MIRROR SYMMETRY AND ITS APPLICATIONS

Strings 2001

K. HORI

C. Vafa & KH th/0002222

A. Iqbal, C. Vafa & KH th/0005247

KH th/0012179

M. Aganagic, A. Karch, D. Tong & KH to appear

c.f. M. Aganagic & C. Vafa th/0012041

Mirror Symmetry

... Duality in (2,2) theories in 1+1 dim.

vector R sym $\longleftrightarrow$ axial R sym

Kähler class $\longleftrightarrow$ Complex structure

Symplectic Geometry $\longleftrightarrow$ Complex Analytic Geometry

Examples
- T^{2n} ... T-duality ✓
- MORE
  Greene-Plesser
  Candelas et al
  Batyrev
  Fendley-Intriligator
  ⋮

  mostly Conjectures

Questions: How they emerge ?

Is there a change of variables ?

Motivations

1. String Theory

Type II on CY^3 : 4d $\mathcal{N}=2$, prepotential ...

+ D-branes : $\mathcal{N}=1$, superpotential ...

2. Duality in SUSY QFT

4d Electric-Magnetic $\mathcal{N}=4$ Montonen-Olive
2 Seiberg-Witten
1 Seiberg

3d Coulomb-Higgs $\mathcal{N}=4$ Intriligator-Seiberg
2 ...

3. Geometry

Symplectic vs Analytic/Algebraic

Set up: Linear σ-Model

U(1) gauge theory

$\Phi_1 \cdots \Phi_N$ charge 1 chiral

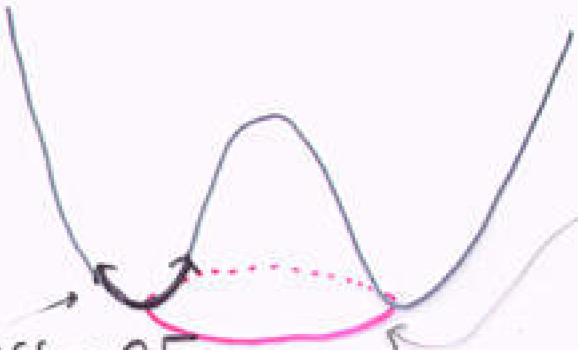
$\Sigma = \bar{D}_+ D_- V$ fieldstrength Twisted chiral

$$L = \int d^4\theta \left(\sum_{i=1}^{N} \bar{\Phi}_i e^V \Phi_i - \frac{1}{2e^2} |\Sigma|^2 \right) + \mathrm{Re} \int d^2\tilde{\theta} \, (-t\Sigma)$$

$$\boxed{t = r - i\theta}$$

$$= \text{Kinetic} - \frac{e^2}{2} \left(\sum_{i=1}^{N} |\phi_i|^2 - r \right)^2 + \theta\, v_{01}$$

$\hookrightarrow$ B-field



mass $= e\sqrt{r}$

$$M_{vac} = \frac{\left\{ \sum_{i=1}^{N} |\phi_i|^2 = r \right\}}{U(1)}$$

$e \to \infty$: $\mathbb{C}P^{N-1}$ NL σ-Model

- More general $\left\{ \begin{array}{l} G_{abel} \\ \text{charges} \end{array} \right.$: $M_{vac} = X_{toric}$
- superpotential for Φ_i : $M_{vac} \subset X_{toric}$

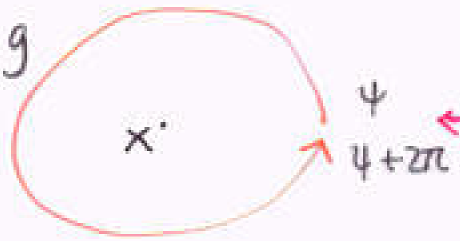
T - Duality

$$L = R^2 |\partial_\mu \psi|^2 \quad \xleftrightarrow{T} \quad \tilde{L} = \frac{1}{R^2} |\partial_\mu \vartheta|^2$$

