Main Points

- Many NC solitons (GMS, fluxons ...) can be constructed in a unified way

- Applied to unstable D-branes w/ $B \neq 0$ this leads to a detailed description of lower D-branes as NC solitons

- The description of D-brane charge in this framework involves an example of K-theory of C^* algebras, BDF extension theory, ... a rich math structure which may play some role in future developments.

Outline

1. NC geometry and field theory

2. Solution generating transformation

3. Application to tachyon condensation in string theory.

4. Eine kleine K-theory

References (partial)

Motivation:	Sen + Zwiebach Gopakumar, Minwalla + Strominger
NC-solitons as D-branes	Dasgupta, Mukhi + Rajesh H, Kraus, Larsen + Martinec Witten
Ext to finite B, gauge soln's	Polychronakos Jatkar, Mandal + Wadia Bak Gross + Nekrasov Aganagic GMS H, Kraus, Larsen
K-theory	Witten H and Moore

will follow ___

(1) NC geometry has its origins in QM:

Weyl: $[\hat{q}, \hat{p}] = i \Rightarrow U(\tau,\sigma) = e^{-i(\tau\hat{q} + \sigma\hat{p})}$

Lie alg. Group element

$$U(\tau_1,\sigma_1)U(\tau_2,\sigma_2) = \underbrace{e^{i(\tau_1\sigma_2 - \sigma_1\tau_2)}}\, U(\tau_1+\tau_2, \sigma_1+\sigma_2)$$

proj. rep. of Abelian group of translation in phase space $\mathbb{R}^2$.

The Weyl transform is a map between classical dynamical variables $f(q,p)$ and QM operator $\hat{O}_f(\hat{q},\hat{p})$:

$$\hat{O}_f = \frac{1}{(2\pi)^2}\int d\sigma\, d\tau\; U(\tau,\sigma)\,\tilde{f}(\tau,\sigma)$$

$$f(q,p) = \int d\sigma'\, e^{-ip\sigma'} \langle q+\sigma'/2 | \hat{O}_f | q-\sigma'/2 \rangle$$

often used in study of semi-classical systems.

This defines NC $\mathbb{R}^2$ via

$$\hat{O}_f \hat{O}_g = \hat{O}_{f*g}$$

w/ $$f*g = e^{\frac{i}{2}\theta^{ij}\partial_i \partial_j'} f(x) g(x')\Big|_{x=x'}$$

$$x^i = (q, p) \qquad \theta^{12} = 1 \quad (\hbar = 1)$$

- a NC algebra of fcn's on $\mathbb{R}^2$ (or $\mathbb{R}^{2n}$ w/ θ = symplectic form).

Derivatives (Weyl)

$$\hat{O}_{\partial_q f} = i[\hat{p}, O_f], \quad \hat{O}_{\partial_p f} = -i[\hat{q}, O_f]$$

or $\partial_i = -i\theta^{-1}_{ij} \,\mathrm{ad}\, x_j$ is natural deriv op.

Classification

$$S^m = \{ f \mid |\partial_x^\alpha \partial_p^\beta f| \le C_{\alpha\beta}(1+x^2+p^2)^{(m-\alpha-\beta)/2} \}$$

$m \le 0 \Rightarrow \hat{O}_f \in B(H)$ (bounded)

$m < 0 \Rightarrow \hat{O}_f \in K(H)$ (compact - ideal in $B(H)$)

We define NC field theories in terms of
- fields on NC $\mathbb{R}^{2n}$ w/ $*$ prod
- $\sim$ ops on $\mathcal{H}$
- $\sim N \to \infty$ limit of $N \times N$ matrices

It is natural (and appears in D-branes) to introduce gauge fields A_μ that gauge unitary transformations on $\mathcal{H}$:

$$\text{If} \quad \phi \to U \phi \bar{U} \qquad U \in U(\mathcal{H})$$

$$D_i \phi = \partial_i \phi - i[A_i, \phi] \to U D_i \phi \bar{U}$$

In complex coor $z = (x^1 + i x^2)/\sqrt{2}$, $a = z/\sqrt{\theta}$

$$D_z \phi = -\theta^{-1/2}[\bar{a} + i\theta^{1/2} A, \phi]$$

$$\equiv -\theta^{-1/2}[C, \phi]$$

and $\qquad i F_{z\bar{z}} = \theta^{-1}([C, \bar{C}] + 1)$

Given an action $S[A, \phi, \cdots]$ w/ $U(H)$ gauge symmetry, the e.o.m. will transform covariantly:

$$\frac{\delta S}{\delta \phi_i} \to U \frac{\delta S}{\delta \phi_i} \bar{U},$$

Since we only need $\phi \cdots D\phi \cdots F \cdots \phi$ $\to U\phi\bar{U} \cdots U D\phi \bar{U} \cdots U\phi\bar{U}$ this only requires $\bar{U}U = 1$. In finite dimensions $\bar{U}U = 1$ $\Rightarrow U\bar{U} = 1$, but this is not true in $U(H)$.

