a nontrivial constraint and leads to $\left(\frac{1}{e^2}\right)$ corrections to the NLSM boundary conditions.

Ooguri; Lawrence, TJ, Sarkar (unpub)

SG, TJ, Sarkar

Vector multiplet has to be treated carefully.

Stability of B-Branes

c.f. Douglas' talk!