$$\psi \equiv \psi + 2\pi \qquad\qquad \vartheta \equiv \vartheta + 2\pi$$

$$* R\, d\psi = \frac{1}{R}\, d\vartheta$$

winding



$\longleftrightarrow \quad e^{i\vartheta(x)}$ momentum

Apply this to $\arg(\Phi_i)$

$N=1$: $\quad \Phi$ chiral $\longleftrightarrow$ Y twisted chiral

Rocek-Verlinde

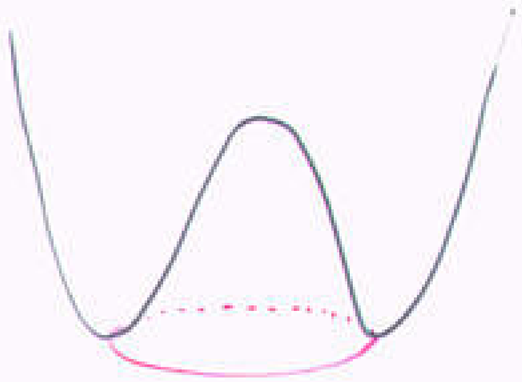
$$\arg \Phi \xleftrightarrow{T} \operatorname{Im} Y \qquad Y \equiv Y + 2\pi i$$

$$\bar{\Phi} e^{V} \Phi = \operatorname{Re} Y$$

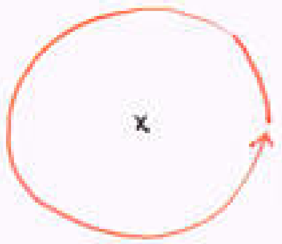
charged $\longleftrightarrow$ dynamical Theta angle

$$\tilde{W}_{\text{dualize}} = \underbrace{\sum (Y - t)}$$

Instantons



$\rightarrow$ Φ - vortex



$\arg\Phi$

$\arg\Phi + 2\pi$

$\Rightarrow$ e^{-Y} generated

$$\widetilde{W} = \Sigma\,(Y - t) + e^{-Y} \quad \text{exact}$$

$U(1)^k$ $\Phi_1, \ldots, \Phi_N$ $\longleftrightarrow$ $Y_1, \ldots, Y_N$

Σ_a $Q_{1a}, \ldots, Q_{Na}$

$t_a = r_a - i\theta_a \quad (a = 1, \ldots, k)$

$$\widetilde{W} = \sum_{a=1}^{k} \Sigma_a \left(\sum_{i=1}^{N} Q_{ia} Y_i - t_a \right) + e^{-Y_1} + \cdots + e^{-Y_N}$$

- $+$ superpotential for Φ_i $\longleftrightarrow$
  - No change in $\widetilde{W}$
  - Change in $K(Y_i, \overline{Y}_i)$

σ-Model limit $e \to \infty$

Σ_a: heavy → integrate out

⇒ constraints

$$\sum_{i=1}^{N} Q_{ia} Y_i = t_a \quad (a=1,\cdots,k)$$

- we are left with

$$\widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_N}$$

NL σ-Model on ↔ LG Model on

$$X^{N-k} = \frac{\left\{ \sum_{i=1}^{N} Q_{ia} |\phi_i|^2 = r_a \right\}}{U(1)^k} \qquad (\mathbb{C}^\times)^{N-k} = \left\{ \sum_{i=1}^{N} Q_{ia} Y_i = t_a \right\}$$

B-field ~ θ_a

$$\widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_N}$$

$r_a - i\theta_a = t_a$

Mirror Symmetry is derived

- σ- Model on $M \subset X^{N-k}_{\text{toric}}$

$\leftrightarrow$ LG model $\begin{cases} \circ \text{ Same } \widetilde{W} \\ \circ\ \underline{(\mathbb{C}^\times)^{N-k} \text{ replaced}} \end{cases}$