A transformation w/ $\bar{U}U = 1$ but $U\bar{U} \neq 1$ will take a solution to a gauge in-equivalent solution since if $U\bar{U} \neq 1$ U is not a gauge symmetry.

Now $\bar{U}U = 1 \Rightarrow U\bar{U}U\bar{U} = U\bar{U}$ so

$$U\bar{U} = P$$

is a projection operator. $\bar{U}U = 1 \Rightarrow$ U is an isometry:

$$\langle x | \psi \rangle \to \langle x | \bar{U} U | \psi \rangle = \langle x | \psi \rangle.$$

So, a non-unitary isometry will generate gauge inequivalent solutions to the e.o.m. of NC gauge theory.

The canonical example is a power of the shift operator

$$S : |n\rangle \to |n+1\rangle \qquad n = 0, 1, 2, \cdots$$

$$\bar{S}^n S^n = 1 \quad , \text{ but}$$

$$S^n \bar{S}^n = (1 - P_n)$$

w/ $P_n = |0\rangle\langle 0| + |1\rangle\langle 1| + \cdots + |n-1\rangle\langle n-1|$

This solution carries flux

$$F = \frac{1}{\theta}\left([c,\bar{c}]+1\right) = \frac{1}{\theta} P_n$$

and has energy

$$E = 2\pi\theta n \left(\frac{1}{2\theta^2} + V(-\phi_*)\right)$$

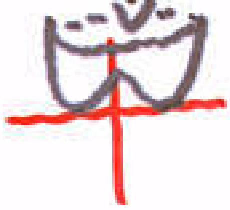
- When $\phi_* = 0$ this reduces to the pure gauge 'fluxon' solution

- Dropping gauge fields the sol'n reduces to GMS soliton at $\theta \to \infty$

- Trivially extends to $\mathbb{R}^{2n}$, the divergence of the energy as $\theta \to 0$ is in accordance with Derrick's th$^{\underline{em}}$

We can generalize to vortex soln's in a theory w/ complex ϕ $(\neq \bar{\phi})$ and NC $U(1) \times U(1)$ $[U(N) \times U(N)]$ gauge symmetry w/ gauge fields $A_{\pm}$

$$S = \int -\tfrac{1}{4}(F^+)^2 - \tfrac{1}{4}(F^-)^2 + \tfrac{1}{4}(D_\mu \phi D^\mu \bar{\phi} + D^\mu \bar{\phi} D_\mu \phi)$$
$$- V(\phi\bar{\phi}-1) - V(\bar{\phi}\phi-1)$$



Gauge sym:
$$\phi \to V \phi \bar{U}$$
$$C^- \to U C^- \bar{U}$$
$$C^+ \to V C^+ \bar{V}$$

Choosing $U = S^n$, $V = S^m$ generates from the vacuum $\phi = 1$

$$\phi = S^m \bar{S}^n$$
$$C^- = S^n \bar{a} \bar{S}^n$$
$$C^+ = S^m \bar{a} \bar{S}^m$$

~ n vortices + m anti-vortices of broken $U(1)$.

These ideas can be applied to the NC field theories which arise from unstable brane systems in a B-field

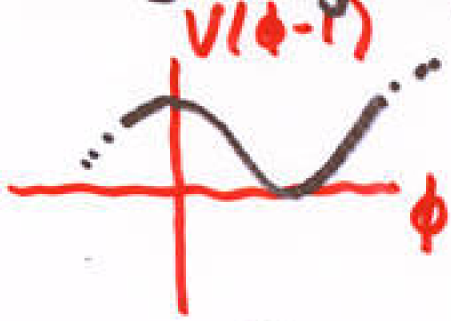
In the $\Phi = -B$ convention (Seiberg, Witten) the open string param. are

NC : $\Theta = 1/B$

Metric : $G = -(2\pi\alpha')^2 B \frac{1}{g} B$

Coupling : $G_s = g_s \det(2\pi\alpha' B g^{-1})^{1/2}$

Consider for example a D25 in bosonic string w/ tachyon ϕ and



as from level trunc.

Turn on $B_{24,25}$ and construct $S_{eff}(\phi, A)$ in operator language.

$$S_{eff} = \frac{2\pi\,\Theta(g_s T_{25})}{G_s} \int d^{24}x \sqrt{G}\ \mathrm{Tr}\,\mathcal{L}$$

$$\left.\begin{aligned}\mathcal{L} = & -\tfrac{1}{4} h(\phi - 1)\left(F^{\mu\nu} + \Phi^{\mu\nu}\right)^2 + \cdots \\ & + \tfrac{1}{2} f(\phi-1)\left(D^\mu \phi\right)^2 + \cdots\end{aligned}\right\}\text{higher derivs}$$

$$- V(\phi - 1)$$

w/ $h(0) = 0$ (CFT, BSFT)

$V(0) = 1$ (Sen)

Acting w/ $U = S^n$ on vac $\phi = 1,\ C = \bar{a},\ A_\mu = 0$ gives

$$\phi = (1 - P_n)$$
$$C = S^n \bar{a} \bar{S}^n$$
$$A_\mu = 0$$

This is a localized sol'n, asymptotic to closed string vacuum.