One can find the holomorphic $(N-k)$-form Ω

Example

Degree d hypersurface $\subset \mathbb{CP}^{N-2} \subset X^{N-1}_{\text{toric}}$

$\leftrightarrow$ LG orbifold $(\mathbb{Z}_d)^{N-2}$

$$\widetilde{W} = X_1^d + \cdots + X_{N-1}^d + e^{t/d} X_1 \cdots X_{N-1}$$

$$\Omega = dX_1 \wedge \cdots \wedge dX_{N-1}$$

Application I : D-branes

boundary

(A) $\left.\begin{matrix}\bar{Q}_+ + Q_- \\ Q_+ + \bar{Q}_-\end{matrix}\right\}$ preserved

(B) $\left.\begin{matrix}\bar{Q}_+ + \bar{Q}_- \\ Q_+ + Q_-\end{matrix}\right\}$ preserved

Ooguri-Oz-Yin

NL σ-M on X — symp. str. ω, complx. str. J

(with sup. pot W)

D-brane wrapped on $\gamma \subset X$

supporting gauge pot A on γ

(A) γ : Lagrangian submfd of (X, ω)

A : flat

(Im W const on γ)

(B) γ : complex submfd of (X, J)

A : holomorphic

(W : const on γ)

Govindarajan-Jayaraman-Sarkar
HIV

A-branes in X^{N-k} $\longleftrightarrow$ B-branes in LG

$\boxed{\begin{array}{lcl} \text{D(N-k)-brane at} & & \text{D0-brane at} \\ T^{N-k} = \{ |\varphi_i|^2 = c_i \} & \longleftrightarrow & Y_i = s_i \\ \text{Wilson line } a_i & & \end{array}}$

$c_i - i a_i = s_i$

Promote $s_i \rightsquigarrow S_i$ boundary chiral

constraint : $e^{-S_1} + \cdots + e^{-S_N} = \text{const.}$

$\Rightarrow$

more general Lagrangian submfds	$\longleftrightarrow$	higher dimensional holomorphic cycles

Disc instantons $\longleftrightarrow$ Classical Computation

- T^{N-k} in massive theory $\longleftrightarrow$ Algebraic computation in Open topological LG

- Special LAG in toric CY^3 $\longleftrightarrow$ Space-time superpotential as Open period integral

Aganagic - Vafa

cf. Witten   Kachru-Katz-Lawrence-McGreevy

B-branes in X $\longleftrightarrow$ A-branes in LG

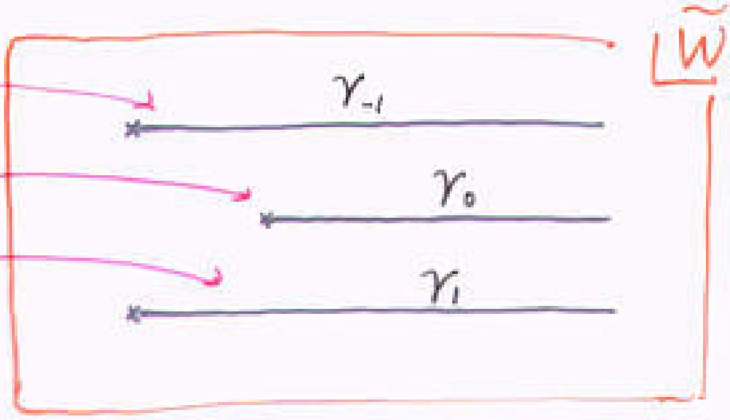
- $X = \mathbb{C}P^2$ $\qquad\qquad$ $\widetilde{W} = e^{-Y_1} + e^{-Y_2} + e^{-t+Y_1+Y_2}$

3 vacua $\qquad\qquad$ $e^{-Y_1} = e^{-Y_2} = e^{-t/3},\ \omega e^{-t/3},\ \omega^2 e^{-t/3}$

$\mathcal{O}(-1)$

$\mathcal{O}(0)$

$\mathcal{O}(1)$



$$\mathrm{Ext}^{\bullet}(\mathcal{O}(i), \mathcal{O}(j)) \xleftrightarrow[\text{Ground States}]{\text{Openstring}} HF^{\bullet}_{W}(\gamma_i, \gamma_j)$$