To compute the energy note

$$D\phi = DF = 0$$

since they vanish in the vacuum and this is preserved by soln gen. transf.

$$h(\phi - 1)[C, \bar{C}]^2 = h(-P_n)(1-P_n) = h(-1)P_n(1-P_n) = 0$$

So this and higher order terms from DBI do not contribute.

Only the potential w/ $V(\phi - 1) = P_n$ contributes and gives

$$T_{23}^{NC} = (2\pi)^2 \alpha' n T_{25} = n T_{23}$$

- the tension of n D23-branes

Comments

- As in HKLM analysis, the spectrum of fluctuations includes $U(n)$ gauge fields and tachyon in adjoint w/ $U(n) \subset U(H)$

- Improves over earlier treatments in that no $\alpha' B \to \infty$ limit is needed - hence this describes D-branes as solitons at finite g_s

- The analysis is easily extended to other systems - e.g. Dp-branes as solitons of D9-$\overline{\text{D9}}$ in IIB using NC ABS extension of vortex configurations.

K-theory aspects

Until recently, the basic tools (functors) of algebraic topology (homotopy, homology, cohomology) have been sufficient to compute the charges of solitons in field theory

$$\text{topological space} \longrightarrow \text{Abelian group (charges)}$$

with the advent of D-branes we have needed to generalize this to

$$\text{spaces w/ vector bundles} \longrightarrow \text{Abelian group}$$

which is one defn of K-theory

(Minasian-Moore, Witten)

In studying NC solitons we need 'noncommutative algebraic topology'

C^* operator algebra $\longrightarrow$ Abelian group

'K-theory of C^* algebras'

Consider for example the construction of a D7 from a D9-$\overline{\text{D9}}$. This involves

$$\phi = S$$
$$c = \cdots$$

In a commutative theory we would have

$$Q_{D7} = \frac{1}{2\pi}\int_{S^1_\infty} \bar{\phi}\, d\phi$$

i.e. $\pi_1(S^1)$ -homotopy

To relate this to ind(S) consider the following generalization of S:

$$\mathcal{H} = L^2(S^1) \qquad \text{ON basis}$$

$$\psi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}} \quad n \in \mathbb{Z}$$

Let $f(\theta) \in C_0(S^1)$ - C^* alg. of fcns on S^1
It determines an op. on $\mathcal{H}$ in the obvious way:

$$M_f \psi(\theta) = f(\theta)\psi(\theta)$$

e.g. $f = \frac{e^{i\theta}}{\sqrt{2\pi}}$ takes $\psi_n \to \psi_{n+1}$ but has index 0 since n runs over all integers.
To generalize S let

$$\mathcal{H}_+ = \text{span}\{\psi_n(\theta),\ n \geq 0\}$$

and

$$P = \text{proj. op. onto } \mathcal{H}_+$$

We can define a (Toeplitz) operator

$$T_f = P M_f : H_+ \to H_+$$

which generalizes S

$$T_{e^{i\ell\theta}} \sim S^\ell$$

The index th$^{\text{em}}$ for Toeplitz ops (Venugopalkrisha) then relates the index of T_f to the winding number of f:

$$\text{ind } T_f = \frac{1}{2\pi}\int \bar{f}\, df$$

This example also provides a simple example of the extension theory of Brown - Douglas - Fillmore which enters into deeper aspects of this subject.

We have

$$T_f T_g - T_{fg} \in K(H)$$

e.g. $T_{f_\ell} T_{f_\ell^*} - T_1 = P_\ell$, $f_\ell = e^{i\ell\theta}$

and thus a map

$$\mathcal{T} \to C_0(S^1)$$

C^* alg. of Toeplitz ops

C^* alg. of fcns on S^1

whose kernel is $K(H)$:

$$0 \to K \to \mathcal{T} \to C_0(S^1) \to 0$$

The classification of these extensions w/ $\mathcal{T}$ replaced by a general C^* alg. $\mathcal{A}$ is related to 'K-homology'

Questions

1. Can NC geometry be used effectivly to study the hard problems in tachyon condensation such as the emergence of closed strings?

2. Can the solution generating idea be used to construct D-branes in OSFT

$$- (Q + A_{D\text{-brane}}) = V * (Q + A_{vac}) V$$

for a 'non-unitary isometry' V ?

3. Are there generalizations of D-branes involving more general C^* algebras?