"Mutation of bundles" $\longleftrightarrow$ Brane Creation

c.f. Govindarajan's talk

- X = toric CY $\longleftrightarrow$ $\widetilde{W} = e^{-Y_0}(\text{———})$

"Stable" bundles $\qquad\qquad$ SLAG

c.f. Douglas's talk $\qquad\qquad$ or String junction

... Useful for Stability analysis

Application II : 3 d Mirror

3 d $\mathcal{N}=4$ R-sym $= SU(2)_V \times SU(2)_H$

Mirror Sym. $SU(2)_H \leftrightarrow SU(2)_V$

Higgs $\leftrightarrow$ Coulomb

- Abelian Mirror Pair (Conjecture : J. de Boer et al)

$U(1)^k \longleftrightarrow U(1)^{N-k}$

$Q_{1a} \cdots Q_{Na}$ hypers $\qquad \hat{Q}^p_1 \cdots \hat{Q}^p_N$ hypers

$$\sum_{i=1}^{N} Q_{ia} \hat{Q}^p_i = 0$$

$$\left(\begin{array}{c} \zeta = Q \cdot \hat{m} \\ \hat{Q} \cdot m = \hat{\zeta} \end{array} \right)$$

$\xrightarrow[\text{D. Tong}]{\text{"gauge" } U(1) \subset SU(2)_H \times SU(2)_V}$ Mirror Pair of 3 d $\mathcal{N}=2$ CS theories (c.f. N. Dorey - D. Tong)

$U(1)^k \longleftrightarrow U(1)^{N-k}$

$Q_{1a} \cdots Q_{Na}$ chirals $\qquad \hat{Q}^p_1 \cdots \hat{Q}^p_N$ chirals

$k = \frac{1}{2} Q \cdot Q \qquad \hat{k} = -\frac{1}{2} \hat{Q} \cdot \hat{Q}$

level

$$\left(\begin{array}{c} \zeta - \frac{1}{2} Q \cdot m = Q \cdot \hat{m} \\ \hat{Q} \cdot m = \hat{\zeta} + \frac{1}{2} \hat{Q} \cdot \hat{m} \end{array} \right)$$

Compactify on S^1_R

AHKT

Higgs $\xleftrightarrow{\text{Conjecture}}$ Coulomb

2d theory with $\Lambda_{UV} = \frac{1}{R}$

NL σ-Model on X^{N-k}_{toric}

$$W = -\sum_{\substack{i \\ n:KK}} \hat{\Sigma}_i(n)\left(\log \hat{\Sigma}_i(n) - 1\right)$$

$$\xrightarrow{R\to 0} \sum_{i=1}^{N} e^{-\hat{Q}_i^p \hat{\Sigma}_p - t_i}$$

derived

A New evidence of 3d Mirror

- Not merely an IR duality
- vortex $\longleftrightarrow$ electron

c.f. Aharony et al

Summary

Mirror Symmetry is derived.

$$\begin{cases} \text{T-duality} \\ \text{Vortex-instanton} \end{cases}$$

Applications :

$$\text{D-branes} \begin{cases} A \leftrightarrow B \\ B \leftrightarrow A \end{cases}$$

3d Mirror Symmetry

Outlook

1 Non-abelian Gauge theories

2 Less SUSY
- (0,2)
- No SUSY

3 More on D-branes / Orientifolds

4 More on higher dim. Gauge dynamics

4d $\mathcal{N}=1$ $SU(2)$ YM

"$W_{eff} = 2S\log S$" Veneziano-Yankielowicz

$\downarrow S^1_R$

3d $\mathcal{N}=2$

$$W = e^{-\Phi} + e^{-\frac{1}{R}+\Phi}$$

Seiberg-Witten

⋮

$\downarrow T^2$

$\mathbb{CP}^1$

//

2d (2,2) σ-model on $\mathcal{M}^{flat}_{T^2}(SU(2))$

$$W = \Sigma(Y_1+Y_2-t) + e^{-Y_1} + e^{-Y_2}$$

$\swarrow$ $\searrow$

$$W = e^{-Y_1} + e^{-t+Y_1} \qquad W = 2\Sigma\log\Sigma$$